AN INVERSE STABILITY RESULT FOR NON COMPACTLY SUPPORTED POTENTIALS BY ONE ARBITRARY LATERAL NEUMANN OBSERVATION

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ABSTRACT. In this paper we investigate the inverse problem of determining the time independent scalar potential of the dynamic Schrödinger equation in an infinite cylindrical domain, from partial measurement of the solution on the boundary. Namely, if the potential is known in a neighborhood of the boundary of the spatial domain, we prove that it can be logarithmic stably determined in the whole waveguide from a single observation of the solution on any arbitrary strip-shaped subset of the boundary.

1. INTRODUCTION

In the present paper we seek global stability in the inverse problem of determining the (non necessarily compactly supported) zero-th order term (the so called electric potential) of the dynamical Schrödinger equation in an infinite cylindrical domain, from a single lateral observation of the solution over the entire time span. But in contrast to [33], where the measurement is performed on a sub-boundary fulfilling the geometric control property expressed by Bardos, Lebeau and Rauch in [3], here we aim for proving that the measurement of the Neumann data can be limited to any arbitrary extended strip designed on the lateral boundary.

1.1. **Inverse problem.**

Let us make this statement a little bit more precise. We stick with the notations of [33]. Namely, ω is an open connected bounded domain in $\mathbb{R}^{n-1}$, $n \geq 3$, with smooth boundary $\partial\omega$, and we consider the infinite straight cylinder $\Omega := \omega \times \mathbb{R}$, in $\mathbb{R}^n$, with cross section ω. Its boundary is denoted by $\Gamma := \partial\omega \times \mathbb{R}$. Given $T > 0$, $p : \Omega \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$, we consider the Schrödinger equation,

$$-i\partial_t u(x,t) - \Delta u(x,t) + p(x)u(x,t) = 0,\ (x,t) \in \Omega \times (0,T), \tag{1.1}$$

associated with the initial data u_0,

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{1.2}$$

and the homogeneous Dirichlet boundary condition,

$$u(x,t) = 0,\ (x,t) \in \Gamma \times (0,T). \tag{1.3}$$

For suitable (real-valued) u_0 and p, and under appropriate compatibility conditions on these two functions, we denote by u_p the unique $\mathcal{C}^0([0,T], H^1(\Omega))$-solution to the initial boundary value problem (abbreviated as IBVP in the sequel) (1.1)–(1.3). Given an arbitrary relatively open subset S_* of $\partial\omega$, with positive Lebesgue measure, we aim for determining the unknown potential $p = p(x)$ from one Neumann observation of the function u_p on $\Sigma_* := \Gamma_* \times (0,T)$, where $\Gamma_* := S_* \times \mathbb{R}$ is an infinitely extended strip designed on the boundary Γ of the waveguide Ω.

The equations (1.1)–(1.3) describe the quantum motion constrained by the waveguide Ω, of a charged particle (in a "natural" system of units where the various physical constants such as the mass and the electric charge are taken equal to one) under the influence of the "electric" potential p. Carbon nanotubes, who have a length-to-diameter ratio up to $10^8/1$, are commonly modelled by infinite cylindrical domains such as $\Omega = \omega \times \mathbb{R}$. These nanostructures exhibit unusual physical properties, which are valuable for electronics, optics and other fields of materials science and technology, but they can be affected by the inevitable presence of electrostatic quantum disorder, see e. g. [17, 29]. This motivates for a closer look into the inverse problem of estimating the strength of the electric impurity potential p from the (partial) knowledge of the wave function u on the boundary Γ of the infinite carbon nanotube Ω.

The uniqueness issue in the inverse problem examined in this paper is to know whether any two admissible potentials p_j, $j=1,2$, are equal, i.e. $p_1(x)=p_2(x)$ for a.e. $x\in\Omega$, if their observation data coincide, that is, if we have

$$\partial_\nu u_{p_1}(x,t)=\partial_\nu u_{p_2}(x,t),\ (x,t)\in\Sigma_*.$$

Here $\nu=\nu(x)$, $x\in\Gamma$, denotes as usual the unit outward normal vector to Γ and $\partial_\nu u=\nabla u\cdot\nu$ stands for the normal derivative of u. We shall give a positive answer to this question provided the two unknown functions p_1 and p_2 coincide in a neighborhood of the boundary Γ. This extra information imposed on the unknown zero-th order coefficient of (1.1) near Γ is technically restrictive, but it is acceptable from a strict practical viewpoint upon admitting that the electric potential can be measured from outside the domain Ω in the vicinity of the boundary.

Actually, the above mentioned uniqueness result follows from a stronger statement claiming logarithmic stability in the determination of the potential p from the observation of $\partial_\nu u_p$ on Σ_*. This amount to saying that $\|p_2-p_1\|_{L^2(\Omega)}$ can be bounded from above in terms of (the logarithm of) a suitable norm of the trace of the function $\partial_\nu u_{p_2}-\partial_\nu u_{p_1}$ on Σ_*. Such stability estimates play a key role in the analysis of ill-posed inverse problems (in the classical sense of [39]), by suggesting regularization parameters and indicating the rate of convergence of the regularized solutions to the exact one.

The main achievement of this paper is that the Neumann data used in this stability estimate can be measured on the unbounded strip-shaped subpart $\Gamma_*=S_*\times\mathbb{R}$ of the whole boundary Γ, where we recall that S_* is an arbitrary non-empty relative open subset of $\partial\omega$. The key idea of the proof is to combine the analysis carried out in [6, 24], which is based on a Carleman estimate specifically designed for the system under consideration, with the Fourier-Bros-Iagolnitzer (FBI) transformation used by Robbiano for sharp unique continuation in [48] (see also [41, 45]). Indeed we take advantage of the fact that the FBI transform of the time derivative of the solution to (1.1) satisfies a parabolic equation in the vicinity of the boundary Γ in order to apply a Carleman parabolic estimate where no geometric condition is imposed on the control domain.

1.2. **Existing papers.** There is a wide mathematical literature on uniqueness and stability issues in inverse coefficients problems of partial differential equations (PDEs), see e.g. [4, 7, 11, 24, 46] and the references therein. However, most of the known results on these two problems require that the corresponding Dirichlet or Neumann data be at least measured on a sufficiently large part $\Gamma_\sharp$ of the boundary Γ of the spatial domain under consideration, if not on the whole boundary itself.

On the other hand, when $\Gamma_\sharp=\{x\in\Gamma,\ (x-x_0).\nu(x)\geq 0\}$, where x_0 denotes a fixed point in the complement set of $\overline{\Omega}$, is a sub-boundary fulfilling the geometric optics condition for the observability derived by Bardos, Lebeau and Rauch in [3], Baudouin and Puel [5] proved uniqueness and Lipschitz stability in the inverse problem of determining the electric potential of the Schrödinger equation from the observation of one Neumann data on $\Gamma_\sharp$. In the present paper we claim logarithmic stability for arbitrarily small boundary parts $\Gamma_\sharp=\Gamma_*$ which do no necessarily comply with the geometric condition of Bardos, Lebeau and Rauch. Nevertheless this is at the expense of a stronger hypothesis on the potential, which is assumed to be known in a neighborhood of Γ.

In the framework of this paper, we are dealing with a single observation of the solution. Uniqueness results for multidimensional inverse problems from a single observation of the solution were first derived by Bukhgeim and Klibanov [15] or Yamamoto [53] when $\Gamma_\sharp=\Gamma$, by means of suitable Carleman estimates. For a general treatment of the analysis of inverse coefficients problems with a finite number of observations, based on Carleman inequalities, we refer to Bukhgeim [13], Isakov [27], Isakov and Yamamoto [28], Khaĭdarov [34], and Klibanov [35]. More specifically, this approach was employed by Bellassoued

[6, 7], Bukhgeim, Cheng, Isakov and Yamamoto [14], Imanuvilov and Yamamoto [24], Puel and Yamamoto [46], and Yamamoto [53], for hyperbolic systems, by Bellassoued and Yamamoto [10], Imanuvilov and Yamamoto [26], and Klibanov and Yamamoto [38], for the acoustic equation, by Bellassoued and Yamamoto [11], and Bellassoued, Imanuvilov and Yamamoto [12], for the elasticity operator, and by Choulli and Yamamoto [18, 19], and Imanuvilov and Yamamoto [25], for parabolic equations.

The stability issue in the inverse problem of determining the time-independent electric potential in the dynamic Schrödinger equation from a single boundary measurement was treated by Baudouin and Puel in [5] and by Mercado, Osses and Rosier in [43]. In these two papers the Neumann data is observed on a sub-boundary satisfying the geometric control condition of Bardos, Lebeau and Rauch. This condition was relaxed in [8] under the assumption that the potential is known near the boundary. In [54], Yamamoto and Yuan established Carleman estimates for Schrödinger equations in Sobolev spaces of negative orders, and derived a result of uniqueness from these estimates.

As for inverse problems for the non-stationary Schrödinger equation by infinitely many boundary observations (i.e. the Dirichlet-to-Neumann map, abbreviated as DN map in the following), we refer to e.g. Avdonin et al. [2], where the real valued electric potential is retrieved from the partial knowledge of the DN map (the observation of the Neumann data is performed on a sub-part of Γ).

In all the above mentioned papers the Schrödinger equation is defined in a bounded spatial domain. In the present work we rather investigate the problem of determining the scalar potential of the Schrödinger equation in an infinite cylindrical domain. Actually, it turns out that mathematical papers dealing with inverse coefficient problems in an unbounded domain are rather sparse. Without being exhaustive, we mention [42], where Li and Uhlmann proved uniqueness in the determination of the compactly supported electric potential in an inifinite slab from partial DN map. In [32], the compactly supported potential of the Schrödinger equation defined in an unbounded waveguide was Lipschitz stably retrieved from one measurement of the solution on a sub-boundary fulfilling the geometric control property of Bardos, Lebeau and Rauch. This result was extended to non compactly supported potentials in [33], but Lipschitz stability degenerated to Hölder stability. Similar uniqueness results for non-compactly supported coefficients of the wave equation are derived by Rakesh in [47] and Nakamura in [44], while the stability issue was treated by Kian in [31].

1.3. **Main results.** We start by examining the direct problem associated with (1.1)–(1.3). To this purpose we consider a fixed natural number $k \in \mathbb{N}^* := \{1, 2, \dots\}$, and given $p_0 \in W^{2(k-1),\infty}(\Omega)$ and $u_0 \in H^{2k}(\Omega)$, we set

$$\mathrm{v}_0 := u_0 \text{ and } \mathrm{v}_j := (-\Delta + p_0)\mathrm{v}_{j-1} \text{ for } j = 1, \dots, k-1.$$

We say that u_0 satisfies the k-th order compatibility conditions with respect to p_0, if the k following identities

$$\mathrm{v}_j(x) = 0,\ x \in \Gamma,\ j = 0, \cdots, k-1,$$

hold simultaneously. Evidently, if u_0 satisfies the k-th order compatibility conditions with respect to p_0, then it satisfies the k-th order compatibility conditions with respect to p for any $p \in W^{2(k-1),\infty}(\Omega)$ verifying $p = p_0$ in the vicinity of Γ.

Further, we introduce the set

$$\mathcal{H}^k = \mathcal{H}^k(\Omega \times (0,T)) := \bigcap_{j=0}^{k} \mathcal{C}^j([0,T], H^{2(k-j)}(\Omega)),$$

where $H^k(\Omega)$ denotes the usual Sobolev space of order k in Ω. Endowed with the norm

$$\|u\|_{\mathcal{H}^k}^2 := \sum_{j=0}^{k} \|\partial_t^j u\|_{\mathcal{C}^0([0,T],H^{2(k-j)}(\Omega))}^2,$$

$\mathcal{H}^k$ is a Banach space, and we recall from [33, Propositions 2.2 and 2.5] the following existence and uniqueness result for the system (1.1)–(1.3).

Proposition 1.1. *For $k \in \mathbb{N}^*$ fixed, assume that $\partial\omega$ is $\mathcal{C}^{2k}$, and pick $p \in W^{2(k-1),\infty}(\Omega)$ in such a way that $\|p\|_{W^{2(k-1),\infty}(\Omega)} \leq M$ for some a priori fixed constant $M \geq 0$. Then for any $u_0 \in H^{2k}(\Omega)$ satisfying the k-th order compatibility conditions with respect to p, there exists a unique solution $u \in \mathcal{H}^k$ to the IBVP* (1.1)–(1.3). *Moreover, the estimate*

$$\|u\|_{\mathcal{H}^k} \leq C\|u_0\|_{H^{2k}(\Omega)}, \tag{1.4}$$

holds for some constant $C > 0$, depending only on ω, T, k and M.

Put $N := [n/4]+1$, where $[s]$ denotes the integer part of $s \in \mathbb{R}$. Then, applying Proposition 1.1 with $k = N+1$, we get that $u \in \mathcal{C}^1([0,T], H^{2N}(\Omega))$ satisfies the estimate $\|u\|_{\mathcal{C}^1([0,T];H^{2N}(\Omega))} \leq C\|u_0\|_{H^{2(N+1)}(\Omega)}$. Further, as the embedding $H^{2N}(\Omega) \hookrightarrow L^\infty(\Omega)$ is continuous, since $2N > n/2$, we deduce from Proposition 1.1 the following result.

Corollary 1.2. *Assume that the conditions of Proposition 1.1 are satisfied with $k = N+1$. Then there exists a positive constant C, depending only on ω, T and M, such that the solution u to* (1.1)–(1.3) *satisfies the estimate:*

$$\|u\|_{\mathcal{C}^1([0,T],L^\infty(\Omega))} \leq C\|u_0\|_{H^{2(N+1)}(\Omega)}.$$

Having seen this, we turn now to introducing the inverse problem associated with (1.1)–(1.3). We consider $p_0 \in W^{2N,\infty}(\Omega;\mathbb{R})$ and pick an open subset ω_0 of ω, such that $\partial\omega \subset \overline{\omega_0}$. Given $b > 0$ and $d > 0$, we aim to retrieve all functions $p : \Omega \to \mathbb{R}$ satisfying

$$\mathtt{N}_{b,d}(p-p_0) := \|e^{b\langle x_n\rangle^d}(p-p_0)\|_{L^\infty(\Omega)} < \infty \text{ and } p(x) = p_0(x) \text{ for } x \in \Omega_0 := \omega_0 \times \mathbb{R}. \tag{1.5}$$

Here and henceforth $\langle s\rangle$ is a short hand for $(1+s^2)^{1/s}$, $s \in \mathbb{R}$. Notice that the assumption (1.5) is weaker than the compactness condition imposed in [32, Theorem 1.1] on the support of the unknown part of p. Further, M being an *a priori* fixed non-negative constant, we define the set of admissible potentials as

$$\mathcal{A}(p_0,\omega_0) := \{p \in W^{2N,\infty}(\Omega);\ p = p_0 \text{ in } \Omega_0,\ \|p\|_{W^{2N,\infty}(\Omega)} \leq M \text{ and } \mathtt{N}_{b,d}(p-p_0) \leq M\}. \tag{1.6}$$

Last, we choose a relatively open subset S_* of $\partial\omega$, put $\Gamma_* := S_* \times \mathbb{R}$, and introduce the norm

$$\|\partial_\nu u\|_* := \|\partial_\nu u\|_{H^1(0,T;L^2(\Gamma_*))},\ u \in \mathcal{H}^2.$$

The main result of this article is as follows.

Theorem 1.3. *Let the conditions of Proposition 1.1 be satisfied with $k = N+1$ and $p = p_0$. Assume moreover that u_0 fulfills $\|u_0\|_{H^{2(N+1)}(\Omega)} \leq M'$ for some constant $M' > 0$, and that*

$$\exists\kappa > 0,\ \exists d_0 \in (0, 2d/3),\ |u_0(x', x_n)| \geq \kappa\langle x_n\rangle^{-d_0/2},\ (x', x_n) \in \Omega\backslash\Omega_0. \tag{1.7}$$

For $p_j \in \mathcal{A}(p_0,\omega_0)$, $j = 1,2$, we denote by u_j the $\mathcal{H}^{N+1}$-solution to (1.1)–(1.3)*, where p_j is substituted for p. Then, for any $\epsilon \in (0, N/2)$, there exists a constant $C = C(\omega,\omega_0,T,M,M',b,d,\epsilon) > 0$, such that we have*

$$\|p_1 - p_2\|_{L^2(\Omega)} \leq C\left(\|\partial_\nu(u_1-u_2)\|_* + |\log\|\partial_\nu(u_1-u_2)\|_*|^{-1}\right)^\epsilon. \tag{1.8}$$

1.4. **Comments.** Thanks to the extra information $p_1 = p_2$ in the vicinity of Γ, the sharp unique continuation result by Robbiano [49], Robbiano and Zuily [50] or Tataru [51, 52], entails $u_1 = u_2$ and $\nabla u_1 = \nabla u_2$ on $\partial(\Omega \setminus \overline{\Omega_0}) \times (0, T)$, provided $T > 0$ is sufficiently large. Therefore the method developed by Baudouin and Puel in [5] yields uniqueness in the inverse problem under consideration. However, since we address the stability issue here, it is worth noticing that Theorem 1.3 cannot be obtained by only combining the results of [25, 49, 51].

The technique carried out in this article may be applied, with appropriate modifications, to the determination of higher order unknown coefficients of the Schrödinger equation, but in order to avoid the inappropriate expense of the size of this paper, we shall not go further into details about the treatment of this specific problem.

The analysis developed in this paper boils down to a new specifically designed Carleman estimate for the Schrödinger equation in the cylindrical domain $\Omega \times (0, T)$, when the classical one is valid only in level sets bounded by the weight function. For a general treatment of Carleman estimates, we refer to Hörmander [22], Isakov [27], Tataru [51], and also to Baudouin and Puel [5], where Carleman estimates are derived in a direct pointwise manner. Due to the extra information $p_1 = p_2$ in the vicinity of the boundary Γ, it is useless to discuss here the uniform Lopatinskii condition (see [51] or [9, Section 1.3]) or Carleman estimates with a reduced number of boundary traces.

We assume in (1.7) that $|u(\cdot, 0)| = |u_0| > 0$ in any subset of Ω where the electric potential is retrieved. This is because the uniqueness of the potential is not known in general, without this specific assumption, even in the case where the set $\{x \in \Omega \setminus \Omega_0; u_0(x) = 0\}$ has zero Lebesgue measure. This non-degeneracy condition is very restrictive but it is still an open question to know how it can be weakened in the context of inverse coefficients problems with a finite number of data observations.

Notice that in the framework of the Bukhgeim-Klibanov method in a bounded spatial domain Ω, it is crucial that $|u_0|$ be bounded from below by a positive constant, uniformly in Ω. But since Ω is infinitely extended here, such a statement is incompatible with the square integrability property satisfied by u_0 in Ω. Therefore the usual non-degeneracy condition imposed on the initial condition function has to be weakened into (1.7). In the same spirit we point out that the derivation of a Carleman estimate in an unbounded domain such as Ω is not straightforward and does not directly follows from the corresponding known results in bounded domains.

The subset $\{x \in \Gamma,\ (x - x_0) \cdot \nu \geq 0\}$, lying in the shadow of the boundary Γ viewed from a point x_0 taken in the complement set of $\overline{\Omega}$, satisfies the geometric control property of Bardos, Lebeau and Rauch, see [3]. This property is essentially a necessary and sufficient condition for exact controllability and stabilization of wave equations. However, due to infinite speed of propagation in the Schrödinger equation, this concept is not completely natural in the context of quantum systems. Nevertheless, Lebeau proved in [40] that the above mentioned condition guarantees the boundary controllability of the Schrödinger equation in $H^{-1}(\Omega)$ with $L^2(\Omega)$ boundary controls.

The remainder of the paper is organized as follows. In Section 2, we establish a Carleman inequality for the Schrödinger equation and we state a stability estimate for unique continuation. These two results are needed in the proof of Theorem 1.3, which is given in Section 3. Finally, Section 4 contains the proof of the logarithmic observation inequality stated in Section 2.

2. Preliminary Estimates

In this section we state two preliminary PDE estimates which are the main ingredients in the analysis of the inverse problem under study. To this end we introduce the following notations used throughout the entire text. We consider three open subsets ω_j, $j = 1, 2, 3$, of ω_0, such that

$$\omega_j \subsetneq \omega_{j-1} \text{ and } \partial\omega \subset \partial\omega_j. \tag{2.1}$$

We put

$$\Omega_j := (\omega \setminus \overline{\omega_j}) \times \mathbb{R} = \Omega \setminus (\overline{\omega_j} \times \mathbb{R}) \text{ and } Q_j := \Omega_j \times (-T,T),\ j=2,3,$$

and for $T' \in (0,T)$ fixed, we set $Q_j' := \Omega_j \times (-T',T')$.

2.1. **A Carleman estimate for the Schrödinger equation.** A Carleman estimate is a weighted L^2-norm inequality for a PDE solution. It is particularly useful for proving uniqueness in Cauchy problems or unique continuation results for PDEs with non-analytic coefficients. Carleman estimates are also well adapted to energy estimation in PDEs, see e.g. Kazemi and Klibanov [30] or Klibanov and Malinsky [36]. An alternative method for the derivation of energy inequalities, which is not applicable to the problem under consideration in this paper, can be found in [3].

It is Carleman who first derived in his pioneering paper [16], a suitable inequality, which was later called a Carleman estimate, for proving uniqueness in a two-dimensional elliptic Cauchy problem. Since then, Carleman estimates have been extensively studied by numerous mathematicians. For the general theory of Carleman inequalities for PDEs with isotropic (resp. anisotropic) symbol and compactly supported functions, we refer to Hörmander [22] (resp. Isakov [27]). For Carleman estimates with non-compactly supported functions, see Tataru [51], Bellassoued [7], Fursikov and Imanuvilov [21], and Imanuvilov [23]. Notice that a direct derivation of pointwise Carleman estimates for hyperbolic equations, which are applicable to non compactly supported functions, is available in Klibanov and Timonov's paper [37].

Although Carleman estimates for Schrödinger operators in a bounded domain are rather classical, see e.g. [1, 5, 52], we seek in the context of this paper, a Carleman inequality for the operator

$$P := L + p \text{ with } L := -i\partial_t - \Delta, \tag{2.2}$$

acting in the infinite cylinder Ω. We start by defining suitable weight functions. To this end, we fix $x_0' \in \mathbb{R}^{n-1} \backslash \overline{\omega}$ and put

$$\tilde{\beta}(x') := \left|x' - x_0'\right|^2,\ x' \in \omega, \tag{2.3}$$

in such a way that $\tilde{\beta} \in \mathcal{C}^4(\overline{\omega})$. Here $|x'|$ denotes the Euclidian norm of $x' \in \mathbb{R}^{n-1}$. Next, for every $x = (x', x_n) \in \Omega$, we set

$$\beta(x) := \widetilde{\beta}(x') + K, \text{ where } K := r\|\tilde{\beta}\|_{L^\infty(\omega)} \text{ for some } r > 1, \tag{2.4}$$

and we define two weight functions associated with the parameter $\lambda > 0$:

$$\varphi(x,t) := \frac{e^{\lambda\beta(x)}}{(T'+t)(T'-t)} \text{ and } \eta(x,t) := \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T'+t)(T'-t)},\ (x,t) \in Q' := \Omega \times (-T',T'). \tag{2.5}$$

Finally, for all $s > 0$, we denote by M_1 (resp. M_2) the adjoint (resp. skew-adjoint) part of the operator $e^{-s\eta} L e^{s\eta}$, acting in $(\mathcal{C}_0^\infty)'(Q')$, i.e.

$$M_1 := i\partial_t + \Delta + s^2|\nabla\eta|^2 \text{ and } M_2 := is(\partial_t\eta) + 2s\nabla\eta\cdot\nabla + s(\Delta\eta), \tag{2.6}$$

where we recall that L is the principal part of the operator P given by (2.2).

Having said that, we now state the following global Carleman estimate for the operator P.

Proposition 2.1. *Let β, φ and η be given by* (2.3)–(2.5)*, and let the operators M_j, $j=1,2$, be defined by* (2.6)*. Then there are two constants $s_0 > 0$ and $C > 0$, each of them depending only on ω and T', such that the estimate*

$$\begin{aligned} & s\|e^{-s\eta}\nabla_{x'} w\|^2_{L^2(Q_2')} + s^3\|e^{-s\eta} w\|^2_{L^2(Q_2')} + \sum_{j=1,2} \|M_j e^{-s\eta} w\|^2_{L^2(Q_2')} \\ \leq \ & C\left(\|e^{-s\eta} P w\|^2_{L^2(Q')} + \|e^{-s\eta}\nabla_{x'} w\|^2_{L^2(Q_3'\backslash Q_2')} + \|e^{-s\eta} w\|^2_{L^2(Q_3'\backslash Q_2')}\right), \end{aligned} \tag{2.7}$$

holds for all $s \geq s_0$ and any function $w \in L^2(-T',T';H_0^1(\Omega))$ verifying $Pw \in L^2(Q')$.

Proof. The proof boils down to [32, Proposition 3.3], which provides two constants $s_0 > 0$ and $C > 0$, both of them depending only on ω, T and M, such that we have

$$s\|e^{-s\eta}\nabla_{x'}\tilde{w}\|_{L^2(Q')}^2 + s^3\|e^{-s\eta}\tilde{w}\|_{L^2(Q')}^2 + \sum_{j=1,2}\|M_j e^{-s\eta}\tilde{w}\|_{L^2(Q')}^2 \leq C\|e^{-s\eta}P\tilde{w}\|_{L^2(Q')}^2, \tag{2.8}$$

for every $s \geq s_0$, and any $\tilde{w} \in L^2(-T', T'; H_0^1(\Omega))$ such that $P\tilde{w} \in L^2(Q')$ and $\partial_\nu \tilde{w} = 0$ on $\Sigma' := \Gamma \times (-T', T')$. Next we pick a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R}^{n-1}, [0,1])$ satisfying

$$\chi(x') = \begin{cases} 1 & \text{if } x' \in \omega \setminus \omega_2, \\ 0 & \text{if } x' \in \omega_3, \end{cases}$$

and apply the estimate (2.8) to $\tilde{w}(x,t) = \chi(x')w(x,t)$. Using that $P\tilde{w} = \chi P w + [P,\chi]w$, where $[A,B]$ stands for the commutator of the operators A and B, and taking into account that $[P,\chi]$ is a first order differential operator whose coefficients are supported in $\Omega_3 \setminus \Omega_2$, we obtain (2.7). □

2.2. **Logarithmic stability of unique continuation.** The unique continuation of a solution to the Schrödinger equation (1.1) from lateral boundary data on Γ_* was proved by Phung in [45]. The coming result claims stability for the same problem.

Lemma 2.2. *Let $p_j \in \mathcal{A}(p_0, \omega_0)$ for $j = 1,2$, let u_j be a solution to the Schrödinger equation* (1.1) *where p_j is substituted for p, and put $u := u_1 - u_2$. Then for all $\mu \in (0,1)$, we may find a constant $C = C(\omega, \omega_0, T, M, M', \mu) > 0$, depending neither on p_1 nor on p_2, such that we have*

$$\|\partial_t u\|_{L^2((\Omega_3\setminus\Omega_2)\times(0,T/6))}^2 + \|\nabla_{x'}\partial_t u\|_{L^2((\Omega_3\setminus\Omega_2)\times(0,T/6))}^2 \leq C\left(\|\partial_\nu u\|_* + |\log\|\partial_\nu u\|_*|^{-1}\right)^{2\mu N}. \tag{2.9}$$

The proof of this result boils down to the analysis carried out by Robbiano in [48, 49] or Phung in [45], by means of the FBI transformation. Since it is rather lengthy, we postpone it to Section 4.

3. Proof of Theorem 1.3

In this section we establish the stability estimate (1.8) by adapting the Bukhgeim-Klibanov method presented in [15], to the context of the infinite waveguide Ω. In light of Lemma 2.2, we set $T' := T/6$. The first step involves linearizing the system (1.1)–(1.3) and symmetrizing its solution with respect to the time variable t.

3.1. **Linearization and time symmetrization.** With reference to the notations of Theorem 1.3, we put $p := p_2 - p_1$ and notice that $u := u_1 - u_2$ is a $\mathcal{H}^{N+1}$-solution to the IBVP

$$\begin{cases} -i\partial_t u - \Delta u + p_1 u = p u_2 & \text{in } \Omega \times (0,T), \\ u(\cdot,0) = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \Gamma \times (0,T). \end{cases} \tag{3.1}$$

In particular we have $u \in \mathcal{C}^1([0,T]; H^{2N}(\Omega))$, hence upon differentiating (3.1) with respect to t, we get that $v := \partial_t u \in \mathcal{H}^N$ is solution to the system

$$\begin{cases} -i\partial_t v - \Delta v + p_1 v = p\partial_t u_2 & \text{in } \Omega \times (0,T), \\ v(\cdot,0) = ipu_0 & \text{in } \Omega, \\ v = 0, & \text{on } \Gamma \times (0,T). \end{cases} \tag{3.2}$$

Further, putting $u_2(x,-t) := \overline{u_2(x,t)}$ for all $(x,t) \in \Omega \times (0,T]$, and bearing in mind that u_0 and p are real-valued, we deduce from (3.2) that the function v, extended on $[-T,0) \times \Omega$ by setting $v(x,t) := -\overline{v(x,-t)}$, is the $\cap_{k=0}^{N} \mathcal{C}^k([-T,T], H^{2(N-k)}(\Omega))$-solution to the system

$$\begin{cases} -i\partial_t v - \Delta v + p_1 v &= p\partial_t u_2 \quad \text{in } Q := \Omega \times (-T,T), \\ v(\cdot,0) &= ipu_0 \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \Sigma := \Gamma \times (-T,T). \end{cases} \tag{3.3}$$

The second step in the derivation of (1.8) is to apply the global Carleman inequality of Proposition 2.1 to v, in order to establish Lemma 3.1 stated below.

3.2. **An *a priori* estimate.** We stick with notations of Section 2 and Subsection 3.1, and we establish the following technical result, which is quite similar to [32, Lemmas 3.3 & 3.4] and [33, Lemma 3.3]. Nevertheless, we include the proof just for the convenience of the reader.

Lemma 3.1. *Let v denote the $\mathcal{C}^1([-T,T], L^2(\Omega)) \cap \mathcal{C}^0([-T,T], H_0^1(\Omega) \cap H^2(\Omega))$-solution to* (3.3). *Then there exists a constant $C > 0$, independent of s, such that we have*

$$\|e^{-s\eta(\cdot,0)} pu_0\|_{L^2(\Omega)}^2 \leq Cs^{-3/2} \left(\|e^{-s\eta(\cdot,0)} p\partial_t u_2\|_{L^2(Q')}^2 + \|e^{-s\eta} \nabla_{x'} v\|_{L^2(Q_3' \setminus Q_2')}^2 + \|e^{-s\eta} v\|_{L^2(Q_3' \setminus Q_2')}^2 \right),$$

uniformly in $s \in (0, +\infty)$.

Proof. Put $\phi(x,t) := e^{-s\eta(x,t)} \xi(x') v(x,t)$ for $(x,t) \in Q'$, where $\xi \in \mathcal{C}_0^\infty(\omega)$ is a cut-off function satisfying

$$\xi(x') = \begin{cases} 1 & \text{if } x \in \omega \setminus \omega_1, \\ 0 & \text{if } x \in \omega_2. \end{cases}$$

According to (2.4)-(2.5), we have $\lim_{t \downarrow (-T')} \eta(x,t) = +\infty$ for every $x \in \Omega$, and hence $\lim_{t \downarrow (-T')} \phi(x,t) = 0$. As a consequence, it holds true that

$$\|\phi(\cdot,0)\|_{L^2(\Omega)}^2 = \int_{\Omega \times (-T',0)} \partial_t |\phi|^2(x,t) dx dt = 2\mathrm{Re} \left(\int_{\Omega \times (-T',0)} (\partial_t \phi) \overline{\phi(x,t)} dx dt \right). \tag{3.4}$$

On the other hand, (2.6) and the Green formula yield

$$\mathrm{Im} \left(\int_{\Omega \times (-T',0)} (M_1 \phi) \overline{\phi(x,t)} dx dt \right) = \mathrm{Re} \left(\int_{\Omega \times (-T',0)} (\partial_t \phi) \overline{\phi(x,t)} dx dt \right) + R,$$

where

$$R = \mathrm{Im} \left(\int_{\Omega \times (-T',0)} (\Delta \phi) \overline{\phi(x,t)} dx dt + s^2 \|(\nabla \eta) \phi\|_{L^2(\Omega \times (-T',0))}^2 \right) = -\mathrm{Im} \left(\|\nabla \phi\|_{L^2(\Omega \times (-T',0))}^2 \right) = 0.$$

We deduce from this, (3.4) and the identity $\|\phi(\cdot,0)\|_{L^2(\Omega)} = \|e^{-s\eta(\cdot,0)} \xi v(\cdot,0)\|_{L^2(\Omega)}$, that

$$\begin{aligned} \|e^{-s\eta(\cdot,0)} \xi v(\cdot,0)\|_{L^2(\Omega)}^2 &= 2\mathrm{Im} \left(\int_{\Omega \times (-T',0)} (M_1 \phi) \overline{\phi(x,t)} dx dt \right) \leq 2 \|M_1 \phi\|_{L^2(Q')} \|\phi\|_{L^2(Q')} \\ &\leq s^{-3/2} \left(s \|e^{-s\eta} \nabla_{x'} v\|_{L^2(Q_2')}^2 + s^3 \|e^{-s\eta} v\|_{L^2(Q_2')}^2 + \|M_1 e^{-s\eta} v\|_{L^2(Q_2')}^2 \right). \end{aligned}$$

Finally, the desired result follows from this upon recalling (3.3), applying Proposition 2.1 to v, and noticing from (2.4)-(2.5) that $\eta(x,t) \geq \eta(x,0)$ for every $(x,t) \in Q'$. $\square$

3.3. **End of the proof.** For any fixed $y>0$, it follows from Lemma 3.1 that

$$\|e^{-s\eta(\cdot,0)}pu_0\|^2_{L^2(\omega\times(-y,y))}\leq Cs^{-3/2}\left(\|e^{-s\eta(\cdot,0)}p\partial_t u_2\|^2_{L^2(Q')}+\varrho^2\right),\ s>0, \tag{3.5}$$

where $\varrho^2:=\|\nabla_{x'}v\|^2_{L^2(Q_3'\setminus Q_2')}+\|v\|^2_{L^2(Q_3'\setminus Q_2')}$. Here and in the remaining part of the proof, C denotes a generic positive constant that is independent of s.

Notice from (1.7) and the vanishing of p in Ω_0, that the inequality $|(pu_0)(x)|\geq\kappa\langle y\rangle^{-d_0/2}|p(x)|$ holds for every $x\in\omega\times(-y,y)$. Furthermore, we have $\|\partial_t u_2\|_{L^\infty(Q')}\leq C$, by Corollary 1.2, and $\eta(x,0)\geq 0$ for every $x\in\Omega$, by (2.4)-(2.5), so we may deduce from (3.5) that

$$\left(\kappa^2\langle y\rangle^{-d_0}-Cs^{-3/2}\right)\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\omega\times(-y,y))}\leq Cs^{-3/2}\left(\|p\|^2_{L^2(\omega\times(\mathbb{R}\setminus(-y,y)))}+\varrho^2\right),\ s>0. \tag{3.6}$$

Thus, taking $s=(\kappa^2/(2C))^{-2/3}\langle y\rangle^{2d_0/3}$ in (3.6), and recalling from (2.4)-(2.5) that $\|\eta(.,0)\|_{L^\infty(\Omega)}\leq(e^{\lambda K}/T')^2$, we obtain that

$$\|p\|^2_{L^2(\omega\times(-y,y))}\leq Ce^{C\langle y\rangle^{2d_0/3}}\left(\|p\|^2_{L^2(\omega\times(\mathbb{R}\setminus(-y,y)))}+\varrho^2\right). \tag{3.7}$$

Moreover, as $\|p\|^2_{L^2(\omega\times(\mathbb{R}\setminus(-y,y)))}\leq C\|e^{-2b\langle\cdot\rangle^d}\|_{L^1(\mathbb{R}\setminus(-y,y))}$ from (1.5), we have for any $\delta\in(0,b)$,

$$\|p\|^2_{L^2(\omega\times(\mathbb{R}\setminus(-y,y)))}\leq C\|e^{-\delta\langle\cdot\rangle^d}\|_{L^1(\mathbb{R})}e^{-(2b-\delta)\langle y\rangle^d}\leq Ce^{-(2b-\delta)\langle y\rangle^d}. \tag{3.8}$$

Putting this together with (3.7), we find that

$$\|p\|^2_{L^2(\omega\times(-y,y))}\leq Ce^{C\langle y\rangle^{2d_0/3}}\left(e^{-(2b-\delta)\langle y\rangle^d}+\varrho^2\right). \tag{3.9}$$

Setting $\varrho_\delta:=e^{-(2b-\delta)}$, we turn now to examining the two cases $\varrho\in(0,\varrho_\delta)$ and $\varrho\in[\varrho_\delta,+\infty)$ separately. First, if $\varrho\in(0,\varrho_\delta)$, we take $y=y(\varrho):=\left(\left(2\frac{\ln\varrho}{\ln\varrho_\delta}\right)^{2/d}-1\right)^{1/2}$ in (3.9), in such a way that $\varrho^2=e^{-(2b-\delta)\langle y\rangle^d}$, and consequently

$$\|p\|^2_{L^2(\omega\times(-y,y))}\leq Ce^{C\langle y\rangle^{2d_0/3}-(2b-\delta)\langle y\rangle^d}. \tag{3.10}$$

Since $d>2d_0/3$, we have $\sup_{t\in(0,1)}e^{Ct^{2d_0/3}-\delta t^d}<+\infty$, whence (3.10) yields

$$\|p\|^2_{L^2(\omega\times(-y,y))}\leq C\left(\sup_{t\in(1,+\infty)}e^{Ct^{2d_0/3}-\delta t^d}\right)e^{-2(b-\delta)\langle y\rangle^d}\leq C\varrho^{2\theta},\ \varrho\in(0,\varrho_\delta), \tag{3.11}$$

with

$$\theta:=\frac{b-\delta}{2b-\delta}\in(0,1/2). \tag{3.12}$$

On the other hand, we have $\|p\|^2_{L^2(\omega\times(\mathbb{R}\setminus(-y,y)))}\leq Ce^{-2(b-\delta)\langle y\rangle^d}\leq C\varrho^{2\theta}$ for all $\varrho\in(0,\varrho_\delta)$, by (3.8). This and (3.11) entail

$$\|p\|^2_{L^2(\Omega)}\leq C\varrho^{2\theta},\ \varrho\in(0,\varrho_\delta). \tag{3.13}$$

In the case where $\varrho\in[\varrho_\delta,+\infty)$, we use the upper bound $\|p\|^2_{L^2(\Omega)}\leq C\|e^{-2b\langle\cdot\rangle^d}\|_{L^1(\mathbb{R})}$, arising from (1.5), and obtain that

$$\|p\|^2_{L^2(\Omega)}\leq C\left(\|e^{-2b\langle\cdot\rangle^d}\|_{L^1(\mathbb{R})}/\varrho_\delta^{2\theta}\right)\varrho^{2\theta}\leq C\varrho^{2\theta},\ \varrho\in[\varrho_\delta,+\infty). \tag{3.14}$$

Now, recalling the above definition of ϱ^2 and the identity $v=\partial_t u$, we find upon applying Lemma 2.2 to u, that

$$\varrho^2\leq C\left(\|\partial_\nu u\|_*+|\log\|\partial_\nu u\|_*|^{-1}\right)^{2\mu N}, \tag{3.15}$$

for any arbitrary $\mu \in (0,1)$. Therefore, θ being any real number in $(0,1/2)$, according to (3.12) and since δ is arbitrary in $(0,b)$, the estimate (1.8) follows from (3.13)–(3.15) upon taking $\epsilon = \theta\mu N$.

4. Logarithmic observability inequality: Proof of Lemma 2.2

In this section we prove the logarithmic observability inequality stated in Lemma 2.2.

Prior to doing that we recall for further reference from the energy inequality (1.4) with $k = N+1$, that for any $p_j \in \mathcal{A}(\omega_0, M)$, $j = 1,2$, the solution $v = \partial_t(u_1 - u_2)$ to the IBVP (3.3), satisfies the estimate

$$\|v\|_{\mathcal{C}^N([-T,T],L^2(\Omega))} + \|v\|_{\mathcal{C}^{N-1}([-T,T],H^2(\Omega))} \leq 2C\|u_0\|_{H^{2(N+1)}(\Omega)}, \tag{4.1}$$

where the positive constant $C = C(\omega, \omega_0, T, M, M') > 0$ is the same as in (1.4).

4.1. A parabolic Carleman estimate for unbounded cylindrical domains.

In this subsection we state a parabolic Carleman estimate in unbounded cylindrical domains, which is needed in the proof of Lemma 2.2. To do that, we start by introducing the two sets

$$S_\sharp := \partial\omega_0 \setminus \partial\omega \text{ and } \Gamma_\sharp := S_\sharp \times \mathbb{R},$$

and we assume without loss of generality (upon possibly smoothening $\partial\omega_0$ by enlarging ω_0), that $S_\sharp$ is $\mathcal{C}^2$. Then, with reference to [25, Lemma 2.3] and its proof, we pick a function $\psi_0 \in \mathcal{C}^2(\overline{\omega}_0)$, obeying the four following conditions:

$$\psi_0(x') > 0, x' \in \omega_0 \quad \text{and} \quad \left|\nabla\psi_0(x')\right| > 0,\ x' \in \overline{\omega}_0, \tag{4.2}$$

$$\psi_0(x') = 0, x' \in S_\sharp \quad \text{and} \quad \partial_\nu\psi_0(x') \leq 0,\ x' \in \partial\omega_0 \setminus S_*. \tag{4.3}$$

Next we put $\ell(\tau) := (1-\tau)(1+\tau)$ for each $\tau \in (-1,1)$, and introduce the two weight functions

$$\varphi_0(x',\tau) := \frac{e^{\lambda(\psi_0(x')+a)}}{\ell(\tau)},\ x \in \omega_0,\ \tau \in (-1,1), \tag{4.4}$$

and

$$\alpha(x',\tau) := \frac{e^{\lambda(\psi_0(x')+a)} - e^{\lambda(\|\psi_0\|_{L^\infty(\omega_0)}+b)}}{\ell(\tau)},\ x \in \omega_0,\ \tau \in (-1,1), \tag{4.5}$$

where $\lambda \in (0,+\infty)$ is a fixed parameter, ψ_0 is the function defined by (4.2)-(4.3), and

$$\|\psi_0\|_{L^\infty(\omega_0)} < a < b < 2a - \|\psi_0\|_{L^\infty(\omega_0)}.$$

Further, in connection with the Schrödinger operator P defined in (2.2), we consider the formal parabolic operator in $\Omega_0 := \omega_0 \times \mathbb{R}$, associated with some fixed parameter $h \in (0,1)$,

$$\mathcal{L}_h := h^{-1}\partial_\tau - \Delta + p_1. \tag{4.6}$$

We are now in position to state the following Carleman estimate for the operator $\mathcal{L}_h$.

Lemma 4.1. *Let φ_0 and α be defined by* (4.4)-(4.5)*, and for $h \in (0,1)$ fixed, let $\mathcal{L}_h$ be defined by* (4.6)*. Then we may find three positive constants λ_0, σ_0 and C_0, such that for every $\lambda \geq \lambda_0$ and $\sigma \geq \sigma_0/h$, the estimate*

$$\begin{aligned} &\sigma\|e^{\sigma\alpha}\nabla_{x'}w\|^2_{L^2(\Omega_0\times(-1,1))} + \sigma^3\|e^{\sigma\alpha}w\|^2_{L^2(\Omega_0\times(-1,1))} \\ \leq\ & C_0\left(\|e^{\sigma\alpha}\mathcal{L}_h w\|^2_{L^2(\Omega_0\times(-1,1))} + \sigma\|\varphi_0^{1/2}e^{\sigma\alpha}\partial_\nu w\|^2_{L^2(\Gamma_*\times(-1,1))}\right), \end{aligned} \tag{4.7}$$

holds for $w \in L^2(-1,1;H_0^1(\Omega_0))$ verifying $\mathcal{L}_h w \in L^2(-1,1;L^2(\Omega_0))$ and $\partial_\nu w \in L^2(-1,1;L^2(\Gamma_*))$. Here the constant $C_0 > 0$ depends continuously on λ, M, M' and h, but is independent of σ.

We stress out that a result similar to Lemma 4.1 can be found in [25, Lemma 2.4] (see also [20, 21]) in the context of bounded spatial domains.

The dependence of the various constants appearing in (4.7), with respect to the parameter $h \in (0,1)$, is made precise in the derivation of Lemma 4.1, which is given in Appendix A.

4.2. **A connection between Schrödinger and parabolic equations.** As pointed out by Lebeau and Robbiano [41], Robbiano [49], Robbiano and Zuily [50] and Phung [45], connections between solutions of different types of PDEs may be useful for examining the controllability of numerous Cauchy problems. In this subsection we prove that the FBI transform of χv, where v is the solution to (3.3) and $\chi = \chi(x')$ is a suitable cut-off function that will be made precise below, is solution to a parabolic Cauchy problem in $\Omega_0 = \omega_0 \times \mathbb{R}$.

Prior to doing that we introduce the FBI transform, as defined by Lebeau and Robbiano in [41]. To this purpose we fix $\mu \in (0,1)$ and choose $m \in \mathbb{N}^*$ so large that

$$2m \geq N \text{ and } \rho := 1 - \frac{1}{2m} > \mu. \tag{4.8}$$

Then, for any $\gamma \in (1, +\infty)$, the function

$$F_\gamma(z) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\eta} e^{-(\eta/\gamma^\rho)^{2m}} d\eta,\ z \in \mathbb{C}, \tag{4.9}$$

is holomorphic in $\mathbb{C}$, and there exist four positive constants C_j, $j = 1, 2, 3, 4$, none of them depending on γ, such that we have

$$|F_\gamma(z)| \leq C_1 \gamma^\rho e^{C_2 \gamma |\mathrm{Im} z|^{1/\rho}},\ z \in \mathbb{C}, \tag{4.10}$$

and

$$|F_\gamma(z)| \leq C_1 \gamma^\rho e^{-C_3 \gamma |\mathrm{Re} z|^{1/\rho}},\ z \in \{z \in \mathbb{C},\ |\mathrm{Im} z| \leq C_4 |\mathrm{Re} z|\}. \tag{4.11}$$

Given $T_0 \in (T/3, +\infty)$, we consider a cut-off function $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$, obeying

$$\theta(\eta) = \begin{cases} 1 & \text{if } |\eta| \leq 2T_0, \\ 0 & \text{if } |\eta| \geq 3T_0, \end{cases} \tag{4.12}$$

and we define the partial FBI transform of $w \in \mathcal{S}(\mathbb{R}^{n+1})$, by

$$w_{\gamma,t}(x,\tau) := \mathscr{F}_\gamma w(x,z) = \int_{\mathbb{R}} F_\gamma(z-\eta)\theta(\eta) w(x, h\eta) d\eta,\ z = t - i\tau, \tag{4.13}$$

for all $t \in (-T_0, T_0)$, $\tau \in (-1,1)$, $\gamma \in (1, +\infty)$ and $x \in \mathbb{R}^n$, where $h := T/(3T_0)$.

Next, taking into account that $\omega_1 \subset \omega_0$, by (2.1), we deduce from the continuity of the function ψ_0 introduced in Subsection 4.1, and from the first part of (4.2), that there exists a constant $\beta_0 > 0$, such that

$$\psi_0(x') \geq 2\beta_0,\ x' \in \omega_2 \setminus \omega_3. \tag{4.14}$$

Moreover, due to the vanishing of ψ_0 on $S_\sharp$, imposed by the first claim of (4.3), we may find a subset $\omega^\sharp \subset \omega_0 \setminus \overline{\omega_1}$, such that

$$S_\sharp \subset \overline{\omega^\sharp} \text{ and } \psi_0(x') \leq \beta_0 \text{ for } x' \in \omega^\sharp. \tag{4.15}$$

Let us now pick $\widetilde{\omega}^\sharp \subset \omega^\sharp$, such that $S_\sharp \subset \overline{\widetilde{\omega}^\sharp}$, and introduce a function $\chi \in \mathcal{C}^\infty(\mathbb{R}^{n-1}, [0,1])$ satisfying

$$\chi(x') = \begin{cases} 1 & \text{if } x' \in \omega_0 \setminus \omega^\sharp, \\ 0 & \text{if } x' \in \widetilde{\omega}^\sharp. \end{cases} \tag{4.16}$$

Thus, bearing in mind that $p = p_1 - p_2$ vanishes in Ω_0 and that v is the solution to (3.3), we easily find that the function $w(x,t) := \chi(x')v(x,t)$ satisfies the IBVP

$$\left\{ \begin{array}{rcll} -i\partial_t w - \Delta w + p_1 w & = & -\left[\Delta, \chi\right] v & \text{in } Q_0 := \Omega_0 \times (-T,T), \\ w(0,\cdot) & = & 0 & \text{in } \Omega_0, \\ w & = & 0 & \text{on } \Sigma_0 := \partial\Omega_0 \times (-T,T). \end{array} \right. \tag{4.17}$$

Moreover, we deduce from (4.1) that

$$\|w\|_{\mathcal{C}^N([-T,T],L^2(\Omega))} + \|w\|_{\mathcal{C}^{N-1}([-T,T],H^2(\Omega))} \leq C\|u_0\|_{H^{2(N+1)}(\Omega)}, \tag{4.18}$$

where C denotes a generic positive constant that is independent of γ. From this, (4.10) and (4.13), we get two positive constants $C = C(\omega, \omega_0, T, T_0, M, M')$ and δ_1, the last one being independent of T_0, such that the estimate

$$\|w_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} + \|\nabla w_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} \leq Ce^{\delta_1\gamma},$$

holds uniformly in $t \in (-T_0, T_0)$ and $\gamma \in (1, +\infty)$.

We turn now to establishing that $w_{\gamma,t}$ is solution to a parabolic Cauchy problem in Ω_0, we shall make explicit below. To do that, we derive from (4.13) upon integrating by parts, that

$$h^{-1}\partial_\tau w_{\gamma,t}(x,\tau) = -i\mathscr{F}_\gamma(\partial_t w)(x,z) - ih^{-1}\int_{\mathbb{R}} F_\gamma(z-\eta)\theta'(\eta)w(x,h\eta)d\eta,\ z = t - i\tau.$$

Next, as we have $\Delta w_{\gamma,t}(x,\tau) = \mathscr{F}_\gamma(\Delta w)(x,z)$ by direct calculation, we get upon applying the FBI transform $\mathscr{F}_\gamma$ to (4.17) and remembering (4.6), that

$$\left\{ \begin{array}{rcll} \mathcal{L}_h w_{\gamma,t}(x,\tau) & = & A_{\gamma,t}(x,\tau) + B_{\gamma,t}(x,\tau), & (x,\tau) \in \Omega_0 \times (-1,1), \\ w_{\gamma,t}(x,\tau) & = & 0, & (x,\tau) \in \partial\Omega_0 \times (-1,1), \end{array} \right. \tag{4.19}$$

where

$$A_{\gamma,t}(x,\tau) := -\int_{\mathbb{R}} F_\gamma(z-\eta)\theta(\eta)\left[\Delta_{x'}, \chi\right] v(x,h\eta)d\eta = -\left[\Delta_{x'}, \chi\right] v_{\gamma,t}(x,\tau), \tag{4.20}$$

and

$$B_{\gamma,t}(x,\tau) := -ih^{-1}\int_{\mathbb{R}} F_\gamma(z-\eta)\theta'(\eta)w(x,\eta h)d\eta,\ z = t - i\tau. \tag{4.21}$$

The next step of the proof is to apply the parabolic Carleman estimate of Lemma 4.1 to the solution $w_{\gamma,t}$ of (4.19) in order to derive the coming result.

Lemma 4.2. *There exist $\varepsilon \in (0,1)$, $\delta_2 > 0$, $\delta_3 > 0$, and $\gamma_0 > 0$, such that any solution $w_{\gamma,t}$ to* (4.19)*, satisfies the estimate*

$$\|w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} + \|\nabla_{x'} w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} \leq C\left(e^{-\delta_2\gamma} + e^{\delta_3\gamma}\|\partial_\nu w_{\gamma,t}\|^2_{L^2(\Gamma_*\times(-1,1))}\right), \tag{4.22}$$

uniformly in $t \in (-T_0, T_0)$ and $\gamma \in [\gamma_0, +\infty)$.

Proof. Fix $\gamma \in (1, +\infty)$ and $t \in (-T_0, T_0)$. In light of (4.19), we apply the Carleman estimate of Lemma 4.1 to $w_{\gamma,t}$, and find for every $\sigma \in [\sigma_0/h, +\infty)$ that

$$\begin{array}{rl} & \sigma\|e^{\sigma\alpha}\nabla_{x'} w_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} + \sigma^3\|e^{\sigma\alpha} w_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} \\ \leq & C_0\left(\|e^{\sigma\alpha}A_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} + \|e^{\sigma\alpha}B_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} + \sigma\|\varphi_0^{1/2}e^{\sigma\alpha}\partial_\nu w_{\gamma,t}\|^2_{L^2(\Gamma_*\times(-1,1))}\right). \end{array} \tag{4.23}$$

Further, we notice from (4.16) that $A_{\gamma,t}(\cdot,\tau)$ is supported in $\Omega_\sharp := \omega_\sharp \times \mathbb{R}$ for every $\tau \in (-1,1)$, and from (4.5) and (4.15) that $\alpha(x',\tau) \leq (-\mu_1)$ for all $(x',\tau) \in \omega^\sharp \times (-1,1)$, with $\mu_1 := e^{\lambda(\|\psi_0\|_\infty + b)} - e^{\lambda(\beta_0 + a)} > 0$. As a consequence we have

$$\|e^{\sigma\alpha} A_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} \leq e^{-2\mu_1\sigma} \|A_{\gamma,t}\|^2_{L^2(\Omega_\sharp\times(-1,1))} \leq e^{-2\mu_1\sigma} \|A_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))}. \tag{4.24}$$

The next step is to choose $\varepsilon \in (0,1)$ so small that $\mu_2 := (e^{\lambda(\|\psi_0\|_\infty+b)} - e^{\lambda(2\beta_0+a)})/\ell(\varepsilon) < \mu_1$. Then, bearing in mind that $\ell(\tau) \geq \ell(\varepsilon) > 0$ for each $\tau \in (-\varepsilon,\varepsilon)$, we see from (4.14) that $\alpha(x',\tau) \geq (-\mu_2)$ for every $(x',\tau) \in (\omega_2 \setminus \omega_3) \times (-\varepsilon,\varepsilon)$. This entails that

$$\begin{aligned} & e^{-2\mu_2\sigma}\left(\|\nabla_{x'} w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} + \|w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))}\right) \\ \leq\ & \sigma \|e^{\sigma\alpha}\nabla_{x'} w_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} + \sigma^3 \|e^{\sigma\alpha} w_{\gamma,t}\|^2_{L^2((\Omega_0\times(-1,1))}, \ \sigma \in [1,+\infty). \end{aligned}$$

Setting $\mu := \mu_1 - \mu_2$, it follows from this and (4.23)-(4.24) that

$$\begin{aligned} & \|\nabla_{x'} w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} + \|w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} \\ \leq\ & C\left(e^{-2\mu\sigma}\|A_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} + e^{2\mu_2\sigma}\|B_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} + \sigma e^{2\mu_2\sigma}\|e^{\sigma\alpha}\varphi_0^{1/2}\partial_\nu w_{\gamma,t}\|^2_{L^2(\Gamma_*\times(-1,1))}\right) \end{aligned} \tag{4.25}$$

whenever $\sigma \in [\sigma_0/h,+\infty)$. Here we assumed upon possibly substituting $\max(1,\sigma_0/h)$ for σ_0, that $\sigma_0 \geq h$.

In view of (4.10) and (4.20), the first term in the right hand side of (4.25) can be treated by the energy estimate $\|v\|_{\mathcal{C}^0([-T,T],H^1(\Omega))} \leq 2C\|u_0\|_{H^{2(N+1)}(\Omega)}$, arising from (4.1): We get a constant $\delta' > 0$, independent of T_0 and γ, such that

$$\|A_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} \leq Ce^{\delta'\gamma},\ t \in (-T_0,T_0). \tag{4.26}$$

For the second term, we take into account the vanishing of θ' in the interval $(-2T_0,2T_0)$, imposed by (4.12), and deduce from (4.11) and (4.21) that

$$\|B_{\gamma,t}\|^2_{L^2(\Omega_0\times(-1,1))} \leq Ce^{-\tilde{\delta}T_0^{1/\rho}\gamma},\ t \in (-T_0,T_0), \tag{4.27}$$

for some constant $\tilde{\delta} > 0$ depending neither on T_0 nor on γ. Here we used the estimate $\|w\|_{\mathcal{C}^0([-T,T],L^2(\Omega))} \leq C\|u_0\|_{H^{2(N+1)}(\Omega)}$, which follows from (4.18).

Last we notice from (4.3)–(4.5) that $\varphi_0^{1/2}e^{\sigma\alpha}$ is bounded on $S_* \times (-1,1)$, and then deduce from (4.25)–(4.27) that

$$\begin{aligned} & \|\nabla_{x'} w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} + \|w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} \\ \leq\ & C\left(e^{-2\mu\sigma+\delta'\gamma} + e^{2\mu_2\sigma-\tilde{\delta}T_0^{1/\rho}\gamma} + \sigma e^{2\mu_2\sigma}\|\partial_\nu w_{\gamma,t}\|^2_{L^2(\Gamma_*\times(-1,1))}\right),\ \sigma \in [\sigma_0/h,+\infty). \end{aligned} \tag{4.28}$$

Now, set $\gamma_0 := \max(1, 3\sigma_0/T)$ and for $\gamma \in [\gamma_0,+\infty)$, take $\sigma := T_0\gamma \geq \sigma_0/h$ in (4.28). As the sum of the two first terms in the right hand side of (4.28) is majorized by $e^{(-2\mu T_0+\delta')\gamma} + e^{(2\mu_2 T_0 - \tilde{\delta}T_0^{1/\rho})\gamma} \leq Ce^{-\delta_2\gamma}$ upon taking T_0 sufficiently large (as we have $1/\rho > 1$), we end up getting for all $t \in (-T_0,T_0)$, that

$$\begin{aligned} & \|\nabla_{x'} w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} + \|w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))} \\ \leq\ & C\left(e^{-\delta_2\gamma} + e^{\delta_3\gamma}\|\partial_\nu w_{\gamma,t}\|^2_{L^2(\Gamma_*\times(-1,1))}\right),\ \gamma \in [\gamma_0,+\infty). \end{aligned}$$

This entails the desired result. $\square$

4.3. **Completion of the proof.** Set $w_\gamma(x,t) := w_{\gamma,t}(x,0)$ and recall from (4.13) that we have

$$w_\gamma(x,t) = (F_\gamma * (\theta\widetilde{w})(x,\cdot))(t), \tag{4.29}$$

for all $\gamma \in [\gamma_0,+\infty)$, $x \in \mathbb{R}^n$ and $t \in (-T_0,T_0)$, where F_γ is defined in (4.9) and $\widetilde{w}(x,\eta) := w(x,h\eta)$.

Let us first deduce from Lemma 4.2 the following estimate on w_γ.

Lemma 4.3. *There exist two positive constants δ_4 and δ_5, such that the estimate*

$$\begin{aligned} & \|\nabla_{x'} w_\gamma\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} + \|w_\gamma\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} \\ \leq \;& C\left(e^{-\delta_4\gamma} + e^{\delta_5\gamma}\|\partial_\nu \tilde{w}\|^2_{L^2(\Gamma_*\times(-3T_0,3T_0))}\right), \end{aligned}$$

holds for all $\gamma \in [\gamma_0,+\infty)$.

Proof. Let $\varepsilon \in (0,1)$ be given by Lemma 4.2 and fix $\kappa \in [T_0-\varepsilon, T_0+\varepsilon]$. We assume, upon possibly shortening ε into the left hand side of the estimate (4.22) in Lemma 4.2, that $\varepsilon < T_0/2$. Since $w_\gamma(x,z) := w_{\gamma,\mathrm{Re}(z)}(x,\mathrm{Im}\,(z))$ is analytic in $z \in \{\zeta \in \mathbb{C},\ \mathrm{Re}\,(\zeta) \in (-T_0,T_0),\ \mathrm{Im}\,(\zeta) \in (-1,1)\}$ for every fixed $x \in \omega_2 \setminus \omega_3$, the Cauchy formula yields

$$w_\gamma(x,\kappa) = \frac{1}{2i\pi}\int_{|z-\kappa|=\varrho} \frac{w_\gamma(x,z)}{z-\kappa}dz = \frac{1}{2\pi}\int_0^{2\pi} w_\gamma(x,\kappa+\varrho e^{i\phi})d\phi,\ \varrho \in (0,\varepsilon).$$

Therefore, we have $|w_\gamma(x,\kappa)|^2 \leq (2\pi)^{-1}\int_0^{2\pi}\left|w_\gamma(x,\kappa+\varrho e^{i\phi})\right|^2 d\phi$, from the Cauchy-Schwarz inequality. Since the above estimate is valid uniformly in $\varrho \in (0,\varepsilon)$, we find that

$$\begin{aligned} |w_\gamma(x,\kappa)|^2 &\leq \frac{1}{2\pi\varepsilon}\int_0^\varepsilon\int_0^{2\pi}\left|w_\gamma(x,\kappa+\varrho e^{i\phi})\right|^2 d\phi d\varrho \\ &\leq \frac{1}{2\pi\varepsilon}\int_{|\tau|\leq\varepsilon}\int_{|t-\kappa|\leq\varepsilon}|w_\gamma(x,t+i\tau)|^2 dtd\tau \\ &\leq \frac{1}{2\pi\varepsilon}\int_{-T_0}^{T_0}\|w_{\gamma,t}(x,\cdot)\|^2_{L^2(-\varepsilon,\varepsilon)}dt, \end{aligned}$$

and hence

$$\|w_\gamma\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} \leq C\int_{-T_0}^{T_0}\|w_{\gamma,t}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-\varepsilon,\varepsilon))}dt,$$

upon integrating with respect to (x,κ) over $(\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2)$. Thus, bearing in mind (4.12)-(4.13) and applying Lemma 4.2, we end up getting that

$$\|w_\gamma\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} \leq C\left(e^{-C\gamma} + e^{C\gamma}\|\partial_\nu\widetilde{w}\|^2_{L^2(\Gamma_*\times(-3T_0,3T_0))}\right). \tag{4.30}$$

Finally, we notice upon arguing in the same way, that $\nabla_{x'}w_\gamma$ may be substituted for w_γ in the left hand side of (4.30), so the desired result follows from this and (4.30). $\square$

We next establish the coming technical result with the help of Lemma 4.3.

Lemma 4.4. *There exists $T_1 \in (0,T)$, such that we have*

$$\|v\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_1,T_1))} + \|\nabla_{x'}v\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_1,T_1))} \leq C\left(\frac{1}{\gamma^{2\mu N}} + e^{\delta_5\gamma}\|\partial_\nu v\|^2_{L^2(\Gamma_*\times(-T,T))}\right), \tag{4.31}$$

for every $\gamma \in [\gamma_0,+\infty)$. Here $C>0$ depends only on ω, ω_0, and T, and the constant δ_5 is the same as in Lemma 4.3.

Proof. Let $\widehat{u}(\cdot,\zeta)$, for $\zeta \in \mathbb{R}$, denote the partial Fourier transform computed at ζ of $t \mapsto u(\cdot,t)$. In light of (4.29), it holds true that

$$\widehat{\theta\widetilde{w}}(\cdot,\zeta) - \widehat{w_\gamma}(\cdot,\zeta) = (1-\widehat{F_\gamma})\widehat{\theta\widetilde{w}}(\cdot,\zeta),\ \zeta \in \mathbb{R}. \tag{4.32}$$

Therefore, taking into account that $\widehat{F_\gamma}(\zeta) = e^{-(\zeta/\gamma^\rho)^{2m}}$, using that $1 - e^{-y^{2m}} \leq Cy^N$ for all $y \in [0,+\infty)$ (since $2m \geq N$ from (4.8)), and recalling that $\rho > \mu$ and $\gamma > 1$, we derive from (4.32) that

$$\|\widehat{\theta\widetilde{w}}(x,\cdot) - \widehat{w_\gamma}(x,\cdot)\|_{L^2(\mathbb{R})} \leq \frac{1}{\gamma^{\mu N}}\|\zeta^N\widehat{\theta\widetilde{w}}(x,\cdot)\|_{L^2(\mathbb{R})},\ x \in \Omega_3 \setminus \Omega_2. \tag{4.33}$$

Since the function θ, defined in (4.12), is supported in $(-3T_0, 3T_0)$, it then follows from (4.33) that

$$\|\theta\widetilde{w}(x,\cdot) - w_\gamma(x,\cdot)\|_{L^2(\mathbb{R})} \leq \frac{1}{\gamma^{\mu N}}\|\partial_t^N(\theta\widetilde{w})(x,\cdot)\|_{L^2(\mathbb{R})} \leq \frac{C}{\gamma^{\mu N}}\|\widetilde{w}(x,\cdot)\|_{H^N(-3T_0,3T_0)},\ x \in \Omega_3 \setminus \Omega_2, \tag{4.34}$$

for some constant $C > 0$ depending neither on x nor on γ. Thus, bearing in mind that $\theta(t) = 1$ for each $t \in [-T_0/2, T_0/2]$, we get upon squaring and integrating both sides of (4.34) with respect to x over $\Omega_3 \setminus \Omega_2$, that

$$\|\widetilde{w} - w_\gamma\|_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} \leq \frac{C}{\gamma^{\mu N}},\ \gamma \in (1,+\infty). \tag{4.35}$$

Here we used (4.18) to bound from above the $L^2(\Omega_3 \setminus \Omega_2, H^N(-3T_0, 3T_0))$-norm of the function $\widetilde{w}(x,t) = w(x, T/(3T_0)t)$, uniformly in $\gamma \in (0,1)$.

Further, we proceed in the same way as in the derivation of (4.35), and obtain that

$$\|\nabla_{x'}\widetilde{w} - \nabla_{x'}w_\gamma\|_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} \leq \frac{C}{\gamma^{\mu N}}.$$

From this, (4.35) and Lemma 4.3, it follows for all $\gamma \in [\gamma_0, +\infty)$ that

$$\begin{aligned}
&\|\widetilde{w}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} + \|\nabla_{x'}\widetilde{w}\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} \\
\leq\ & \frac{C}{\gamma^{2\mu N}} + \|w_\gamma\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} + \|\nabla_{x'}w_\gamma\|^2_{L^2((\Omega_3\setminus\Omega_2)\times(-T_0/2,T_0/2))} \\
\leq\ & C\left(\frac{1}{\gamma^{2\mu N}} + e^{\delta_5\gamma}\|\partial_\nu\widetilde{w}\|^2_{L^2(\Gamma_*\times(-3T_0,3T_0))}\right),
\end{aligned}$$

provided T_0 is sufficiently large. As a consequence we have

$$\begin{aligned}
&\|w\|^2_{L^2((\Omega_2\setminus\Omega_3)\times(-hT_0/2,hT_0/2))} + \|\nabla_{x'}w\|^2_{L^2((\Omega_2\setminus\Omega_3)\times(-hT_0/2,hT_0/2))} \\
\leq\ & C\left(\frac{1}{\gamma^{2\mu N}} + e^{\delta_5\gamma}\|\partial_\nu w\|^2_{L^2(\Gamma_*\times(-3hT_0,3hT_0))}\right),
\end{aligned} \tag{4.36}$$

since $\tilde{w}(\cdot,t) = w(\cdot,ht)$ for every $t \in \mathbb{R}$. Finally, bearing in mind that $h = T/(3T_0)$ and recalling from (4.16) that $w(x,t) = \chi(x')v(x,t) = v(x,t)$ for every $(x,t) \in (\Omega_3 \setminus \Omega_2) \times (-T,T)$, we end up getting from (4.36) that

$$\|v\|^2_{L^2((\Omega_2\setminus\Omega_3)\times(-T/6,T/6))} + \|\nabla_{x'}v\|^2_{L^2((\Omega_2\setminus\Omega_3)\times(-T/6,T/6))} \leq C\left(\frac{1}{\gamma^{2\mu N}} + e^{\delta_5\gamma}\|\partial_\nu v\|^2_{L^2(\Gamma_*\times(-T,T))}\right).$$

This yields (4.31) with $T_1 = T/6$. □

Armed with Lemma 4.4, we are now in position to complete the proof of Lemma 2.2. This can be done upon applying (4.31) with $\gamma = |\log\|\partial_\nu v\|_*|/\delta_5$, which is permitted when $\|\partial_\nu v\|_* \in (0, e^{-\delta_5\gamma_0}]$, in such a way that $\gamma \geq \gamma_0$. We find that

$$\|v\|^2_{L^2((\Omega_2\setminus\Omega_3)\times(-T_1,T_1))} + \|\nabla_{x'}v\|^2_{L^2((\Omega_2\setminus\Omega_3)\times(-T_1,T_1))}$$

$$\leq \quad C\left(\left|\log\|\partial_\nu v\|_*\right|^{-2\mu N}+\|\partial_\nu v\|_*\right)\leq C'\left|\log\|\partial_\nu v\|_*\right|^{-2\mu N}, \tag{4.37}$$

where C' is a suitable positive constant. On the other hand, when $\|\partial_\nu v\|_* > e^{-\delta_5\gamma_0}$, it is clear from the estimate $\|v\|_{L^2(-T,T;H^1(\Omega))} \leq 2C\|u_0\|_{H^{2(N+1)}(\Omega)}$, arising from (4.1), that

$$\|v\|^2_{L^2((\Omega_2\setminus\Omega_3)\times(-T_1,T_1))}+\|\nabla_{x'}v\|^2_{L^2((\Omega_2\setminus\Omega_3)\times(-T_1,T_1))}\leq C\,\|\partial_\nu v\|_*^{2\mu N},$$

so we get (2.9) directly from this and from (4.37).

5. Appendix

In this appendix we prove the parabolic Carleman estimate stated in Lemma 4.1. Incidentally we make precise the dependence with respect to λ and h, of the constant C_0 appearing in the right hand side of (4.7).

We stick with the notations of Section 4 and start by gathering several useful straightforward properties of the weight functions φ_0 and α, defined by (4.4)-(4.5), in the coming lemma.

Lemma 5.1. *We may find three constants $\lambda_0 \geq 1$, $c > 0$ and $c' > 0$, all of them depending only on ω_0, such that for each $\lambda \geq \lambda_0$ and all $(x',\tau) \in \omega_0 \times (-1,1)$, the following estimates hold simultaneously:*

$$\begin{aligned}
D^2_{x'}\alpha(\nabla_{x'}\alpha,\nabla_{x'}\alpha)(x',\tau) &\geq c\lambda^4\varphi_0(x',\tau)^3, & (5.1)\\
\left|\Delta_{x'}\alpha(x',\tau)\right| &\leq c'\lambda^2\varphi_0(x',\tau)^2, & (5.2)\\
\left|\Delta^2_{x'}\alpha(x',\tau)\right| &\leq c'\lambda^4\varphi_0(x',\tau)^3, & (5.3)\\
\left|(\partial_\tau\alpha)(\Delta_{x'}\alpha)(x',\tau)\right| &\leq c'\lambda^2\varphi_0(x',\tau)^3, & (5.4)\\
\left|\partial^2_\tau\alpha(x',\tau)\right| &\leq c'\varphi_0(x',\tau)^3, & (5.5)\\
\left|\nabla_{x'}(\partial_\tau\alpha)(x',\tau)\cdot\nabla_{x'}\alpha(x',\tau)\right| &\leq c'\lambda^2\varphi_0(x',\tau)^3, & (5.6)\\
D^2_{x'}\alpha(\xi',\xi')(x',\tau) &\geq -c\lambda\varphi_0\left|\xi'\right|^2,\ \xi'\in\mathbb{R}^{n-1}. & (5.7)
\end{aligned}$$

In the sequel C denotes a generic positive constant which depends only on ω_0, whose value can change from line to line.

Put $z := e^{\sigma\alpha}w$ and notice for further reference from (4.4)-(4.5) that

$$z(x,\pm 1)=0,\ x\in\Omega_0=\omega_0\times\mathbb{R}. \tag{5.8}$$

Next, setting $f_\sigma := e^{\sigma\alpha}(h^{-1}\partial_\tau - \Delta)w$, we find through direct computation that

$$L_1z+L_2z=g_\sigma:=f_\sigma-\sigma(\Delta_{x'}\alpha)z, \tag{5.9}$$

where

$$\begin{aligned}
L_1z &:= h^{-1}\partial_\tau z+2\sigma\nabla_{x'}\alpha\cdot\nabla_{x'}z, & (5.10)\\
L_2z &:= -\Delta z-\sigma\left(h^{-1}(\partial_\tau\alpha)+\sigma\left|\nabla_{x'}\alpha\right|^2\right)z. & (5.11)
\end{aligned}$$

Due to (5.9), we have the identity

$$\sum_{j=1,2}\|L_jz\|^2_{L^2(\Omega_0\times(-1,1))}+2\mathrm{Re}\left(\int_{-1}^1\int_{\Omega_0}L_1z\overline{L_2z}dxd\tau\right)=\|g_\sigma\|^2_{L^2(\Omega_0\times(-1,1))}, \tag{5.12}$$

so we are left with the task of estimating the $L^2(\Omega_0\times(-1,1))$-scalar product of L_1z and L_2z, appearing in the left hand side of (5.12). To do that, we notice from (5.10)-(5.11) that

$$2\int_{-1}^1\int_{\Omega_0}L_1z\overline{L_2z}dxd\tau=\sum_{j=1,2,3}I_j, \tag{5.13}$$

where

$$
\begin{aligned}
I_1 &:= 2h^{-1}\int_{-1}^{1}\int_{\Omega_0}\partial_\tau z\left(-\Delta\overline{z}-\sigma\left(h^{-1}(\partial_\tau\alpha)+\sigma\left|\nabla_{x'}\alpha\right|^2\right)\overline{z}\right)dxd\tau, && (5.14)\\
I_2 &:= -4\sigma\int_{-1}^{1}\int_{\Omega_0}\left(\nabla_{x'}\alpha\cdot\nabla_{x'}z\right)\Delta\overline{z}dxd\tau, && (5.15)\\
I_3 &:= -4\sigma^2\int_{-1}^{1}\int_{\Omega_0}\left(\nabla_{x'}\alpha\cdot\nabla_{x'}z\right)\left(h^{-1}(\partial_\tau\alpha)+\sigma\left|\nabla_{x'}\alpha\right|^2\right)\overline{z}dxd\tau. && (5.16)
\end{aligned}
$$

Bearing in mind that $z_{|\partial\Omega_0\times(-1,1)}=0$ from the very definition of z (since the same is true for the function w, according to (4.17)), we integrate by parts with respect to x in (5.14), and obtain that

$$
\mathrm{Re}\,(I_1)=h^{-1}\int_{-1}^{1}\int_{\Omega_0}\left(\partial_\tau\left|\nabla z\right|^2-\sigma(h^{-1}\partial_\tau\alpha+\sigma\left|\nabla_{x'}\alpha\right|^2)\partial_\tau\left|z\right|^2\right)dxd\tau.
$$

Recalling (5.8), we next integrate by parts with respect to τ, and find that

$$
\mathrm{Re}\,(I_1)=\sigma h^{-1}\int_{-1}^{1}\int_{\Omega_0}\left(h^{-1}(\partial_\tau^2\alpha)\left|z\right|^2+2\sigma\nabla_{x'}(\partial_\tau\alpha)\cdot\nabla_{x'}\alpha\right)|z|^2dxd\tau. \tag{5.17}
$$

The second term, I_2, is handled in a similar way. Namely, we integrate by parts with respect to x in the right hand side of (5.15), use the identity $\nabla z=(\partial_\nu z)\nu$ on $\partial\Omega_0\times(-1,1)$, and get

$$
\begin{aligned}
\mathrm{Re}\,(I_2) &= 4\sigma\int_{-1}^{1}\int_{\Omega_0}D_{x'}^2\alpha(\nabla_{x'}z,\nabla_{x'}\overline{z})dxd\tau+2\sigma\int_{-1}^{1}\int_{\Omega_0}\nabla\alpha\cdot\nabla|\nabla z|^2dxd\tau\\
&\quad -4\sigma\int_{-1}^{1}\int_{\partial\Omega_0}(\partial_\nu\alpha)\left|\partial_\nu z\right|^2dxd\tau.
\end{aligned}
$$

Therefore, taking into account that

$$
\int_{-1}^{1}\int_{\Omega_0}\nabla\alpha\cdot\nabla|\nabla z|^2dxd\tau=-\int_{-1}^{1}\int_{\Omega_0}(\Delta\alpha)|\nabla z|^2dxd\tau+\int_{-1}^{1}\int_{\partial\Omega_0}(\partial_\nu\alpha)\left|\partial_\nu z\right|^2dxd\tau,
$$

we see that

$$
\begin{aligned}
\mathrm{Re}\,(I_2) &= 4\sigma\int_{-1}^{1}\int_{\Omega_0}D_{x'}^2\alpha(\nabla_{x'}z,\nabla_{x'}\overline{z})dxd\tau-2\sigma\int_{-1}^{1}\int_{\Omega_0}(\Delta\alpha)|\nabla z|^2dxd\tau\\
&\quad -2\sigma\int_{-1}^{1}\int_{\partial\Omega_0}(\partial_\nu\alpha)\left|\partial_\nu z\right|^2dxd\tau. && (5.18)
\end{aligned}
$$

Let us now compute the real part of I_3. To this end, we notice upon integrating by parts with respect to x, that

$$
\begin{aligned}
&2\mathrm{Re}\left(\int_{-1}^{1}\int_{\Omega_0}\left|\nabla_{x'}\alpha\right|^2(\nabla_{x'}\alpha\cdot\nabla_{x'}z)\overline{z}dxd\tau\right)\\
&= -\int_{-1}^{1}\int_{\Omega_0}\left(\left|\nabla_{x'}\alpha\right|^2\Delta_{x'}\alpha+2D_{x'}^2\alpha(\nabla_{x'}\alpha,\nabla_{x'}\alpha)\right)|z|^2dxd\tau, && (5.19)
\end{aligned}
$$

and that

$$
\begin{aligned}
&2\mathrm{Re}\left(\int_{-1}^{1}\int_{\Omega_0}(\partial_\tau\alpha)(\nabla_{x'}\alpha\cdot\nabla_{x'}z)\overline{z}dxd\tau\right)\\
&= -\int_{-1}^{1}\int_{\Omega_0}\left((\partial_\tau\alpha)(\Delta_{x'}\alpha)+(\nabla_{x'}(\partial_\tau\alpha)\cdot\nabla_{x'}\alpha)\right)\left|z\right|^2dxd\tau.
\end{aligned}
$$

It follows from this, (5.16) and (5.19) that

$$\begin{aligned}\mathrm{Re}\,(I_3) &= 2\sigma^3\int_{-1}^1\int_{\Omega_0}\Big(\left|\nabla_{x'}\alpha\right|^2(\Delta_{x'}\alpha)+2D^2_{x'}\alpha(\nabla_{x'}\alpha,\nabla_{x'}\alpha)\Big)\left|z\right|^2dxd\tau\\ &\quad+2\sigma^2h^{-1}\int_{-1}^1\int_{\Omega_0}\left((\partial_\tau\alpha)(\Delta_{x'}\alpha)+\nabla_{x'}(\partial_\tau\alpha)\cdot\nabla_{x'}\alpha\right)\left|z\right|^2dxd\tau. \end{aligned}\tag{5.20}$$

Finally, putting (5.13), (5.17)-(5.18) and (5.20) together, we end up getting that

$$\begin{aligned}2\mathrm{Re}\left(\int_{-1}^1\int_{\Omega_0}L_1z\overline{L_2z}dxd\tau\right) &= -2\sigma\int_{-1}^1\int_{\Omega_0}(\Delta_{x'}\alpha)\left(\left|\nabla z\right|^2-\sigma^2\left|\nabla_{x'}\alpha\right|^2\left|z\right|^2\right)dxd\tau\\ &\quad+4\sigma\int_{-1}^1\int_{\Omega_0}D^2_{x'}\alpha(\nabla_{x'}z,\nabla_{x'}\overline{z})dxd\tau+\sum_{j=1,2,3}J_j, \end{aligned}\tag{5.21}$$

where have set

$$J_1 := 4\sigma^3\int_{-1}^1\int_{\Omega_0}D^2_{x'}\alpha(\nabla_{x'}\alpha,\nabla_{x'}\alpha)\left|z\right|^2dxd\tau, \tag{5.22}$$

$$J_2 := -2\sigma\int_{-1}^1\int_{\partial\Omega_0}(\partial_\nu\alpha)\left|\partial_\nu z\right|^2dxd\tau, \tag{5.23}$$

$$\begin{aligned}J_3 &:= 2\sigma^2h^{-1}\int_{-1}^1\int_{\Omega_0}\left(2\nabla_{x'}(\partial_\tau\alpha)\cdot\nabla_{x'}\alpha+(\partial_\tau\alpha)\Delta_{x'}\alpha\right)\left|z\right|^2dxd\tau\\ &\quad+\sigma h^{-2}\int_{-1}^1\int_{\Omega_0}(\partial_\tau^2\alpha)\left|z\right|^2dxd\tau. \end{aligned}\tag{5.24}$$

The next step of the proof is to bound from below each of the three terms J_j, $j=1,2,3$, appearing in the right hand side of (5.21). In view of (5.1), J_1 is easily treated by (5.22), as we have

$$J_1 \geq 4c\sigma^3\lambda^4\|\varphi_0^{3/2}z\|^2_{L^2(\Omega_0\times(-1,1))},\ \lambda\geq\lambda_0, \tag{5.25}$$

where c is the constant defined in Lemma 5.1. Similarly, we deduce from (5.4)–(5.6) and (5.24), that

$$|J_3| \leq (6\sigma^2h^{-1}+\sigma h^{-2})C_2\lambda^2\|\varphi_0^{3/2}z\|^2_{L^2(\Omega_0\times(-1,1))},\ \lambda\geq\lambda_0,$$

where c' is the same as in Lemma 5.1. Therefore, for all $\sigma\geq\sigma_0/h$, it follows readily from this and (5.25), that

$$J_1+J_3 \geq C\sigma^3\lambda^4\|\varphi_0^{3/2}z\|^2_{L^2(\Omega_0\times(-1,1))},\ \lambda\geq\lambda_0, \tag{5.26}$$

for some constant $C>0$, depending only on ω_0. On the other hand, since $\partial_\nu\alpha=\lambda(\partial_\nu\psi_0)\varphi_0$ on $\partial\Omega_0\times(-1,1)$, by (4.4), the identity (5.23) yields

$$J_2 \geq -2\sigma\lambda\|\varphi_0^{1/2}(\partial_\nu\psi_0)^{1/2}\partial_\nu z\|^2_{L^2(\Gamma_*\times(-1,1)}.$$

Now, putting this together with (5.7), (5.21) and (5.26), we obtain for all $\lambda\geq\lambda_0$ and all $\sigma\geq\sigma_0/h$, that

$$\begin{aligned}2\mathrm{Re}\left(\int_{-1}^1\int_{\Omega_0}L_1z\overline{L_2z}dxd\tau\right) &\geq C\sigma^3\lambda^4\|\varphi_0^{3/2}z\|^2_{L^2(\Omega_0\times(-1,1))}-4c\sigma\lambda\|\varphi_0^{1/2}\nabla_{x'}z\|_{L^2(\Omega_0\times(-1,1))}\\ &\quad-2\sigma\lambda\|\varphi_0^{1/2}(\partial_\nu\psi_0)^{1/2}\partial_\nu z\|^2_{L^2(\Gamma_*\times(-1,1))}-2S, \end{aligned}\tag{5.27}$$

where

$$S := \sigma\int_{-1}^1\int_{\Omega_0}(\Delta_{x'}\alpha)\left(\left|\nabla z\right|^2-\sigma^2\left|\nabla_{x'}\alpha\right|^2\left|z\right|^2\right)dxd\tau. \tag{5.28}$$

The rest of the proof involves bounding S from above. To this purpose we recall from (5.11), that

$$|\nabla z|^2 - \sigma^2 |\nabla_{x'}\alpha|^2 |z|^2 = \mathrm{Re}\left((L_2 z)\overline{z}\right) + \frac{\Delta |z|^2}{2} + \sigma h^{-1}(\partial_\tau \alpha)|z|^2,$$

and then deduce from (5.28) that

$$\begin{aligned} S &= \sigma \int_{-1}^{1}\int_{\Omega_0} (\Delta_{x'}\alpha)\left(\Delta|z|^2/2 + \mathrm{Re}\left((L_2 z)\overline{z}\right) + \sigma h^{-1}(\partial_\tau\alpha)|z|^2\right) dx d\tau \\ &= \sigma \int_{-1}^{1}\int_{\Omega_0} \left((\Delta_{x'}^2 \alpha/2)|z|^2 + (\Delta_{x'}\alpha)\left(\mathrm{Re}\left((L_2 z)\overline{z}\right) + \sigma h^{-1}(\partial_\tau\alpha)|z|^2\right)\right) dx d\tau. \end{aligned}$$

As a consequence we have

$$\begin{aligned} |S| \leq\ & \|L_2 z\|_{L^2(Q)}^2 /4 + \||\Delta_{x'}^2\alpha|^{1/2} z\|_{L^2(\Omega_0\times(-1,1))}^2 \\ & + \sigma^2 \left(\|(\Delta_{x'}\alpha) z\|_{L^2(\Omega_0\times(-1,1))}^2 + h^{-1}\| \, |\partial_\tau \alpha| \, z\|_{L^2(\Omega_0\times(-1,1))}^2 \right). \end{aligned}$$

This, together with (5.2)-(5.3) and (5.6), yields

$$|S| \leq \|L_2 z\|_{L^2(Q)}^2 /4 + C\left(\sigma\lambda^4 + \sigma^2\lambda^2(\lambda^2 + h^{-1})\right) \|\varphi_0^{3/2} z\|_{L^2(\Omega_0\times(-1,1))}^2.$$

It follows from this and (5.27) upon taking $\sigma \geq \sigma_0/h$, that

$$\begin{aligned} & 2\mathrm{Re}\left(\int_{-1}^{1}\int_{\Omega_0} L_1 z \overline{L_2 z} dx d\tau\right) + \frac{\|L_2 z\|_{L^2(Q)}^2}{2} \\ \geq\ & \sigma^3\lambda^4 \|\varphi_0^{3/2} z\|_{L^2(\Omega_0\times(-1,1))}^2 - 4c\sigma \|\varphi_0^{1/2}\nabla_{x'} z\|_{L^2(\Omega_0\times(-1,1))}^2 - 2\sigma\lambda \|\varphi_0^{1/2}(\partial_\nu\psi_0)^{1/2}\partial_\nu z\|_{L^2(\Gamma_*\times(-1,1))}^2. \end{aligned}$$

Having estimated all the contributions depending on h, we proceed as in [20, Appendix], and obtain the desired result.

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