STABLE RECONSTRUCTION OF THE VOLATILITY IN A REGIME-SWITCHING LOCAL-VOLATILITY MODEL

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Abstract. Prices of European call options in a regime-switching local-volatility model can be computed by solving a parabolic system which generalizes the classical Black and Scholes equation, giving these prices as functionals of the local-volatilities. We prove Lipschitz stability for the inverse problem of determining the local-volatilities from quoted call option prices for a range of strikes, if the calls are indexed by the different states of the continuous Markov chain which governs the regime switches.

1. **Introduction and main result.** As is known since the fundamental work of Black, Scholes and Merton, prices of European options and other financial derivatives can be computed by solving a final value problem for a parabolic linear partial differential equation (PDE) known as the Black and Scholes equation. The coefficients of this PDE will depend on the stochastic model, in the form of a stochastic differential equation or SDE driven by one or more Brownian motions, for the underlying asset of the derivative. Conversely, for derivative contracts which are liquidly traded, one would like to use the market-quoted derivative prices to infer information about the (parameters of the) underlying stochastic model. Once the latter are known, the model can be used to price less liquidly traded derivatives, including derivatives which are not market-quoted but privately exchanged between two parties (the so-called 'Over-the-Counter' contracts). Inferring the model from

observed prices is known as the *calibration problem*. By what we have just said, it is clear that the calibration problem for derivatives can be interpreted as an *inverse PDE problem*, that of determining one or more coefficients of a parabolic PDE from the observation of certain of its solutions. Consequently, the various techniques that have been developed to study inverse PDE problems can be made to bear on calibration. This point of view was first elaborated by Bouchouev and Isakov [7] and [8], who examined the problem of reconstructing the, by assumption time-independent, local-volatility function of a so-called *Local-Volatility model* from European call option prices of a fixed maturity but arbitrary strike, these prices being observed at some given point in time (and hence for some given value of the price of the underlying asset). We recall that a Local-Volatility (LV) model for the price of a risky asset is Brownian motion driven SDE for which the diffusion coefficient is itself a function of the asset-price and possibly also of time. Bouchouev and Isakov established uniqueness and, under additional assumptions, stability in Hölder norms. An essential ingredient in their work was Dupire's PDE [19] for European calls as functions of their strikes and maturities.

In this paper we will prove a Sobolev-norm version of Bouchouev's and Isakov's stability estimate. Moreover, we will do so for a larger class of models called *Regime Switching Local-Volatility* or RSLV models, also known as *Markov Modulated Local-Volatility models*. These generalise the Local-Volatility models by allowing the local-volatility function to jump according to an independent continuous-time Markov chain, thereby incorporating features of a Stochastic Volatility or SV model.

The stochastic ingredients of an RSLV model are a Brownian motion $(W_t)_{t\geq 0}$ and an independent finite state continuous time Markov process $(X_t)_{t\geq 0}$ with state space[1] $E = \{e_1, \ldots, e_n\}$. The Markov chain is specified through its generator $B = (b_{ij})_{1\leq i,j\leq n}$ by

$$\mathbb{P}\left(X_{t+dt} = e_i | X_t = e_j\right) = \delta_{ij} + b_{ij}dt. \tag{1}$$

Here B is a matrix with non-negative off-diagonal elements whose column-sums are equal to zero: $\sum_i b_{ij} = 0$. We will call the b_{ij} the transition probability rates, or more briefly, transition rates, of the continuous-time Markov chain X_t. Please note that some authors call b_{ji} what we have called b_{ij}, which amounts to replacing B by its transpose.

To specify the RSLV model, we are further given functions $r, q : E \to \mathbb{R}$ and $\sigma : \mathbb{R}_{\geq 0} \times E \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, to be interpreted as, respectively, the, Markov-state-dependent, interest and dividend rate[2] and the, equally state-dependent, local-volatility function. If we let S_t be the price at time t of the underlying asset of the derivative, then our model for S_t is:

$$dS_t = (r(X_t) - q(X_t))S_{t-}dt + \sigma(S_{t-}, X_t, t)S_{t-}dW_t, \tag{2}$$

where $S_{t-} := \lim_{\varepsilon\to 0+} S_{t-\varepsilon}$, complemented by the following pay-out rule for the dividend: the holder of the asset S_t will receive $q(X_t)S_t dt$ in dividends over the infinitesimal interval $[t, t+dt]$. It is important to note that we are working directly

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[1]Some authors, following [23], realise the states e_i as the canonical basis vectors of $\mathbb{R}^n$ (which amounts to identifying e_i with the function ϵ_i introduced below). Functions on E can then be identified with vectors in $\mathbb{R}^n$, the value of the function on e_i being the inner product with of the corresponding vector with e_i, and the process $(X_t)_{t\geq 0}$ itself can be given the form of a semi-martingale (which would not make sense for an arbitrary set E).

[2]Both can in principle be negative: a negative q would be a cost-of-carry, applicable if we are dealing commodities, and negative interest rates have occured in the recent past.

under a given risk-neutral probability measure, and not under the objective (or statistical) probability: indeed, under (2) the total expected return of the asset given S_t due to price change plus dividend over an infinitesimal time interval $[t, t + dt]$ is $r(X_t)dt$, the risk-free return in that interval. We will often simply write S_t instead of S_{t-}.

The model (2) is interesting even when the volatility function σ does not depend on S and t, in which case we are dealing with a regime switching geometric Brownian motion model. Such a model can be regarded as a simple type of stochastic volatility model, in which the stochastic driver of the volatility is a Markov chain instead of the Brownian motion which is traditionally used for stochastic volatility models, such as the models of Hull and White, of Stein and Stein and of Heston or the SABR model. Option pricing in regime-switching diffusion models is attracting an increasing interest in the literature: we mention [6] as an early reference, and as more recent ones [10], [11], [24], [25], [26], [29], [40] and [41].

One possible, and popular, economic interpretation of the model is that the state space E of the Markov chain represents different global states of the economy, for example, different phases of a business cycle. In each of these phases, which correspond to the different states $X_t = e_i$ of the Markov chain, the price S_t of the underlying asset evolves according to a Local-Volatility model with volatility function $\sigma_i(S, t) := \sigma(S, e_i, t)$ and interest rate and dividend rates $r_i := r(e_i)$, $q_i := q(e_i)$. For example, the risk-free base rate, as set by a Central Bank, tends to be low during a recession, and high during an economic boom, and one would expect the opposite behaviour for the dividend rate. In case $r : E \to \mathbb{R}$ is injective, investors can effectively observe the different states of the Markov chain. If not, we are dealing with a Hidden Markov model. In this paper we will assume that the Markov chain is observable. We will moreover assume that there exist European call options with a state dependent pay-off which again allows one to distinguish the different states. This last assumption will enable us to deduce a Dupire-type system of PDEs for European call options with coefficients given by the σ_i, r_i, q_i and b_{ij}.

Turning therefore to derivatives, consider a European derivative paying its holder an amount of $F(S_T, X_T)$ at maturity T, where $F : \mathbb{R}_{>0} \times E \to \mathbb{R}$ is a given function called the derivative's pay-off. (If $F(S_T, X_T)$ takes on negative values these are interpreted as the amount the holder of the derivative will have to pay to the seller of the derivative, at maturity). By the fundamental theorem of asset pricing, and since the process (S_t, X_t) is Markovian, the value of the derivative at an earlier time $t < T$ is given by

$$V(S_t, X_t, t) := \mathbb{E}\left(e^{-\int_t^T r(X_u)du} F(S_T, X_T) \,|\, S_t, X_t \right), \tag{3}$$

where the expectation is with respect to the risk-neutral measure and where V is a real-valued function on $\mathbb{R}_{>0} \times E \times \mathbb{R}$. If we let $V_j(S, t) := V(S, e_j, t)$, then an application of Ito's lemma shows that the V_j's satisfy the following Black and Scholes-type system of PDEs for $t < T$:

$$\partial_t V_j + \frac{1}{2}\sigma_j(S, t)^2 S^2 \partial_S^2 V_j + (r_j - q_j)S\partial_S V_j + \sum_i b_{ij} V_i = r_j V_j, \quad j = 1, \dots, n. \tag{4}$$

It is convenient to introduce the *row*-vector $(= 1 \times n$ matrix)

$$\underline{V}(S, t) := (V_1(S, t), \dots, V_n(S, t)),$$

as well as the diagonal $n \times n$-matrices $\Sigma(S,t) := \operatorname{diag}(\sigma_1(S,t),\ldots\sigma_n(S,t))$, $R := \operatorname{diag}(r_1,\ldots,r_n)$ and $Q := \operatorname{diag}(q_1,\ldots,q_n)$ and write the system as

$$\underline{V}_t + \frac{1}{2}S^2\underline{V}_{SS}\,\Sigma(S,t)^2 + S\underline{V}_S\,(R-Q) + \underline{V}\,(B-R) = 0, \tag{5}$$

where $\underline{V}_t := \partial_t\underline{V}$ and similarly for $\underline{V}_S$ and $\underline{V}_{SS}$, and where B is the matrix of transition probabilities of the Markov chain which was already defined above. (Note that these matrices multiply $\underline{V}$ and its derivatives on the right.) This system has to be supplemented by the final boundary condition

$$\underline{V}(S,T) = \underline{F}(S), \tag{6}$$

where $\underline{F}(S)$ is the row-vector $(F_1(S),\ldots,F_n(S))$ with $F_j(S) := F(S,e_j)$. If $n=1$, the Markov chain is trivial, and (4) reduces to the classical Black and Scholes equation for a Local-Volatility model.

We briefly discuss unique solvability of (5) - (6). We will assume that there exists two positive constants σ_{min} and σ_{max} such that

$$0 < \sigma_{\min} \leq \sigma_j(S,t) \leq \sigma_{\max} < \infty, \tag{7}$$

for all $j = 1,\cdots,n$ and $S,t > 0$. In particular, the system (5) will be parabolic. Since S is constrained to be positive we would, at first sight, need an additional boundary condition on $S = 0$. Note, however, that the principal symbol of (5) is not uniformly parabolic on $\{S > 0\}$, but degenerates at the boundary, and a logarithmic transformation $S = e^x$ and time-reversal $\tau := T - t$ transforms our problem into a Cauchy problem for a uniformly parabolic system on $\mathbb{R}^n \times \mathbb{R}_{>0}$: if we let $\underline{v}(x,\tau) := \underline{V}(e^x, T-\tau)$ and $A(x,\tau) := \frac{1}{2}\Sigma^2(e^x, T-\tau) = \operatorname{diag}(a_1(x,t),\ldots,a_n(x,t))$ with $a_j(x,\tau) = \sigma_j(e^x, T-\tau)$, then

$$\begin{cases} \underline{v}_\tau = \underline{v}_{xx}\,A(x,\tau) + \underline{v}_x\,(R-Q-A(x,\tau)) + \underline{v}\,(B-R) = 0, \\ \underline{v}(x,0) = \underline{f}(x), \end{cases} \tag{8}$$

where $\underline{f}(x) := \underline{F}(e^x)$. By the classical theory of parabolic systems (see for example [21] or [28]) if, in addition to (7), the $a_j(x,\tau)$ are continuous, bounded and C^α in x for some $0 < \alpha < 1$, uniformly in (x,τ), and if there exist constants $c,C > 0$ such that

$$|f_j(x)| \leq Ce^{c|x|^2}, \quad j = 1,\ldots,n, \tag{9}$$

then (8) will have a unique[3] solution subject to the components of $\underline{v}$ also satisfying (9), uniformly in τ. We note that assuming (7), the Hölder condition on $A(x,\tau)$ is equivalent to

$$\sup_{S_1,S_2,\tau>0} \frac{|\sigma_j(S_1,\tau) - \sigma_j(S_2,\tau)|}{|\log(S_2/S_1)|^\alpha} < \infty, \quad j = 1,\ldots,n.$$

We next specialise to a class of generalised European call options whose pay-offs are, by definition, of the form $F(S,X) := \max(S-K,0)\pi(X)$, where $\pi : E \to \mathbb{R}$ is a given function. The pay-off at T therefore depends on the Markov chain state the economy will be in at T. Including the pay-off parameters K, π and T in the notation, we will denote the resulting call option prices by row-vectors

$$\underline{C}(K,T,\pi;S,t) = (C_j(K,T,\pi;S,t))_{1\leq j\leq n},$$

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[3]Since our system is diagonal in the first and second order terms, we can use the maximum principle (more precisely, the Phragmen - Lindelöf principle) to prove uniqueness. For more general systems, uniqueness follows from solvability of the adjoint system, but for that the coefficients would need to be $C^{2,\alpha}$.

whose j-th component is the price at t when the state of the Markov chain at t is e_j:

$$C_j(K, T, \pi; S, t) := C(K, T, \pi; S, e_j, t).$$

Here, the option parameters K, T, π should be thought of as forward variables. As a function of S and t, the vector $\underline{C}(K, T, \pi; S, t)$ solves (5)-(6) with $F_j(S) := \pi_j \max(S - K, 0)$, where $\pi_j := \pi(e_j)$. Since the pay-off of the option at T is between $(\min_j \pi_j)S$ and $(\max_j \pi_j)S$, the same has to be true for each of the C_j, since otherwise the market would present arbitrage opportunities. In particular, the $|C_j(e^x, t)| \leq \text{Const. } e^x$ and $\underline{C}(K, T, \pi; S, t)$ is the unique solution of (5) with final pay-off $\max(S - K, 0)\underline{\pi}$, where $\underline{\pi}$ is the row vector with components π_j.

We now suppose that sufficiently many of such generalised call options are traded, in the sense that the associated $\underline{\pi}$'s will span $\mathbb{R}^n$. By taking suitable linear combinations, we can then take $\pi = \epsilon_i$, $i = 1, \ldots, n$, where $\epsilon_i : E \to \mathbb{R}$ is defined by

$$\epsilon_i(e_j) = \delta_{ij};$$

equivalently, ϵ_i is the indicator function of the singleton subset $\{e_i\} \subset E$. The associated call option will pay out $\max(S_T - K, 0)$ if the economy is in state i at T, and nothing if the economy finds itself in one of the other states. Such securities effectively provide a set of Arrow-Debreu securities for the economy, allowing investors to not only differentiate between different price levels of the underlying asset S at maturity T, but also between the different Markov-chain states. A classical European call, which corresponds to taking for π the function which is identically 1, does not allow this, since it will pay off the identical amount $\max(S_T - K, 0)$ regardless of the state X_T.

It is convenient to collect all possible call option values into a matrix-valued function $\mathcal{C}(K, T; S, t) = (C_{ij}(K, T; S, t))_{1 \leq i,j \leq n}$, where

$$C_{ij}(K, T; S, t) = C(K, \epsilon_i, T; S, e_j, t), \tag{10}$$

is the value at t of the generalised call with pay-off $\max(S - K, 0)\epsilon_j$ at T, when $S_t = S$ and $X_t = e_j$. This function $\mathcal{C}$ then satisfies

$$\begin{cases} \mathcal{C}_t + \frac{1}{2}S^2\mathcal{C}_{SS}\Sigma(S,t)^2 + S\mathcal{C}_S(R - Q) + \mathcal{C}(B - R) = 0, \\ \mathcal{C}(K, T; S, T) = \max(S - K, 0)I_n, \end{cases} \tag{11}$$

with I_n the $n \times n$ identity matrix. We will see in Section 2 below that, as function of K and T, $\mathcal{C}$ satisfies a closely related system which we will call the Dupire system and which reduces to the well-known Dupire equation when $n = 1$.

The instantaneous volatility function $\sigma(S, X, t)$ (momentarily reverting to the notation (2)) cannot be directly observed, though we can in principle infer the realised volatilites $\int_t^{t+h} \sigma(S_u, X_u, u)^2 du$, as a process in t, from historical data, for different time-windows h. Inferring $\sigma(S, X, t)^2$ from this is a non-trivial statistical problem and would, at best, provide information about this function for values of $S > 0$ and $X \in E$ which have been realised sufficiently often. This might not include values which are relevant for the, by their nature future-looking, call options. As regards the latter, the Regime-Switching Black-Scholes model (11) defines European call prices as a (non-linear) function of Σ, and if European call prices are liquidly quoted in the market, one may hope that such call prices determine the volatility. This is indeed the message of the classical Dupire equation, and of its generalisation to a RSLV model in Section 2 below, in the ideal case that European call option

prices quoted for the full range of possible maturities, strikes, and final Markov-chain states. In mathematical terms this means that the volatility can be identified from call option prices in a RSLV model. In practice, call prices will of course only be available for a finite number of strikes and maturities, and a naive use of Dupire's equation, in combination with some interpolation procedure to construct prices for the missing strikes and maturities, often leads to numerically unstable reconstructed volatility functions. This is not surprising since we are dealing with a typical ill-posed inverse PDE problem. The local-volatility function can be approximated by the Tikhonov regularization method for which several convergence and stability results, which take into account both the discreteness and the noisiness of the data, are available in the literature: see for instance [2], [3], [13], [16] and [20]. In the present paper we rather work in an idealized continuous setting, and use PDE techniques to establish a Lipschitz-type stability estimate for the unknown volatility in terms of a suitable Sobolev norm on the observed data, analogous to the one of [7] and [8] using Hölder norms. Such stability inequalities are helpful for designing iteration methods for the numerical reconstruction of the unknown volatility matrix, see e.g. [17].

We will suppose from now on, following [7] and [8], that the volatility is a function of S only, $\Sigma(S) = \operatorname{diag}\left((\sigma_1(S), \ldots, \sigma_n(S))\right)$: available maturities for which option prices are available are typically sparse, and one can concentrate on option prices with a single maturity, to construct volatility functions which piece-wise constant as function of time: cf. [7] and [8].

To state the necessary smoothness conditions on Σ, we introduce the space $C_b^k(\mathbb{R})$ of k-times continuously differentiable functions with uniformly bounded derivatives and its subspace $C_b^{k,\alpha}(\mathbb{R})$ of functions whose derivative of order k in addition satisfies a uniform Hölder condition of order $\alpha \in (0,1)$. We also need the Sobolev spaces $H^1(I), H^2(I)$ where $I \subset \mathbb{R}$ is an open interval. Vector- and matrix-valued functions will be said to belong to these spaces if all of their components do. Concerning the norms of the latter, we will use the following notational convention: for any Hilbert or Banach space X equipped the norm $\| \cdot \|_X$, and any $\mathbf{z} = (z_1, \ldots, z_n) \in X^n$ we put

$$\|\mathbf{z}\|_X^2 := \sum_{j=1}^n \|z_j\|_X^2, \tag{12}$$

instead of the more precise but more cumbersome $\|\mathbf{z}\|_{X^n}$.

We will also need the continuous-time Markov chain $(X_t)_{t \geq 0}$ to be *irreducible*, which means that for all $i \neq j$ there exists a $t > 0$ such the transition probability $\mathbb{P}(X_t = e_i | X_0 = e_j) > 0$: see for example [15]. The matrix of transition probabilities is known to be e^{tB} (as follows by integrating the forward Kolmogorov equation), and irreducibility of the Markov chain can be shown to be equivalent to all of the matrix elements of e^{tB} being strictly positive for all $t > 0$, which is the form in which we will use it. A Markov chain will certainly be irreducible if $b_{ij} > 0$ for all $i \neq j$, but this condition is not necessary.

Let $C_\Sigma(K, T; S, t)$ be the matrix of generalised European call options defined by (11), where we make the dependence on the volatility matrix Σ explicit. We seek to stably reconstruct $\Sigma(S)$ on some given interval $I \subset (0, \infty)$ from observed call prices of a fixed maturity T and with strikes K varying in a slightly larger interval $J \supseteq I$. If we observe these call prices at some fixed time t^* at which the Markov chain will be state $X_{t^*} = e_{j^*}$, then we will know the j^*-th column of $C_\Sigma(K, T; S^*, t^*)$ where

$S^* := S_{t^*}$, that is,

$$C_\Sigma^*(K, \epsilon_i) := C_\Sigma(K, T, \epsilon_i; S^*, e_{j^*}, t^*), \quad i = 1, \ldots, n. \tag{13}$$

The first main result of this paper is the following local stability estimate which in the case of $n = 1$ should be compared with Theorem 1 of [8], which gives a similar estimate for the C^α-norm of $\Sigma_1 - \Sigma_2$ in terms of the $C^{2,\alpha}$-norm of $C_{\Sigma_1}^* - C_{\Sigma_2}^*$. We set $\tau_* := T - t^*$ and recall our simplified notation (12) for norms of elements of vector-valued function spaces.

Theorem 1.1. *Suppose that the Markov chain $(X_t)_{t \geq 0}$ is irreducible, and that the coefficients of the (diagonal) matrix $\Sigma_1(e^y)$ are in $C_b^1(\mathbb{R})$, while those of $\Sigma_2(e^y)$ are in $C_b^{2,\alpha}(\mathbb{R})$ for some $\alpha > 0$, with both satisfying (7). Assume that*

$$\Sigma_1 = \Sigma_2 \ on \ \mathbb{R} \setminus I, \tag{14}$$

for some bounded interval $I \Subset (0, \infty)$. Then for any bounded open interval $J \Subset (0, \infty)$ fulfilling $I \Subset J$, there exists a constant $C > 0$, depending only on $\sigma_{\min}$, $\|\Sigma_1\|_{C^1(J)}$, $\|\Sigma_2\|_{C^{2,\alpha}(\mathbb{R})}$, I, J and τ_*, such that we have

$$\|\Sigma_1 - \Sigma_2\|_{L^2(I)}^2 \leq C \sum_{i=1}^n \left\| C_{\Sigma_1}^*(\cdot, \epsilon_i) - C_{\Sigma_2}^*(\cdot, \epsilon_i) \right\|_{H^2(J)}^2. \tag{15}$$

Note the different regularity assumptions on Σ_1 and Σ_2. The reason is technical: at certain points in the proof we use Gaussian bounds for the fundamental solution of a parabolic PDE which is essentially the adjoint of (8) with $A = \frac{1}{2}\Sigma_2^2$, and which are only known for parabolic PDEs with coefficients which are uniformly C^α. We can think of (15) as a first order Taylor estimate for the inverse function $C_\Sigma^* \to \Sigma$ at a point $C_{\Sigma_2}^*$ which is the image of a sufficiently regular Σ_2. It may be that both Σ_i having coefficients in $C_b^1(\mathbb{R})$ would suffice, but C^1 is the minimum regularity required, at least for our proof which is based on a Carleman estimate.

We also note that some condition on the Markov chain is necessary, as shown by the trivial example of $B = 0$, for which the system decouples. In that case, $C_\Sigma^*(K, T)$ will at best allow reconstruction of $\sigma_{j^*}(S)$, but not of the other elements of $\Sigma(S)$, as can be seen from the generalised Dupire equation (20) below.

If we do not want to assume that Σ_1 and Σ_2 coincide outside of an interval I, we have the following version of (15).

Theorem 1.2. *Suppose that Σ_1, Σ_2 and $(X_t)_{t \geq 0}$ satisfy the conditions of Theorem 1.1 except for (14), which is replaced by*

$$\Sigma_1 - \Sigma_2 \in L^2 \left(\mathbb{R}_{>0}, \frac{dS}{S} \right)^n. \tag{16}$$

Let $I \Subset J \Subset (0, \infty)$ be bounded intervals with J open, such that $S^* \notin I$. Then, there exists a constant $C > 0$, with the same dependence as in Theorem 1.1, such that

$$\|\Sigma_1 - \Sigma_2\|_{L^2(I)}^2 \ \leq \ C \left(\sum_{i=1}^n \left\| C_{\Sigma_1}^*(\cdot, \epsilon_i) - C_{\Sigma_2}^*(\cdot, \epsilon_i) \right\|_{H^2(J)}^2 + \|\Sigma_1 - \Sigma_2\|_{H^1(J \setminus I)}^2 \right.$$

$$\left. + \|\Sigma_1 - \Sigma_2\|_{L^2(\mathbb{R}_{>0} \setminus I, dS/S)}^2 \right). \tag{17}$$

Theorem 1.2 implies for example that (15) will be true modulo an error of the order of ε, if both $\|\Sigma_1 - \Sigma_2\|_{H^1(J \setminus I)} \|\Sigma_1 - \Sigma_2\|_{L^2(\mathbb{R}_{>0} \setminus I, dS/S)}$ are smaller than ε. It also shows that if, for a given $\Sigma(e^y) \in C_b^{2,\alpha}(\mathbb{R})$, we can find a sequence $\Sigma_\nu(e^y) \in C_b^1(\mathbb{R})$

such that $\left\|C_\Sigma^*(\cdot,\epsilon_i) - C_{\Sigma_\nu}^*(\cdot,\epsilon_i)\right\|_{H^2(J)}$, $\|\Sigma - \Sigma_\nu\|_{H^1(J\setminus I)}$ and $\|\Sigma - \Sigma_\nu\|_{L^2(\mathbb{R}_{>0}\setminus I, dS/S)}$ all tend to 0, while $\|\Sigma_\nu\|_{C^1(J)}$ remains bounded, then $\Sigma_\nu \to \Sigma$ in $L^2(I)$ as $\nu \to \infty$.

We end this introduction with some general remarks to situate our work within the existing literature. We already noted that this paper contributes to the growing literature on Markov-modulated diffusion processes and their applications in Finance. The problem of reconstructing a local-volatility function from observed option prices has attracted a lot of attention in the mathematical finance literature, ever since the publication of Dupire's paper [19]. An important strand of the literature focusses on solutions using Tikhonov regularisation: without claiming to be exhaustive we cite Crépey [12, 13], Egger and Engl [20], De Cezaro, Scherzer and Zubelli [16], and Albani, De Cezaro and Zubelli [2, 3]. The papers [36] by Lishang and Youshan and [18] by Deng, Yu and Yang, use an optimal control approach, while [9] by Bouchouev, Isakov and Valdiviaand, and [32] by Isakov, use linearization around a constant volatility. In [7, 8] Bouchouev and Isakov prove uniqueness and stability in Hölder norm. We finally mention Cristofol and Roques [14], who show existence of a one-to-one relationship between the diffusion coefficient of a diffusion process X_t defined by a stochastic differential equation and the expectation of a strictly concave or convex function of X_t observed during a small time interval.

All these papers only consider the scalar case (corresponding to $n = 1$), without regime switching. Our paper is closest to, and inspired by, Bouchouev and Isakov [7] and [8], but besides examining the regime-switching model we establish estimates in Sobolev norms instead of Hölder norms. Our proof of Theorems 1.1 and 1.2 uses the Bukhgeim-Klibanov method, as in Imanuvilov and Yamamoto's [31], though we will only need a Carleman estimate which is local in space. The system of PDEs we have to work with is only coupled in the 0-th order terms, and diagonal in the first and second order. This makes that much of the analysis for the scalar Dupire PDE carries through automatically, but we do need at one point strict positivity of the matrix coefficients of the fundamental solution, which is true iff the underlying Markov chain of the model is irreducible. This result, which we prove in Appendix B, seems of some interest in its own right. Another technical problem occurs because of the non-smoothness of the call option pay-off at the strike K which, after linearization of the inverse problem, leads to non-homogeneous parabolic problem whose right hand side has a fundamental-solution type singularity at time 0: see Section 4. This singularity prevents us from applying standard parabolic energy estimates where we would have liked to, and we have had to replace these by certain weighted energy estimates derived in Section 3, as well as by Schur -type estimates for the norms of certain integral operators in Appendix C.

2. **The Dupire equation.** The inverse problem of determining the local volatilities $\sigma_i(S)$ from observed option prices can be reformulated as a coefficient identification problem for a parabolic system of PDEs in K and T. This is a consequence of the following result, which generalises the well-known Dupire equation [19] to the regime-switching case.

Theorem 2.1. *Suppose that the function $A(x,\tau) := \Sigma(e^x, T - \tau)$ is uniformly elliptic, continuous and bounded and C^1 in x, with uniformly bounded derivative. Then, as function of K and T, the matrix of call option prices $\mathcal{C}$ satisfies the forward parabolic system*

$$\mathcal{C}_T = \frac{1}{2}K^2\Sigma(K,T)^2\mathcal{C}_{KK} - K(R-Q)\mathcal{C}_K + (B-Q)\mathcal{C}, \quad T > t, \qquad (18)$$

with boundary condition

$$\mathcal{C}(K, t; S, t) = \max(S - K, 0) I_n, \tag{19}$$

at $T = t$.

We refer to Appendix A for a proof. We note that in [40] the authors consider the inverse problem in the case where the volatility matrix is a function $\Sigma = \Sigma(t)$ of time only and which, in a certain sense, is complementary to the case considered in this paper. They established a Dupire equation for classical call options with state-independent pay-offs that is different from ours, and which can be derived by observing that if the volatilities σ_i do not depend on S, then the call option prices $C_j = C(K, T, \mathbf{1}; S, t, e_i)$ are homogeneous of degree 1 in (S, K). The Euler relations for homogeneous functions then enables one to re-write the Black and Scholes system (11) as a system of PDEs in K and T. Multiplying on the left by the row-vector $\mathbf{1}$ we then find for this case a Dupire-type system satisfied by *classical call* options $C_j = C(K, T, \mathbf{1}; S, t, e_i)$. For general dependend $\Sigma(S, t)$ this does not seem to be possible: in equation (18), the matrix $\mathcal{C}$ and its derivatives multiply the coefficients on the left instead of on the right, so left-contraction with the row-vector $\mathbf{1}$ is no longer possible.

If we now fix $S^* > 0$, $t^* > 0$ and $j^* \in \{1, \ldots, n\}$, then the column vector

$$\mathbf{u}(K, T) := (C(K, T, \epsilon_i; S^*, t^*, e_{j^*})_{1 \le i \le n},$$

(which is simply the j^*-th column of $\mathcal{C}$) satisfies the parabolic system

$$\partial_T \mathbf{u} = \frac{1}{2} K^2 \Sigma(K, T)^2 \mathbf{u}_{KK} - K(R - Q)\mathbf{u}_K + (B - Q)\mathbf{u}, \quad T > t^*, \tag{20}$$

with boundary condition $\mathbf{u}(K, t^*) = \max(S^* - K)e_{j^*}$, where from now we will identify e_j with the j-th canonical basis vector of $\mathbb{R}^n$. The logarithmic substitution

$$y = \ln(K/S^*), \ \tau = T - t^*, \ \mathbf{w}_A(y, \tau) = \mathbf{u}(K, T),$$

transforms the initial value problem for $\mathbf{u}(K, T)$ into

$$\begin{cases} (\partial_\tau - \mathcal{L}_A)\mathbf{w}_A = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ \mathbf{w}_A(y, 0) = S^* \max(1 - e^y, 0)e_{j^*}, & y \in \mathbb{R}, \end{cases} \tag{21}$$

where

$$\mathcal{L}_A \mathbf{w} = A(y)\mathbf{w}_{yy} - (A(y) + R - Q)\mathbf{w}_y + (B - Q)\mathbf{w}, \tag{22}$$

with $A(y) := \frac{1}{2}\Sigma^2(e^y)$, and where we made the dependence of the solution on the coefficient A explicit. This initial value problem is uniquely solvable if we require that the solution $\mathbf{w}_A$ be bounded (note that the solution is unique if it satisfies the bound (9), but since the initial value is bounded, $\mathbf{w}_A$ will then automatically be bounded, by the maximum principle for weakly coupled parabolic systems [38]).

Our inverse problem can therefore be rephrased as follows: letting

$$\Omega_1 := \{\ln(K/S^*), \ K \in I\} \Subset \Omega := \{\ln(K/S^*), \ K \in J\},$$

can we stably retrieve $A(y)$ on Ω_1 from knowing $\mathbf{w}_A(y, \tau_*)$ on Ω, where $\tau_* := T - t^*$ and $\mathbf{w}_A$ is the unique bounded solution of (21)?

3. **Parabolic estimates for the Dupire equation.** Our proof of Theorem 1.1 uses a Carleman estimate for parabolic equations which is local in the space-variable y. To state this estimate, let $\psi \in C^2(\overline{\Omega})$ with $\psi'(y) \neq 0$ on $\overline{\Omega}$. We put $T^* := 2\tau_*$ and $\ell(\tau) := \tau(T^* - \tau)$. Next, for $L > \|\psi\|_\infty$, define

$$\eta(y, \tau) := \eta_\lambda(y, \tau) := \frac{e^{\lambda\psi(y)} - e^{\lambda L}}{\ell(\tau)}, \ y \in \Omega, \ \tau \in (0, T^*),$$

where λ will be chosen sufficiently large below. Note that $\eta(y, \tau) < 0$ on $Q := \Omega \times (0, T^*)$, and that $\eta(y, \tau) \to -\infty$ as $\tau \to 0, T^*$.

Let

$$H^{2,1}(Q) = \left\{ z \in L^2(Q), \ z_y, \ z_{yy}, z_\tau \in L^2(Q) \right\},$$

be the usual Sobolev parabolic space, endowed with the norm

$$\|z\|^2_{H^{2,1}(Q)} = \|z\|^2_{L^2(Q)} + \|z_y\|^2_{L^2(Q)} + \|z_{yy}\|^2_{L^2(Q)} + \|z_\tau\|^2_{L^2(Q)},$$

which we extend in the natural way to vector-valued functions via (12). We next state the parabolic Carleman estimate which we will use and which, since the principal symbol of $\mathcal{L}_A$ is diagonal, is an immediate consequence of a scalar Carleman estimate established in [30] and [31]; see also [35], Theorem 7.3 for an approach using semi-classical analysis.

Theorem 3.1. *Suppose that $A = (a_i \delta_{ij})_{i,j}$ is diagonal with $a_i \in C^1(\Omega)$, for all i, and let $K \subset \Omega$ be compact. Then, there exists a $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$, we may find constants $s_0 > 0$ and $C > 0$ such that, for all $\mathbf{z} = (z_1, \ldots, z_n) \in H^{2,1}(Q)^n$ with $\operatorname{supp}(z_i) \subset K \times [0, T^*]$ and all $s \geq s_0$, we have*

$$\| \left(\frac{s}{\ell}\right)^{3/2} e^{s\eta} \mathbf{z} \|^2_{L^2(Q)} + \| \left(\frac{s}{\ell}\right)^{1/2} e^{s\eta} \mathbf{z}_y \|^2_{L^2(Q)} + \| \left(\frac{s}{\ell}\right)^{-1/2} e^{s\eta} \mathbf{z}_\tau \|^2_{L^2(Q)}$$
$$\leq C \quad \|e^{s\eta}(\partial_\tau - \mathcal{L}_A)\,\mathbf{z}\,\|_{L^2(Q)}, \tag{23}$$

where the constant C depends on the C^1-norm of A and on λ but is independent of s.

We will also need a technical result in the form of a local parabolic regularity estimate with Carleman weights, which we state in slightly greater generality than needed. Consider a parabolic $n \times n$ system

$$\partial_\tau \mathbf{u} - \partial_y(C_2 \partial_y \mathbf{u}) - C_1 \partial_y \mathbf{u} - C_0 \mathbf{u} = \mathbf{f}, \tag{24}$$

on $Q = \Omega \times (0, T^*)$ with real-matrix valued coefficients $C_i = C_i(y)$ and real symmetric positive-definite $C_2 = C_2(y)$. Solutions will be interpreted in the either the strong or the weak sense, depending on context, and we will limit ourselves to solutions with real-valued components.

Lemma 3.2. *Suppose that the coefficients C_0 and C_1 of (24) are locally bounded on Ω and that C_2 is of class C^1 there, while $\mathbf{f} \in L^2_{\mathrm{loc}}(Q)^n$. Let $\gamma \in [0, +\infty)$ and let $\Delta \Subset D \Subset \Omega$. Then, there exists a constant $C = C(\gamma, \Delta, D) > 0$ such that for all $\mathbf{u} \in H^{2,1}(Q)^n$ solution of (24) and all $s \geq s_0 > 0$,*

$$\|e^{s\eta}\ell^{-\gamma}\mathbf{u}_y\|^2_{L^2(\Delta \times (0,T^*))}$$
$$\leq \quad C \left(s^2 \|e^{s\eta}\ell^{-\gamma-1}\mathbf{u}\|^2_{L^2(D \times (0,T^*))} + s^{-2}\|e^{s\eta}\ell^{-\gamma+1}\mathbf{f}\|^2_{L^2(D \times (0,T^*))} \right). \tag{25}$$

If, moreover, $\mathbf{f}_y \in L^2_{\text{loc}}(Q)^n$ *and* C_0 *is* C^1 *also, then*

$$\|e^{s\eta}\ell^{-\gamma}\mathbf{u}_\tau\|^2_{L^2(\Delta\times(0,T^*))}$$

$$\leq\ C\left(s^4\|e^{s\eta}\ell^{-\gamma-2}\mathbf{u}\|^2_{L^2(D\times(0,T^*))} + \|e^{s\eta}\ell^{-\gamma}\mathbf{f}\|^2_{L^2(D\times(0,T^*))}\right.$$

$$\left. +s^{-2}\|e^{s\eta}\ell^{-\gamma+1}\mathbf{f}_y\|^2_{L^2(D\times(0,T^*))}\right). \tag{26}$$

Proof. Let $0 \leq \chi \in C^\infty_c(\mathbb{R})$ be supported in D with $\chi(y) = 1$ for all $y \in \Delta$, and put $w := \ell^{-2\gamma}\chi e^{2s\eta}$. Then, with $(\cdot,\cdot)$ denoting the inner product of $L^2(Q)^n$,

$$-((C_2\mathbf{u}_y)_y, w\mathbf{u}) = (\mathbf{f}, w\mathbf{u}) - (\mathbf{u}_\tau, w\mathbf{u}) + (C_1\mathbf{u}_y, w\mathbf{u}) + (C_0\mathbf{u}, w\mathbf{u}), \tag{27}$$

and we estimate the different terms using integration by parts. For the term on the left we note that $-((C_2\mathbf{u}_y)_y, w\mathbf{u}) = (C_2\mathbf{u}_y, w\mathbf{u}_y) + (C_2\mathbf{u}_y, w_y\mathbf{u})$, while $(C_2\mathbf{u}_y, w_y\mathbf{u}) = -(C_2\mathbf{u}, w_y\mathbf{u}_y) - ((w_yC_2)_y\mathbf{u}, \mathbf{u})$, so that by the symmetry of C_2,

$$2|(C_2\mathbf{u}_y, w_y\mathbf{u})| = |(\mathbf{u}, (w_yC_2)_y\mathbf{u})| \leq Cs^2\|\ell^{-\gamma-1}e^{s\eta}\mathbf{u}\|^2_{L^2(D\times(0,T^*))},$$

where we used that $|\partial^\alpha_y w| \leq (s/\ell)^\alpha w$ for $\alpha \in \mathbb{N}$. Since $C_2(y)$ is continuous and positive-definite for all $y \in \Omega$ and since D is compactly contained in Ω, it follows that there exists a $c > 0$ such that

$$-((C_2\mathbf{u}_y)_y, w\mathbf{u}) \geq c(\mathbf{u}_y, \mathbf{u}_y w) - Cs^2\|e^{s\eta}\ell^{-\gamma-1}\mathbf{u}\|^2_{L^2(D\times(0,T^*))}. \tag{28}$$

Turning to the terms on the right hand side of (27), we can trivially bound the 0-th order term by a constant times $(\mathbf{u}, w\mathbf{u})$, while

$$|(C_1\mathbf{u}_y, w\mathbf{u})| \leq C(\mathbf{u}, w\mathbf{u})^{1/2}(\mathbf{u}_y, w\mathbf{u}_y)^{1/2} \leq C\varepsilon(\mathbf{u}_y, w\mathbf{u}_y) + C\varepsilon^{-1}(\mathbf{u}, w\mathbf{u}),$$

where $\varepsilon > 0$ is arbitrary. Next, using that $\lim_{\tau\to 0,T^*}\ell^{-2\gamma}e^{2s\eta} = 0$ for any $s > 0$, integration by parts with respect to the τ-variable shows that $(\mathbf{u}_\tau, w\mathbf{u}) = -(\mathbf{u}, w\mathbf{u}_\tau) - (\mathbf{u}, w_\tau\mathbf{u})$, and hence

$$2|(\mathbf{u}_\tau, w\mathbf{u})| = |(\mathbf{u}, w_\tau\mathbf{u})| \leq Cs\|\ell^{-\gamma-1}e^{s\eta}\mathbf{u}\|^2_{L^2(D\times(0,T^*))},$$

since $|\eta_\tau| \leq c\ell^{-2}$ and $\partial_\tau(\ell^{-2\gamma}) \leq c\ell^{-2\gamma-1}$. Finally, if we write

$$(\mathbf{f}, w\mathbf{u}) = ((\ell/s)w^{1/2}\mathbf{f}, (s/\ell)w^{1/2}\mathbf{u}),$$

and use Young's inequality (for products), we see that $|(\mathbf{f}, w\mathbf{u})|$ can be bounded by $s^2(\mathbf{u}, \ell^{-2}w\mathbf{u}) + s^{-2}(\mathbf{f}, \ell^2 w\mathbf{f})$. Combining all these estimates with (28) and choosing ε sufficiently small, (25) follows.

Next, if we differentiate the PDE with respect to y we find that $\mathbf{u}_y$ is a solution in the weak sense of

$$\partial_\tau\mathbf{u}_y - \partial_y(C_2\partial_y\mathbf{u}_y) - \partial_y\left(\widetilde{C}_1\mathbf{u}_y\right) - C_0\mathbf{u}_y = \mathbf{f}_y + (\partial_yC_0)\mathbf{u},$$

with $\widetilde{C}_1 := C_1 + \partial_yC_2$. This is a system for $\mathbf{u}_y$ of the type (24) but with first-order term $C_1\partial_y\mathbf{u}$ replaced by $\partial_y(C_1\mathbf{u})$. It is clear from the proof that the bound (25) still remains valid for such a system and for weak solutions. Applying this bound with D replaced by $\Delta \Subset \Delta_1 \Subset D$, we find that

$$\|e^{s\eta}\ell^{-\gamma}\mathbf{u}_{yy}\|^2_{L^2(\Delta\times(0,T^*))}$$

$$\leq\ C\left(s^2\|e^{s\eta}\ell^{-\gamma-1}\mathbf{u}_y\|^2_{L^2(\Delta_1\times(0,T^*))} + s^{-2}\|e^{s\eta}\ell^{-\gamma+1}\mathbf{f}_y\|^2_{L^2(\Delta_1\times(0,T^*))}\right.$$

$$\left. +s^{-2}\|e^{s\eta}\ell^{-\gamma+1}\mathbf{u}\|^2_{L^2(\Delta_1\times(0,T^*))}\right).$$

If we next apply (25) again, with Δ_1 instead of Δ and $\gamma + 1$ instead of γ, we find that

$$\|e^{s\eta}\ell^{-\gamma}\mathbf{u}_{yy}\|^2_{L^2(\Delta \times (0,T^*))}$$
$$\leq C\left(s^4\|e^{s\eta}\ell^{-(\gamma+2)}\mathbf{u}\|^2_{L^2(D\times(0,T^*))} + \|e^{s\eta}\ell^{-\gamma}\mathbf{f}\|^2_{L^2(D\times(0,T^*))}\right.$$
$$\left. + s^{-2}\|e^{s\eta}\ell^{-\gamma+1}\mathbf{f}_y\|^2_{L^2(D\times(0,T^*))}\right).$$

Since $\mathbf{u}_\tau = A_2(y)\mathbf{u}_{yy} + P_1(y,\partial_y)\mathbf{u} + \mathbf{f}$ with P_1 a first order operator, this together with (25) implies (26). □

4. **Stability estimate for the linearized inverse problem.** Recall that $\Omega_1 = \{y = \log(K/S^*) : K \in I\} \Subset \Omega = \{\log(K/S^*) : K \in J\}$. We will from now on assume, without essential loss of generality, that $S^* = 1$. Let $A_i(y) := \frac{1}{2}\Sigma_i(e^y)^2$ for $i = 1, 2$. It follows from (21) that $\mathbf{w} = \mathbf{w}_{A_1} - \mathbf{w}_{A_2}$ is the unique bounded solution of the linearized system

$$\begin{cases} (\partial_\tau - \mathcal{L}_{A_1})\mathbf{w} = G\mathbf{v}, & \text{in } \mathbb{R} \times (0,T), \\ \mathbf{w}(y,0) = 0, & \text{in } \mathbb{R}, \end{cases} \tag{29}$$

with $G = A_1 - A_2$ supported on Ω_1, and $\mathbf{v} = (\mathbf{w}_{A_2})_{yy} - (\mathbf{w}_{A_2})_y$. Conversely, the solution of (29) can be written as such a difference, by taking $A_2 = A_1 - G$. We want to bound G in terms of $\mathbf{w}(\cdot, \tau_*)$. Note that G, although in principle matrix-valued, is diagonal, and so is given by n functions, as is $\mathbf{w}(\cdot, \tau_*)$. If we interpret this as an inverse problem, it is not underdetermined.

As a preparation, we first examine the properties of $\mathbf{v}(y, \tau)$ which, as we will now explain, is related to the fundamental solution of an auxilliary parabolic system. Indeed, it is straightforward to verify that

$$\partial_\tau \mathbf{v} - (\partial_y^2 - \partial_y)(A_2(y,\tau)\mathbf{v}) - (R - Q)\partial_y\mathbf{v} + (B - Q)\mathbf{v} = 0, \tag{30}$$

with initial condition (recalling that $S^* = 1$)

$$\mathbf{v}(y,0) = (\partial_y^2 - \partial_y)\max(1 - e^y, 0)e_{j^*} = \delta(y)\,e_{j^*}. \tag{31}$$

Hence $\mathbf{v}(y, \tau)$ is the j^*-th column of the fundamental solution of the parabolic system (30), and as such has the following properties.

Lemma 4.1. *Suppose that the coefficients of the matrix A_2 are in $C_b^{2,\alpha}(\mathbb{R})$ for some $\alpha \in (0,1)$. Then, $\mathbf{v} \in C^{2,1}(\mathbb{R} \times \mathbb{R}_{>0})$ and there exist for any $T > 0$ two positive constants c and C such that for $y \in \mathbb{R}$ and $0 < \tau \leq T$,*

$$|\partial_y^\nu \mathbf{v}(y,\tau)| \leq \frac{C}{\tau^{(\nu+1)/2}}e^{-c|y|^2/\tau}, \quad \nu = 0, 1, 2. \tag{32}$$

If, moreover, $\left(e^{\tau B}\right)_{ij^*} > 0$ for all i, then all components of $\mathbf{v}(y, \tau)$ remain strictly positive on compacta: there exists, for any compact subset $K \subset \mathbb{R}$ and any $\tau > 0$, a constant $C_\tau > 0$ such that

$$\inf_{y \in K} v_i(y,\tau) > C_\tau, i = 1, \ldots, n. \tag{33}$$

For the first part of the theorem, see [21] and [28] (in particular Theorem 2 of Chapter 9 plus Equation (4.19)), or [22], Theorem 2.2. The constants will depend on the constant $\sigma_{\min}$ of (7) and on $\|A_2\|_{C^{2,\alpha}}$, as well as, of course, on B, R and Q. Note that since $\mathbf{v}$ satisfies (30), the estimate (32) implies that

$$|\partial_\tau \mathbf{v}(y,\tau)| \leq \frac{C}{\tau^{3/2}}e^{-c|y|^2/\tau}. \tag{34}$$

The strict lower bound (33) will be proved in Appendix B; it sharpens a result of Otsuka [37]. The condition $\left(e^{\tau B}\right)_{ij^*} > 0$ means that any state i can be reached, with positive probability, from the initial state j^*. The validity of this condition for arbitrary j^*, that is, the non-vanishing of all matrix elements of $e^{\tau B}$ for arbitrary $\tau > 0$, is equivalent to the irreducibility of the continuous time Markov chain with transition-density matrix B.

Remark 1. It is because of this lemma that we have to impose the condition that Σ_2, and therefore A_2, is of class $C^{2,\alpha}$ in Theorem 1.1. If A_2 is only C^1, we can write $(\partial_y^2 - \partial_y)A_2 = \partial_y A_2 \partial_y - \partial_y (A_2 + [A_2, \partial_y])$, and interpret $\mathbf{v}$ as the weak solution of a parabolic system of the form $\partial_\tau \mathbf{v} - \partial_y A_2 \partial_y \mathbf{v} + \partial_y (E\mathbf{v}) + F\mathbf{v} = 0$, with E continuous, but we do not know whether the Gaussian estimates for in particular the derivatives of $\mathbf{v}$ still remain valid then.

The following lemma is the main technical step in the proof of our stability estimate. Recall that, by hypothesis, $G(y) = 0$ outside of Ω_1. Let $\chi \in C_c^\infty(\Omega)$ be equal to 1 on a neighborhood of $\overline{\Omega}_1$, and let $\omega \subset \Omega \backslash \overline{\Omega}_1$ be an arbitrary neighborhood of $\operatorname{supp}(\chi_y)$. Note that ω will be disconnected, but that we can, and will, arrange for ω to have exactly two components, ω_1 and ω_2.

Lemma 4.2. *There exists a constant $C > 0$, that depends only on B, Ω_1, Ω_2, τ^*, n, $\|A_1\|_{C^1(\mathbb{R})}$ and $\|A_2\|_{C^{2,\alpha}(\mathbb{R})}$, such that for any $G \in L^2(\mathbb{R})^n$ supported in Ω_1, the solution $\mathbf{w}$ of (29) satisfies the estimate*

$$\|G\|_{L^2(\Omega_1)}^2 \leq C \left(\|\mathbf{w}(\cdot, \tau_*)\|_{H^2(\Omega)}^2 + \|\mathbf{w}\|_{L^2(\omega \times (0,T))}^2 \right). \tag{35}$$

Proof. Since $\chi G = G$ and since G and $\mathbf{w}$ are related by (29), we have

$$
\begin{aligned}
& \|e^{s\eta(y,\tau_*)} G(y)\mathbf{v}(y,\tau_*)\|_{L^2(\mathbb{R})}^2 \\
= \;& \|e^{s\eta(y,\tau_*)}\chi(y)G(y)\mathbf{v}(y,\tau_*)\|_{L^2(\mathbb{R})}^2 \\
\leq \;& 2\|e^{s\eta(y,\tau_*)}\chi(y)\mathbf{w}_\tau(y,\tau_*)\|_{L^2(\mathbb{R})}^2 + 2\|\chi(y)\mathcal{L}_{A_1}\mathbf{w}(y,\tau_*)\|_{L^2(\mathbb{R})}^2 \\
\leq \;& 2\|e^{s\eta(y,\tau_*)}\chi(y)\mathbf{w}_\tau(y,\tau_*)\|_{L^2(\mathbb{R})}^2 + C\|\mathbf{w}(y,\tau_*)\|_{H^2(\Omega)}^2,
\end{aligned}
\tag{36}
$$
$$\tag{37}$$

with C depending on $\|A_1\|_\infty$ and on $\|R\|$, $\|B\|$ and $\|Q\|$. We next seek to control the first term on the right by applying the Carleman estimate of Theorem 3.1 to the function $\mathbf{z} = \chi \mathbf{w}_\tau$, which satisfies the PDE

$$\mathbf{z}_\tau - \mathcal{L}_{A_1}\mathbf{z} = G\mathbf{v}_\tau + \mathcal{Q}_1 \mathbf{w}_\tau, \tag{38}$$

where $\mathcal{Q}_1 := [\chi, \mathcal{L}_{A_1}]$ is a first order partial differential operator in y with coefficients supported in $K := \operatorname{supp}(\chi_y) \subset \omega$. Since $\lim_{\tau \downarrow 0} e^{2s\eta(y,\tau)} = 0$, the first term on the right hand side of (37) can be bounded by

$$
\begin{aligned}
& \|e^{s\eta(y,\tau_*)}\mathbf{z}(y,\tau_*)\|_2^2 = \int_0^{\tau_*} \partial_\tau \|e^{s\eta}\mathbf{z}\|_2^2 d\tau \\
= \;& 2\int_0^{\tau_*} (s\eta_\tau \mathbf{z} + \mathbf{z}_\tau, \mathbf{z})_{L^2(\mathbb{R})} \, e^{2s\eta(y,\tau)} d\tau \\
\leq \;& Cs^{-1} \left(\|\frac{s}{\ell}\mathbf{z}e^{s\eta}\|_{L^2(\Omega \times (0,\tau_*))}^2 + |(s^{-1/2}\mathbf{z}_\tau, s^{3/2}e^{2s\eta}\mathbf{z})_{L^2(\Omega \times (0,\tau_*))}| \right),
\end{aligned}
$$

where we used that $|\eta_\tau(y,\tau)| \le C/\ell(\tau)^2$. Since $\ell(\tau)$ is bounded on $(0,T^*)$,

$$|(s^{-1/2}\mathbf{z}_\tau, s^{3/2}e^{2s\eta}\mathbf{z})_{L^2(\Omega\times(0,\tau_*))}|$$

$$\le C\left|\left((s/\ell)^{-1/2}\mathbf{z}_\tau, (s/\ell)^{3/2}e^{2s\eta}\mathbf{z}\right)_{L^2(\Omega\times(0,\tau_*))}\right|$$

$$\le C\left(\|(s/\ell)^{-1/2}e^{s\eta}\mathbf{z}_\tau\|^2_{L^2(\Omega\times(0,\tau_*))} + \|(s/\ell)^{3/2}e^{s\eta}\mathbf{z}\|_{L^2(\Omega\times(0,\tau_*))}\right).$$

Replacing the L^2-norms in the last line by those of $L^2(Q)$, the Carleman estimate (23) and Equation (38) then imply that for sufficiently large s,

$$\|e^{s\eta(y,\tau_*)}\mathbf{z}(y,\tau_*)\|^2_{L^2(Q)}$$

$$\le Cs^{-1}\left(\|e^{s\eta}G\mathbf{v}_\tau\|^2_{L^2(Q)} + \|e^{s\eta}\mathbf{w}_\tau\|^2_{L^2(K\times(0,T^*))} + \|e^{s\eta}\mathbf{w}_{\tau y}\|^2_{L^2(K\times(0,T^*))}\right),$$

since $\mathcal{Q}_1$ is a first order partial differential operator supported on K. Inserting this into (37) we find that

$$\|e^{s\eta(y,\tau_*)}G(y)\mathbf{v}(y,\tau_*)\|^2_{L^2(\Omega_1)}$$

$$\le Cs^{-1}\left(\|e^{s\eta}G\mathbf{v}_\tau\|^2_{L^2(\Omega_1\times(0,T^*))} + \|e^{s\eta}\mathbf{w}_\tau\|^2_{L^2(K\times(0,T^*))}\right.$$

$$\left. + \|e^{s\eta}\mathbf{w}_{\tau y}\|^2_{L^2(K\times(0,T^*))}\right) + C\|\mathbf{w}(y,\tau_*)\|^2_{H^2(\Omega)}. \qquad (39)$$

We next show that the first term on the right can be absorbed into the left hand side if s is sufficiently large. We claim that there exists a constant $C > 0$ such that for all $y \in \Omega$,

$$\sup_{\tau\in(0,T^*)} |\mathbf{v}_\tau(y,\tau)|^2 e^{2s\eta(y,\tau)} \le Ce^{2s\eta(y,\tau_*)}. \qquad (40)$$

This is basically a consequence of the fact that, by construction, $\eta(y,\tau)$ has its maximum on $(0,T^*)$ in $\tau = T^*/2 = \tau_*$ but we have to take some care with the singularity of $\mathbf{v}_\tau(\cdot,\tau)$ in $\tau = 0$. Write $\eta(y,\tau) = -\widetilde{\eta}(y)/\ell(\tau)$, with $\widetilde{\eta}(y) = e^{\lambda L} - e^{\lambda\psi}$ and let $0 < a < \tau_*$. Since $\ell(\tau)^{-1}$ assumes its absolute minimum on $(0,T)$ in $\tau = \tau_*$, there exists an $\varepsilon > 0$ such that $(1-\varepsilon)\inf_{(0,a]}\ell(\tau)^{-1} > \ell(\tau_*)^{-1}$. By (34), $|\mathbf{v}_\tau(y,\tau)| \le C\tau^{-3/2}$ on Ω. Since $\inf_\Omega \widetilde{\eta} > 0$, there exists a constant C_ε such that $\tau^{-3} \le C_\varepsilon e^{2s\varepsilon\widetilde{\eta}/\ell(\tau)}$ for $s \ge s_0$, and therefore

$$\max_{\tau\in(0,a]} |\mathbf{v}_\tau(y,\tau)|^2 e^{2s\eta(y,\tau)} \le C_\varepsilon \max_{\tau\in(0,a]} e^{-2s(1-\varepsilon)\widetilde{\eta}(y)/\ell(\tau)}$$

$$\le C_\varepsilon e^{-2s\widetilde{\eta}(y)/\ell(\tau_*)} = C_\varepsilon e^{2s\eta(y,\tau_*)}.$$

Since $|\mathbf{v}_y(y,\tau)|$ is bounded on $[a,T^*]$, a similar estimate holds on $[a,T^*]$, and (40) follows.

As a consequence, we have $s^{-1}\|e^{s\eta}G\mathbf{v}_\tau\|_{L^2(\Omega_1\times(0,T^*))} \le Cs^{-1}T^*\|e^{s\eta(\cdot,\tau_*)}G\|_{L^2(\Omega_1)}$. Hence by (33), (39) implies that for sufficiently large s,

$$\|e^{s\eta(\cdot,\tau_*)}G\|^2_{L^2(\Omega_1)} \qquad (41)$$

$$\le C\|\mathbf{w}(y,\tau_*)\|^2_{H^2(\Omega)} + Cs^{-1}\left(\|e^{s\eta}\mathbf{w}_\tau\|^2_{L^2(K\times(0,T^*))} + \|e^{s\eta}\mathbf{w}_{\tau y}\|^2_{L^2(K\times(0,T^*))}\right).$$

To finish the proof, we use Lemma 3.2 to show that the last two terms on the right can be bounded by $\|\mathbf{w}\|_{L^2(\omega\times(0,T^*))}$, where we recall that $\omega \subset \Omega \setminus \Omega_1$ is an arbitrarily small neighborhood of K. Lemma 3.2 applies since A_1 and G are both C^1, by assumption. Choose ω' such that $K \subset \omega' \Subset \omega$. By (25) with $\mathbf{u} = \mathbf{w}_\tau$, $\gamma = 0$, $\Delta = K$ and $D = \omega'$, we find that

$$\|e^{s\eta}\mathbf{w}_{\tau y}\|_{L^2(K\times(0,T^*))} \le Cs\|e^{s\eta}\ell^{-1}\mathbf{w}_\tau\|_{L^2(\omega'\times(0,T^*))},$$

since $(\partial_\tau - \mathcal{L}_{A_1})\mathbf{w}_\tau = G\mathbf{v}_\tau$ vanishes on ω'. Applying (26) with $\mathbf{u} = \mathbf{w}$, $\gamma = 0$ or 1, $\Delta = \omega'$ and $D = \omega$, we then see (again using the vanishing of G, on ω this time), that

$$\|e^{s\eta}\mathbf{w}_\tau\|^2_{L^2(K\times(0,T))} + \|e^{s\eta}\mathbf{w}_{\tau y}\|^2_{L^2(K\times(0,T))} \le s^6 \|e^{s\eta}\ell^{-3}\mathbf{w}\|^2_{L^2(\omega\times(0,T))},$$

which is bounded by $C\|\mathbf{w}\|_{L^2(\omega\times(0,T^*))}$ for any $s \ge 1$ upon taking into account that $\sup_{(y,\tau)\in\omega\times(0,T^*)} s^3\ell^{-3}(\tau)e^{s\eta(y,\tau)} < \infty$, for any s. Since $e^{s\eta(\cdot,\tau_*)}$ is bounded from below by a strictly positive constant on Ω_1, the lemma follows. $\qquad\square$

If we do not suppose that G has compact support, we can generalise Lemma 4.2 as follows:

Corollary 1. *Let $\Omega_1 \Subset \Omega \subset \mathbb{R}$ be bounded open intervals and let $\omega \Subset \Omega \setminus \overline{\Omega}_1$ be an open subset such that ω has non-zero intersection with each component of $\Omega \setminus \Omega_1$ and such that, moreover, $0 \notin \overline{\omega}$. Then, there exists a constant $C > 0$ such that if $\mathbf{w}$ solves (29) on Ω, then*

$$\|G\|^2_{L^2(\Omega_1)} \le C\left(\|\mathbf{w}(\cdot,\tau_*)\|^2_{H^2(\Omega)} + \|\mathbf{w}\|^2_{L^2(\omega\times(0,T^*))} + \|G\|^2_{H^1(\omega)}\right). \qquad (42)$$

Proof. Let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi = 1$ on a neighborhood of $\overline{\Omega}_1$, while $\mathrm{supp}(\chi_y) \subset \omega$. Then the proof of (41) shows that

$$\|e^{s\eta(y,\tau_*)}\chi G\|^2_{L^2(\mathbb{R})} - C\|\mathbf{w}\|^2_{H^2(\mathrm{supp}(\chi))}$$
$$\le Cs^{-1}\left(\|e^{s\eta}\mathbf{w}_\tau\|^2_{L^2(\mathrm{supp}(\chi_y)\times(0,T^*))} + \|e^{s\eta}\mathbf{w}_{\tau y}\|^2_{L^2(\mathrm{supp}(\chi_y)\times(0,T^*))}\right).$$

An application of Lemma 3.2 then shows that since ω is a bounded open neighborhood of the $\mathrm{supp}(\chi_y)$, then

$$\|e^{s\eta(y,\tau_*)}\chi G\|^2_{L^2(\mathbb{R})} \le C\left(\|\mathbf{w}\|^2_{H^2(\mathrm{supp}(\chi))} + s^6\|e^{s\eta}\ell^{-3}\mathbf{w}\|^2_{L^2(\omega\times(0,T^*))}\right.$$
$$\left. + s\|e^{s\eta}\ell^{-1}G\mathbf{v}_\tau\|^2_{L^2(\omega\times(0,T^*))} + s^{-1}\|e^{s\eta}(G\mathbf{v})_y\|^2_{L^2(\omega\times(0,T^*))}\right),$$

where the norms of G and G_y on the right occur since G is no longer 0 on a neighborhood of $\mathrm{supp}(\chi_y)$. The corollary follows by observing that $\mathbf{v}$, $\mathbf{v}_\tau$ and $\mathbf{v}_y$ are bounded on $\omega \times (0,T^*)$, since $0 \notin \overline{\omega}$ and that $\chi = 1$ on $\overline{\Omega}_1$. $\qquad\square$

We observe in passing that the $H^1(\omega)$-norm of G in (42) can be replaced by $C_\varepsilon\|G\|^2_{L^2(\omega)} + \varepsilon\|G_y\|^2_{L^2(\omega)}\|$ for arbitrary $\varepsilon > 0$, by choosing s sufficiently large.

5. **Completion of the proof of Theorem 1.1.** The final step in the proof is to get rid of the last term on the right hand side of (35). We proceed in a succession of lemmas. Inspired by [39] we start by establishing

Lemma 5.1. *Let X, $Y_1 \subset Y$ and Z be four Banach spaces such that the inclusion $\mathrm{id} : Y_1 \to Y$ is continuous. Let $\mathcal{A} : X \to Y$ be a bounded injective linear operator, $\mathcal{K} : X \to Z$ be a compact linear operator and let $X_1 \subset X$ be a linear subspace which is mapped into Y_1 by $\mathcal{A}$ and for which there exists a constant $C > 0$ such that*

$$\|f\|_X \le C\|\mathcal{A}f\|_{Y_1} + \|\mathcal{K}f\|_Z, \qquad (43)$$

for all $f \in X_1$. Then, there exists a (in general different) constant $C > 0$ such that for all $f \in X_1$ we have that

$$\|f\|_X \le C\|\mathcal{A}f\|_{Y_1}. \qquad (44)$$

Proof. We argue by contradiction. Assume that (44) does not hold. Then, there exists a sequence $(f_n)_n$ in X_1 such that $\|f_n\|_X = 1$ for all n and $\|\mathcal{A}f_n\|_{Y_1} \to 0$ as n goes to infinity. Since $\mathcal{K} : X \to Z$ is compact, there is a subsequence, still denoted by f_n, such that $(\mathcal{K}f_n)_n$ converges in Z. Hence this is a Cauchy sequence in Z and, by applying (43) to $f_n - f_m$, we get that $\|f_n - f_m\|_X \to 0$, as $n, m \to \infty$; $(f_n)_n$ is therefore a Cauchy sequence in X and $f_n \to f$ in X as $n \to \infty$ for some $f \in X$. Since $\|f_n\|_X = 1$ for all n, it follows that $\|f\|_X = 1$ also. Since $\mathcal{A}$ is continuous, we have $\mathcal{A}f_n \to \mathcal{A}f$ in Y. On the other hand, $\mathcal{A}f_n \to 0$ in Y since this is true in Y_1. Hence $\mathcal{A}f = 0$, which is a contradiction with $\mathcal{A}$ being injective.   □

In the light of (35), and identifying the diagonal matrix G with the vector of its diagonal elements, we will apply this lemma with the Banach spaces $X = \{G \in L^2(\mathbb{R})^n, \; \text{supp}(G) \subset \Omega_1\}$, $Y_1 = H^2(\Omega)^n \hookrightarrow Y = H^1(\Omega)^n$ and $Z = L^2(\omega \times (0, T))^n$, and with the operators $\mathcal{A}$ and $\mathcal{K}$ defined by

$$\mathcal{A} : X \to Y, \; \mathcal{A}G = \mathbf{w}(\cdot, \tau_*)|_\Omega,$$

and

$$\mathcal{K} : X \to Z, \; \mathcal{K}G = \mathbf{w}|_{\omega \times (0, T^*)},$$

where $\mathbf{w}$ denotes the unique solution to (29). The following lemma will imply that $\mathcal{A}$ is well-defined; here and below, C denotes a generic constant which is independent of τ_*.

Lemma 5.2. *For $G \in L^2(\mathbb{R})^n$, let $\mathcal{A}_1 G$ be the function $\mathbf{w}(\cdot, \tau_*)$ on all of $\mathbb{R}$. Then, $\mathcal{A}_1 : L^2(\mathbb{R})^n \to H^1(\mathbb{R})^n$ is bounded, with norm majorized by $C \max(1, \sqrt{\tau_*})$. Moreover, if $G \in H^1(\mathbb{R})^n$, then $\mathcal{A}_1 G \in H^2(\mathbb{R})^n$, with*

$$\|\mathcal{A}_1 G\|_{H^2(\mathbb{R})^n} \leq C \sqrt{\max(\tau_*, \tau_*^{-1})} \, \|G\|_{H^1(\mathbb{R})}.$$

In particular, since $\mathcal{A}G = \mathcal{A}_1 G|_\Omega$ for $G \in X$, $\mathcal{A} : X \to Y$ is continuous, and $\mathcal{A}G \in Y_1$ if $G \in X$ is C^1, which is true for $G = A_1 - A_2$.

The somewhat technical proof of this lemma is given in Appendix C. It uses Schur's bound for the L^2-norm of an integral operator. Here we just note that, because of the singular behavior of $\mathbf{v}$ in the right hand side of (29), the boundedness of $\mathcal{A}$ in L^2-norm does not follow directly from the standard energy estimates for parabolic equations, unless one would for example assume that $0 \notin \text{supp}(G)$,

Lemma 5.3. *The operator $\mathcal{A}$ is injective.*

Proof. Since $\mathbf{w}$ is solution of (29), we deduce from the identities $\mathbf{w}(\cdot, \tau_*) = 0$ and $G = 0$ on $\omega \subset \Omega \backslash \Omega_1$ that $\mathbf{w}_\tau(\cdot, \tau_*) = 0$ on ω. Arguing in the same way we get that the successive derivatives of $\mathbf{w}$ with respect to τ vanish on $\omega \times \{\tau_*\}$. Since $\mathbf{w}$ is the difference of two solutions to an initial value problem with time independent coefficients, $\mathbf{w}$ is analytic in τ, so we necessarily have $\mathbf{w} = 0$ on $\omega \times (0, \tau_*)$. Therefore $G = 0$ on Ω_1 by (35), and the proof is complete.   □

To prove compacticity of $\mathcal{K}$ we will use the following classical parabolic regularity estimate: cf. [27, §7.1, Thm 5]; the lemma is stated there for scalar parabolic equations, but the proof remains valid for parabolic systems.

Lemma 5.4. *Let $I \subset \mathbb{R}$ be a bounded interval, let $F \in L^2(I \times (0, T^*))$ and let $\mathbf{u} \in H^{2,1}(I \times (0, T^*))$ be solution to*

$$\begin{cases} (\partial_\tau - \mathcal{L}_{A_1})\,\mathbf{u} = F, & y \in I, \ \tau \in (0, T^*), \\ \mathbf{u}(y, 0) = 0, & y \in I, \\ \mathbf{u}(y, \tau) = 0 & y \in \partial I, \ \tau \in (0, T^*). \end{cases}$$

Then, there exists a positive constant C such that

$$\|\mathbf{u}\|_{H^{2,1}(I \times (0, T^*))} \le C \, \|F\|_{L^2(I \times (0, T^*))}.$$

Lemma 5.5. $\mathcal{K} : X \to Z$ *is a compact operator.*

Proof. Recall that $\omega = \omega_1 \cup \omega_2$ with ω_1 and ω_2 open and disjoint. Let $I_j \subset \mathbb{R} \setminus \Omega_1$ be an open interval such that $\overline{\omega}_j \subset I_j$, and let $\chi_j \in C_c^\infty(I)$ be such that $\chi_j = 1$ on ω_j for $j = 1, 2$. Then since $G = 0$ on I_j, the function $\mathbf{u}_j := \chi_j \mathbf{w}$ satisfies the boundary value problem of Lemma 5.4 with $F = [\chi_j, \mathcal{L}_{A_1}] \mathbf{w}$. Hence

$$\|\mathbf{w}\|_{H^{2,1}(\omega_j \times (0, T))} \le \|\mathbf{u}_j\|_{H^{2,1}(I \times (0, T))} \le C \left(\|\mathbf{w}\|_{L^2(I_j \times (0, T))} + \|\mathbf{w}_y\|_{L^2(I_j \times (0, T))} \right).$$

If we apply the first statement of Lemma 5.2 and integrate over τ_* from 0 to T^*, we find that the right hand side is bounded by $C \max(\sqrt{T^*}, T^*) \|G\|_{L^2(\mathbb{R})}$, and the lemma follows from the compactness of the injection $H^1(\omega \times (0, T))^n \hookrightarrow Z = (L^2(\omega \times (0, T)))^n$. $\square$

Note that we needed Lemma 5.4 to get a suitable bound for the L^2-norm of $\mathbf{w}_\tau$: those for $\mathbf{w}$ and $\mathbf{w}_y$ already follow from Lemma 5.2.

Remark 2. The operator $\mathcal{K}$ extends naturally to an operator $\mathcal{K} : L^2(\mathbb{R})^n \to Z = L^2(\omega \times (0, T^*))^n$. We claim that if $0 \notin \overline{\omega}$ then $\mathcal{K}$ is still compact, where ω may be an otherwise arbitrary bounded open subset of $\mathbb{R}$. To prove this, we may assume that ω is connected. Let I be an open interval containing $\overline{\omega}$ but not containing 0, and let χ and $\mathbf{u}$ be as above. Since G is no longer 0 on I, we now find that $\mathbf{u}$ is a solution to the system of Lemma 5.4 with right hand side $F = [\chi, \mathcal{L}_{A_1}] \mathbf{w} + \chi G \mathbf{v}$. But $\chi G \mathbf{v} \in L^2(I \times (0, T))^n$ since $0 \notin \mathrm{supp}(\chi)$ and it follows as before that $\mathcal{K}$ sends $L^2(\mathbb{R})^n$ into $H^{2,1}(\omega \times (0, T))$.

If we finally combine Lemmas 4.2, 5.1, 5.2 and 5.5, we conclude that if $G \in H^1(\mathbb{R})^n$ is supported in Ω, then

$$\|G\|_{L^2(\Omega_1)} \le C \, \|\mathcal{A}G\|_Y = C \, \|\mathbf{w}(\cdot, \tau_*)\|_{H^2(\Omega)}. \tag{45}$$

This applies in particular to $G = A_1 - A_2$, which therefore proves Theorem 1.1.

6. **Extension to non-compactly supported G's.** We can extend (45) to non-compactly supported G by allowing suitable norms of G on $\mathbb{R} \setminus \Omega_1$ on the right hand side of the inequality. An easy extension is the following:

Corollary 2. *For any pair of bounded open intervals $\Omega_1 \Subset \Omega$ and any $\tau_* > 0$, there exist a constant $C > 0$, depending on these subsets and on τ^*, as well as on $\|A_1\|_{C^1(\mathbb{R})}$ and $\|A_2\|_{C^{2,\alpha}(\mathbb{R})}$, such that for all $G \in H^1(\mathbb{R})^n$,*

$$\|G\|_{L^2(\Omega_1)} \le C \left(\|\mathbf{w}(\cdot, \tau_*)\|_{H^2(\Omega)} + \|G\|_{H^1(\mathbb{R} \setminus \Omega_1)} \right). \tag{46}$$

Proof. Let $\chi \in C_c^\infty(\Omega)$ such that $\chi(y) = 1$ on Ω_1. Write $G = G_1 + G_2$ with $G_1 := \chi G$ and $G_2 := (1 - \chi)G$. Let $\mathbf{w}_j$, $j = 1, 2$, be the solution of the boundary

value problem (29) with right hand side $G_j\mathbf{v}$ and let $\mathbf{w}$ be the one with right hand side $G\mathbf{v}$. Then, we have $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, by uniqueness, and by (45) above,

$$\|G_1\|_{L^2(\Omega_1)} \le C\|\mathbf{w}_1(\cdot, \tau_*)\|_{H^1(\Omega)} \le C \left(\|\mathbf{w}(\cdot, \tau_*)\|_{H^1(\Omega)} + \|\mathbf{w}_2(\cdot, \tau_*)\|_{H^2(\Omega)} \right),$$

while by the second statement of Lemma 5.2, $\|\mathbf{w}_2(\cdot, \tau_*)\|_{H^1(\mathbb{R})} \le C\|G_2\|_{H^1(\mathbb{R})} \le C\|G\|_{H^1(\mathbb{R}\setminus\Omega_1)}$. Inequality (46) then follows. $\square$

Going back to our original problem, the corollary implies that if we take $\varepsilon > 0$ and $I \Subset J$, and if $\Sigma_1 - \Sigma_2$ has a sufficiently small H^1-norm on the complement of I, then we can achieve "stability up to ε":

$$\|\Sigma_1 - \Sigma_2\|^2_{L^2(I)} \le C \sum_{i=1}^{n} \left\| C^*_{\Sigma_1}(\cdot, \epsilon_i, T) - C^*_{\Sigma_2}(\cdot, \epsilon_i, T) \right\|^2_{H^2(J)} + \varepsilon. \qquad (47)$$

If we would be content to work within a certain numerical precision ε, it would therefore not be necessary that $\Sigma_1 = \Sigma_2$ outside of I, only that it has a sufficiently small H^1-norm there.

We can improve the corollary by re-examining the proof of (45).

Theorem 6.1. *Let $\Omega_1 \Subset \Omega$ be two bounded intervals and let $\omega \Subset \Omega \setminus \Omega_1$ be an open subset with $0 \notin \omega$, such that ω intersects each of the two components of $\Omega \setminus \Omega_1$ in an open interval. Finally, let $\tau_* > 0$. Then there exists a constant C such that*

$$\|G\|_{L^2(\Omega_1)} \le C \left(\|\mathbf{w}(\cdot, \tau_*)\|_{H^2(\Omega)} + \|G\|_{H^1(\omega)} + \|G\|_{L^2(\mathbb{R}\setminus\Omega_1)} \right). \qquad (48)$$

Proof. By contradiction: suppose that (48) does not hold. Then, there exists a sequence $(G_n)_n$ such that $\|G_n\|_{L^2(\Omega_1)} = 1$ and such that

$$\|\mathbf{w}_n(\cdot, \tau_*)\|_{H^2(\Omega)}, \|G_n\|_{H^1(\omega)}, \|G_n\|_{L^2(\mathbb{R}\setminus\Omega_1)} \to 0, \quad n \to \infty,$$

where w_n is the solution of (29) with right hand side $G_n\mathbf{v}$. By Corollary 1,

$$\|G_n\|_{L^2(\Omega_1)} \le C \left(\|\mathbf{w}_n(\cdot, \tau_*)\|_{H^2(\Omega)} + \|\mathbf{w}_n\|_{L^2(\omega\times(0,T))} + \|G_n\|_{H^1(\omega)} \right). \qquad (49)$$

The sequence $(G_n)_n$ is clearly norm-bounded in $L^2(\mathbb{R})$, so by Remark 2 there exists a subsequence, still denoted by $(G_n)_n$ such that $(\mathbf{w}_n)_n$ converges in $L^2(\omega \times (0, \tau_*))$. It then follows from (49) that $(G_n|_{\Omega_1})_n$ is a Cauchy sequence in $L^2(\Omega_1)^n$, which therefore converges to a function G on Ω_1. Extending G by 0 on the complement of Ω_1 and recalling that $\|G_n\|_{L^2(\mathbb{R}\setminus\Omega_1)} \to 0$, we see that $G_n \to G$ in $L^2(\mathbb{R})^n$. Clearly, $\|G\|_{L^2(\mathbb{R})} = 1$. If $\mathbf{w}$ is the solution of (29) with right hand side $G\mathbf{v}$, then by Lemma 5.2, $\mathbf{w}_n(\cdot, \tau_*) \to \mathbf{w}(\cdot, \tau_*)$ in H^1. Hence $w(\cdot, \tau_*) = 0$ on Ω_1, while $\mathrm{supp}(G) \subset \Omega_1$ and $\|G\|_{L^2(\Omega_1)} = 1$. But this gives a contradiction with Lemma 5.3. $\square$

We can always choose ω such that $0 \notin \omega$. Theorem 1.2 is an immediate corollary of Theorem 6.1, on using the trivial bound $\|G\|_{H^1(\Omega)} \le \|G\|_{H^1(\Omega_1)}$. One can slightly strengthen (17) by replacing the H^1-norm of $\Sigma_1 - \Sigma_2$ on $J \setminus I$ by one on a compactly contained open subset which intersects each component of $J \setminus I$, but we opted for the simpler formulation.

Appendix A. **Proof of Theorem 2.1.** Let $E(K, T; S, t) = (e_{ij}(K, T; S, t))_{i,j}$ be the fundamental solution of the system (11):

$$\partial_t e_{ij} + \frac{1}{2}\sigma_j(S, t)^2 S^2 \partial^2_S e_{ij} + (r_j - q_j)S\partial_S e_{ij} + \sum_k b_{kj}e_{ik} = r_j e_{ij}, \quad t \le T, \qquad (50)$$

with final boundary condition $e_{ij}(K, T; S, T) = \delta(S - K)\delta_{ij}$. The existence of such a fundamental solution follows from standard results for parabolic systems after a

logarithmic change of variables $S = e^x$: see [21], [28] and also Appendix B below. It is known that the e_{ij} are C^2 in S and K, and C^1 in t and T, for $t < T$. We can express the generalised call prices in terms of this fundamental solution as

$$\underline{C}(K, T, \pi; S, t) = \int_{\mathbb{R}_{>0}} \underline{\pi}^t E(y, T; S, t) \max(y - K, 0) \, dy, \tag{51}$$

or, component-wise,

$$C_j(K, \pi, T; S, t) = \int_K^\infty \left(\sum_i \pi_i e_{ij}(y, T; S, t) \right) (y - K) \, dy, \quad j = 1, \dots, n,$$

where the integral converges absolutely. Differentiating twice with respect to K then yields the extension to regime-switching models of the well-known Breeden-Litzenberger formula,

$$\partial_K^2 C_j(K, T, \pi, S, t) = \sum_i \pi_i e_{ij}(K, T; S, t), \tag{52}$$

or, in vector notation, $\partial_K^2 \underline{C}(K, \pi, T; S, t) = \underline{\pi}^t E(K, T; S, t)$.

Under our assumption of market completeness we dispose of call options with $\pi = \epsilon_i$ for $i = 1, \dots, n$, and the Breeden-Litzenberg relation (52) then allows, at a time t^*, to recover the j-th column of $P(K, T; S_{t^*}, t^*)$, where $e_j = X_{t^*}$.

In matrix notation, (50) reads

$$\partial_t E + S^2(\partial_S^2 E)\Sigma + S(\partial_S E)(R - Q) + E(B - R) = 0, \quad t < T. \tag{53}$$

The usual argument for deriving a PDE in the forward variables (K, T) for E also apply to the system case: we recall the derivation for convenience of the reader while paying attention to the regularity needed for Σ. Let the row-vector $\underline{V}(S, t)$ be a solution of (5) with arbitrary final value $\underline{F} \in C_c^\infty (\mathbb{R}_{>0})^n$. If $t < u < T$, then, by uniqueness of solution,

$$\underline{V}(S, t) = \int_{\mathbb{R}_{>0}} \underline{V}(y, u) E(y, u; S, t) \, dy.$$

If we differentiate this relation with respect to u, use (5) for $\underline{V}$ and, assuming momentarily that Σ^2 is C^2, integrate by parts twice, we find that

$$0 = \int_{\mathbb{R}_{>0}} \underline{V}(y, u)^t \left(\partial_u E - \frac{1}{2} \partial_y^2 \left(y^2 \Sigma^2 E \right) + S \partial_y (y(R - Q)E) + (R - B)E \right) dy,$$

where $E = E(y, u; S, t)$. Letting $u \to T$ and using that $\underline{F}$ is arbitrary, we find the forward equation

$$\partial_T E - \frac{1}{2} \partial_y^2 \left(y^2 \Sigma(y, T)^2 E \right) + (R - Q)\partial_y (yE) + (R - B)E = 0, \tag{54}$$

for $E(y, T; S, t)$ or, component-wise,

$$\partial_T e_{ij} - \frac{1}{2}\partial_y^2(y^2 \sigma_i(y, T)^2 e_{ij}) + (r_i - q_i)\partial_y(ye_{ij}) + r_i e_{ij} - \sum_k b_{ik}e_{kj} = 0, \quad i, j = 1, \dots, n.$$

If Σ^2 is only C^1, we can only integrate by parts once, but we still find that E satisfies the forward equation in the weak sense:

$$0 = \int_0^\infty \underline{F}(y) \left(\partial_T E + (R - Q)\partial_y(yE) + (R - B)E \right) dy + \frac{1}{2} \int_0^\infty (\partial_y \underline{F}) \, \partial_y \left(y^2 \Sigma^2 E \right) dy, \tag{55}$$

where, in view of the Gaussian estimates for $E(K, T; S, t)$ (see Appendix B) we can, by a density argument, take $\underline{F}$ to be any function in $H^1_{\text{loc}}(\mathbb{R})^n$ which together with its derivative, is of polynomial growth.

We next use this to derive a Dupire-type system of PDEs for the column-vector of call prices

$$(C_{1j}(K, T; S, t), \dots, C_{nj}(K, T; S, t))$$
$$= (C_j(K, \epsilon_1, T; S, e_j, t), \dots C_j(K, \epsilon_n, T; S, e_j, t)),$$

where we recall that $\epsilon_i(e_j) = \delta_{ij}$. We get from (51) and (55) with $\underline{F}(y) := (\max(y - K, 0)\delta_{ik})_{1 \le k \le n}$, that

$$\partial_T C_j(K, \epsilon_i, T) = \int_{\mathbb{R}_{>0}} \partial_T e_{ij}(y, T; S, t) \max(y - K, 0) \, dy$$

$$= \int_{\mathbb{R}_{>0}} \left(-\frac{1}{2} \partial_y(y^2 \sigma_i^2 e_{ij}) + (r_i - q_i)y e_{ij} \right) H(y - K) dy$$

$$- \int_{\mathbb{R}_{>0}} \left(r_i e_{ij} - \sum_k b_{ik} e_{kj} \right) \max(y - K, 0) \, dy$$

$$= \frac{1}{2} K^2 \sigma_i(K, T)^2 e_{ij}(K, T) + (r_i - q_i) \int_{\mathbb{R}_{>0}} y e_{ij} H(y - K) dy - r_i C_j(K, \epsilon_i, T)$$

$$+ \sum_k b_{ik} C_j(K, \epsilon_k, T),$$

where $H(x)$ is the Heaviside function and where we used (51) again for the last two terms. Now, for the first term we use the Breeden-Litzenberger relation (52), and rewrite the second term as

$$\int_{\mathbb{R}_{>0}} y \, e_{ij}(y, T) \, H(y - K) \, dy$$

$$= \int_{\mathbb{R}_{>0}} e_{ij}(y, T) \max(y - K, 0) \, dy + K \int_0^\infty e_{ij}(y, T) \, H(y - K) \, dy$$

$$= C_j(K, \epsilon_i, T) - K \partial_K C_j(K, \epsilon_i, T),$$

we find the (i, j)-th component of equation (18), thereby proving Theorem 2.1.

Appendix B. **Proof of Lemma 4.1.** We examine the non-vanishing of the matrix coefficients of the fundamental solution of a strictly parabolic Cauchy problem:

$$L(\tau, y; \partial)\mathbf{u}(y, \tau) := \left(\partial_\tau - \sum_{k=0}^2 \mathbb{A}_k(y, \tau)\partial_y^k \right) \mathbf{u}(y, \tau) = 0, \quad 0 < \tau < T, \, y \in \mathbb{R},$$

$$\mathbf{u}(y, 0) = \mathbf{u}_0(y), \qquad\qquad\qquad\qquad\qquad y \in \mathbb{R},$$

(56)

where $\mathbf{u}(y, \tau) = {}^t(u_1(y, \tau), \dots, u_d(y, \tau))$ and $\mathbf{u}_0(y) = {}^t(u_{0,1}(y), \dots, u_{0,d}(y))$ are d-dimensional column vectors and where

$$\mathbb{A}_k(y, \tau) = (a_{ij}^k(y, \tau))_{1 \le i,j \le d}, \quad k = 0, 1, 2,$$

are $d \times d$ matrix-valued functions, which are uniformly continuous in (y, τ) and Hölder continuous with exponent $\alpha > 0$ in y, with $\mathbb{A}_2(y, \tau)$ strictly positive definite on $\mathbb{R}$. We use blackboard bold letters for the coefficients to avoid confusion with the

A_k of Sections 2 to 5. By Eidel'man [21] and [28], there exists a unique fundamental solution $E(y, \tau; y_0, s) = (e_{i,j}(y, \tau; y_0, s))_{1 \le i,j \le d}$ characterised by

$$L(\tau, y, \partial)E := \left(\partial_\tau - \sum_{k=0}^{2} \mathbb{A}_k(y, \tau)\partial_y^k \right) E(y, \tau; y_0, s) = 0, \quad 0 \le s < \tau < T, \ y, y_0 \in \mathbb{R},$$
$$\lim_{\tau \to s^+} E(y, \tau; y_0, s) = \delta(y - y_0)I_{d \times d}, \qquad\qquad y, y_0 \in \mathbb{R},$$
(57)

with $I_{d \times d}$ the $d \times d$ unit matrix, and whose matrix elements satisfy the Gaussian estimates

$$|\partial_y^k e_{ij}(y, \tau; y_0, s)| \le \frac{C}{(\tau - s)^{(k+1)/2}} \exp\left(-c\frac{|y - y_0|^2}{\tau - s} \right), \quad k = 0, 1, 2,$$

and

$$|\partial_\tau e_{ij}(y, \tau; y_0, s)| \le \frac{C}{(\tau - s)^{3/2}} \exp\left(-c\frac{|y - y_0|^2}{\tau - s} \right).$$

for suitable constants $c, C > 0$. Otsuka [37] determined necessary and sufficient conditions for the matrix elements of the fundamental solution to be non-negative:

Theorem B.1. *The fundamental solution of a strictly parabolic second order system* (56) *has the positivity property*

$$e_{ij}(y, \tau; y_0, s) \ge 0, \quad 0 \le s < \tau < T, \ x, y \in \mathbb{R}, \quad i, j = 1, \ldots, d.$$

if and only if

  1. *The first and second order coefficients* $\mathbb{A}_k(y, \tau)$ *(k = 1, 2) are diagonal.*
  2. *For all* $i \ne j$, $a_{ij}^0(y, \tau) \ge 0$.

We next examine strict positivity of the matrix coefficients of $E(y, \tau; y_0, s_0)$, and will therefore assume the above conditions 1 and 2 to hold. We first decompose the operator L into a diagonal part, L_{diag} and a 0-th order non-diagonal part, $\mathbb{B}$,

$$L = L_{\text{diag}} + \mathbb{B}.$$

In particular, $\mathbb{A}_0 - \mathbb{B}$ is diagonal, so that if $\mathbb{B}(y, \tau) = ((b_{ij}(y, \tau))_{i,j}$ then $b_{ij}(y, \tau) = a_{ij}^0(y, \tau)$ for $i \ne j$; one can, but does not need to, take $b_{ii}(y, \tau) = 0$. Let

$$E_{\text{diag}}(y, \tau; y_0, s) = \text{Diag}(e_1(y, \tau; y_0, s), \ldots, e_d(y, \tau; y_0, s))$$

be the fundamental solution of L_{diag}. It is known that the e_i satisfy a Gaussian lower bound: there exist positive constants δ_0 and ϵ_0 such that

$$e_i(y, \tau; y_0, s) \ge \delta_0(\tau - s)^{-1/2} \exp\left(-\epsilon_0 \frac{(y - y_0)^2}{\tau - s} \right) : \qquad (58)$$

see for example [4]. Recall that, by assumption, $b_{ij}(y, \tau) \ge 0$ for all i and j, and let $B_* = (b_{*,ij})_{i,j}$ be the matrix of the coefficient-wise infinimae of $B(y, \tau)$:

$$b_{*,ij} := \inf_{y \in \mathbb{R}, \tau > 0} b_{ij}(y, \tau).$$

The following lower bound can be considered to be a sharpening of the sufficient part of Otsuka's theorem above. If we apply it to the system (30) while taking $\mathbb{B} = B$, it immediately implies (33), thereby completing the proof of this lemma.

Theorem B.2. *There exists constants $c, C > 0$ such that for all i, j,*

$$e_{ij}(y, \tau; y_0, s) \geq C \left(e^{c(\tau - s)B_*} \right)_{ij} \frac{e^{-\epsilon_0 (y-z)^2/(\tau - s)}}{\sqrt{\tau - s}}, \tag{59}$$

where we can in fact take $c = \delta_0 (\epsilon_0^{-1} \pi)^{1/2}$ and $C = \delta_0$.

Proof. Following [37], the fundamental solution $E(\tau, s, y, y_0)$ can be written as

$$E(y, \tau; y_0, s) = E_{\text{diag}}(y, \tau; y_0, s) + E_1(y, \tau; y_0, s),$$

where E_{diag} is the fundamental solution of the diagonal operator introduced above, and where the second part of the fundamental solution $E_1 = (e_{1,ij}(y, \tau; y_0, s))_{1 \leq i,j \leq d}$ is given by

$$E_1(y, \tau; y_0, s) = \int_s^\tau \int_{\mathbb{R}} E_{\text{diag}}(y, \tau; z, t) \Phi(z, t; y_0, s) \, dz dt, \tag{60}$$

where

$$\Phi(y, \tau; y_0, s) = \sum_{p \geq 1} \Phi_p(y, \tau; y_0, s), \tag{61}$$

with

$$\Phi_1(y, \tau; y_0, s) = B(y, \tau) E_{\text{diag}}(y, \tau; y_0, s), \tag{62}$$

and

$$\Phi_p(y, \tau; y_0, s) = \int_s^\tau \int_{\mathbb{R}} \Phi_1(y, \tau; z, t) \Phi_{p-1}(z, t; y_0, s) dz dt. \tag{63}$$

The series is absolutely convergent.

Lemma B.3. *For $p \geq 1$,*

$$(\Phi_p)_{ij}(y, \tau, y_0, s) \geq \delta_0^p \left(\frac{\pi}{\epsilon_0} \right)^{(p-1)/2} \frac{(\tau - s)^{p-1}}{(p-1)!} (B_*^p)_{i,j} \frac{\exp\left(-\epsilon_0 \frac{|y-y_0|^2}{\tau - s} \right)}{\sqrt{\tau - s}}, \tag{64}$$

for $1 \leq i, j \leq n$.

Proof. By induction on p. If $p = 1$, the stated inequality is an immediate consequence of (58) and (62). Next, suppose that (64) holds for $p-1$. Then, by (58) and (63),

$$(\Phi_p)_{ij}(y, \tau; y_0, s) = \int_s^\tau \int_{\mathbb{R}} \sum_k b_{ik}(y, \tau) e_{0,k}(y, \tau; z, t) (\Phi_{p-1})_{kj}(z, t; y_0, s) \, dz dt$$

$$\geq \delta_0^p \left(\epsilon_0^{-1} \pi \right)^{(p-2)/2} \int_s^\tau \frac{(t-s)^{p-2}}{(p-2)!} \sum_k b_{*,ik} \left(B_*^{p-1} \right)_{kj} \times$$

$$\left(\int_{\mathbb{R}} \frac{e^{-\epsilon_0 (y-z)^2/(\tau - t)}}{\sqrt{\tau - t}} \frac{e^{-\epsilon_0 (z-y_0)^2/(t-s)}}{\sqrt{t - s}} dz \right) dt$$

$$= \delta_0^p (\epsilon_0^{-1} \pi)^{(p-1)/2} \left(B_*^p \right)_{ij} \frac{(\tau - s)^{p-1}}{(p-1)!} \frac{e^{-\epsilon_0 (y-y_0)^2/(\tau - s)}}{\sqrt{\tau - s}}.$$

where we used the semi-group property of the classical heat-kernel[4]. $\qquad \square$

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[4]In the form: if $p_t(z) := t^{-1/2} e^{-\epsilon_0 z^2}$, then the convolution $p_{t_1} * p_{t_2} = \sqrt{\epsilon^{-1}\pi} p_{t_1 + t_2}$.

Returning to the proof of Theorem B.2, let us take $s = 0$ to simplify notations. Then, by (60), (61) and (64), putting $c := \delta_0(\epsilon_0^{-1}\pi)^{1/2}$, we get

$$e_{1,ij}(y,\tau;y_0,0) \geq \delta_0 \sum_{p \geq 1} \int_0^\tau \int_\mathbb{R} e_{0,i}(y,\tau;z,t) c^{p-1} \frac{\tau^{p-1}}{(p-1)!} (B_*^p)_{ij} \frac{e^{-\epsilon_0(z-y_0)^2/t}}{\sqrt{t}} \, dzdt$$

$$\geq \delta_0 \left(\sum_{p \geq 1} \frac{(c\tau)^p}{p!} B_*^p \right)_{ij} \cdot \frac{e^{-\epsilon_0(y-y_0)^2/\tau}}{\sqrt{\tau}},$$

where we used (58) and the semi-group property of the classical heat-kernel again. Adding $E_0(y,\tau;y_0,0)$, estimated from below by (58) times the identity matrix, the theorem follows. $\qquad\square$

Appendix C. Proof of Lemma 5.2. Let us write the fundamental solution $E_{A_1}(y,\tau;z,s)$ of the time-homogeneous operator $\partial_\tau - \mathcal{L}_{A_1}$ as $E(y,z,\tau-s) = (e_{ij}(y,z,\tau-s))_{1 \leq i,j \leq n}$. Then, by Duhamel's principle,

$$\mathbf{w}(y,\tau) = \int_0^\tau \int_\mathbb{R} E(y,z,s)G(z)\mathbf{v}(z,\tau-s)dzds \qquad (65)$$

where we know that

$$|E(y,z,s)| \leq Cs^{-1/2}e^{-c(y-z)^2/s} \text{ and } |E_y(y,z,s)| \leq Cs^{-1}e^{-c(y-z)^2/s}, \qquad (66)$$

since $A_1(y)$ is C^1 and therefore certainly Hölder continuous. Here E and $G = A_1 - A_2$ are n by n matrices, and $\mathbf{v} = (v_1, \ldots, v_n)$ is a vector-valued function which is a column vector of the fundamental solution with pole in 0 of an auxilliary parabolic system: see (30). Since G is in fact diagonal, $G = \text{diag}(G_1, \ldots, G_n)$, the i-th component of the integrand is $\sum_j e_{ij}(y,z,s)v_j(z,\tau-s)G_j(z)$, and if we now identify G with the vector of its diagonal elements, we can interpret (65) as

$$\mathbf{w}(y,\tau) = \int_0^\tau K_s(G)(y)\, ds, \qquad (67)$$

where K_s is the integral operator on $\mathbb{R}$ with matrix-valued kernel

$$k_s(y,z) = (e_{ij}(y,z,s)v_j(z,\tau-s))_{i,j},$$

acting on $L^2(\mathbb{R})^n$:

$$K_s(G)(y) := \int_\mathbb{R} k_s(y,z)G(z)dz. \qquad (68)$$

1. We begin by estimating the L^2-norm of $\mathbf{w}(\cdot,\tau)$. Taking into account that $\|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} \leq \int_0^\tau \|K_s(G)\|_{L^2(\mathbb{R})}\, ds$, it suffices to estimate the L^2-norm of each the operators K_s. We will do using the following classical lemma of Schur, in a version for integral operators acting on vector-valued functions:

Lemma C.1. *Let K be an integral operator on $L^2(\mathbb{R})$ with matrix-valued kernel $(k_{ij}(y,z))_{1 \leq i,j \leq n}$. Then the operator-norm $\|K\|$ of K can be bounded by $\|K\| \leq n\sqrt{C_1 C_2}$, where*

$$C_1 := C_1(k) := \max_{i,j} \sup_y \int_\mathbb{R} |k_{ij}(y,z)|\, dz, \quad C_2 := C_2(k) := \max_{i,j} \sup_z \int_\mathbb{R} |k_{ij}(y,z)|\, dy. \qquad (69)$$

By the Gaussian upper bounds for fundamental solutions (cf. Appendix B) we know that

$$|k_s(y,z)| \le C \frac{e^{-c(y-z)^2/s}}{\sqrt{s}} \frac{e^{-cz^2/(\tau-s)}}{\sqrt{\tau-s}}.$$

Using the semi-group property of the fundamental solution of the classical heat equation, we then see that

$$C_1(k_s) = \sup_y C \frac{e^{-cy^2/\tau}}{\sqrt{\tau}} = C/\sqrt{\tau},$$

while an easy estimate shows that $C_2(k_s) \le C/\sqrt{\tau-s}$. Hence $\|K_s\| \le C(\tau(\tau - s))^{-1/4}$, and therefore

$$\|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} \le C \left(\int_0^\tau (\tau(\tau-s))^{-1/4}\, ds \right) \|G\|_{L^2(\mathbb{R})} = C\sqrt{\tau}\,\|G\|_{L^2(\mathbb{R})}, \qquad (70)$$

with C independent of τ.

2. Next,

$$\mathbf{w}_y(y,\tau) = \int_0^\tau K_s^1(G)(y)\, ds,$$

where K_s^1 is the integral operator with kernel

$$k_s^1(y,z) = E_y(y,z,s)v(z,\tau-s).$$

We now have that

$$|k_s^1(y,z)| \le C \frac{e^{-c(y-z)^2/s}}{s} \frac{e^{-cz^2/(\tau-s)}}{\sqrt{\tau-s}},$$

and therefore (simply multiply the previous estimates by $s^{-1/2}$)

$$C_1(k_s^1) \le \frac{C}{\sqrt{s\tau}}, \quad C_2(k_s^1) \le \frac{C}{\sqrt{s(\tau-s)}}.$$

It follows that

$$\|\mathbf{w}_y(\cdot,\tau)\|_{L^2(\mathbb{R})} \le C \left(\int_0^\tau \frac{ds}{(\tau(\tau-s))^{1/4}\sqrt{s}} \right) \|G\|_{L^2(\mathbb{R})} = C\|G\|_{L^2(\mathbb{R})}, \qquad (71)$$

with C again independent of τ. Together with (70), and with $\tau = \tau^*$, this proves the first part of Lemma 5.2.

3. We cannot in the same way show that $\|\mathbf{w}_{yy}(\cdot,\tau)\|_{L^2(\mathbb{R})}$ or $\|\mathbf{w}_\tau\cdot,\tau)\|_{L^2(\mathbb{R})}$ are bounded by $\|G\|_{L^2(\mathbb{R})}$, since the singularity in $s = 0$ becomes too strong: for example, $E_{yy}(y,z,s) \le Cs^{-3/2}e^{-c(y-z)^2/s}$ we would end up with a $1/s$-singularity in (71). We therefore proceed differently by differentiating the PDE for $\mathbf{w}$, which gives

$$(\mathbf{w}_y)_\tau - \mathcal{L}(\mathbf{w}_y) = G_y\mathbf{v} + G\mathbf{v}_y + Q_1(\mathbf{w}),$$

with $Q_1 = [\partial_y, \mathcal{L}]$ a first order differential operator. Write $\mathbf{w}_y = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$, where $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ solve the inhomogeneous PDE with right hand side $G_y\mathbf{v}$, $G\mathbf{v}_y$ and $\mathbf{w}_3$ and $Q_1(\mathbf{w})$, respectively, and initial value 0 for $\tau = 0$. Then, by what we have already proven,

$$\|\mathbf{w}_1(\cdot,\tau)\|_{H^1(\mathbb{R})} \le C\|G_y\|_{L^2(\mathbb{R})}.$$

We next show that

$$\|\mathbf{w}_2(\cdot,\tau)\|_{H^1(\mathbb{R})} \le C\tau^{-1/2}\|G\|_{L^2(\mathbb{R})}. \qquad (72)$$

Indeed, by Duhamel's formula,

$$\mathbf{w}_2(y,\tau) = \int_0^\tau \widetilde{K}_s(G)ds,$$

where $\widetilde{K}_s$ is the integral operator with kernel $\widetilde{k}_s(y,z) = (e_{ij}((y,z,s)v_{j,z}(z,\tau-s))_{i,j}$. Hence, the norm of the kernel and of its derivative with resepct to y can be bounded by

$$|\widetilde{k}_s(y,z)| \le C\frac{e^{-c(y-z)^2/s}}{\sqrt{s}}\frac{e^{-cz^2/(\tau-s)}}{\tau-s},$$

and

$$|\partial_y\widetilde{k}_s(y,z)| \le C\frac{e^{-c(y-z)^2/s}}{s}\frac{e^{-cz^2/(\tau-s)}}{\tau-s}.$$

Proceeding as before, we find that

$$\|\mathbf{w}_2(\cdot,\tau)\|_{L^2(\mathbb{R})} \le C\int_0^\tau \frac{ds}{\tau^{1/4}(\tau-s)^{3/4}}\|G\|_{L^2(\mathbb{R})} = C\|G\|_{L^2(\mathbb{R})}$$

and

$$\|\mathbf{w}_{2,y}\|_{L^2(\mathbb{R})} \le C\int_0^\tau \frac{ds}{\tau^{1/4}s^{1/2}(\tau-s)^{3/4}}\|G\|_{L^2(\mathbb{R})} = (C/\sqrt{\tau})\|G\|_{L^2(\mathbb{R})},$$

with constants C which are independent of τ, which proves (72).

Finally, to bound the norm of $\mathbf{w}_3$ we use the classical energy estimate:

$$\begin{aligned}\|\mathbf{w}_3(\cdot,\tau)\|_{H^1(\mathbb{R})}^2 &\le C\|Q_1(w)\|_{L^2((0,T)\times\mathbb{R})}^2 \\ &\le C\left(\int_0^\tau \|\mathbf{w}(\cdot,s)\|_{H^1}\,ds\right) \\ &\le C\tau\|G\|_{L^2(\mathbb{R})}^2,\end{aligned}$$

by Subsection 2.3 again. In conclusion, we have shown that

$$\|\mathbf{w}_y\|_{H^1(\mathbb{R})} \le C\max(\tau^{-1/2},\tau)\left(\|G\|_{L^2(\mathbb{R})} + \|G_y\|_{L^2(\mathbb{R})}\right),$$

which completes the proof of Lemma 5.2 when specializing to $\tau = \tau^*$.

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