Spectral Properties of a Magnetic Quantum Hamiltonian on a Strip

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Dedicated to the memory of Volodya Geyler (1943 - 2007)

Abstract

We consider a 2D Schrödinger operator H_0 with constant magnetic field, on a strip of finite width. The spectrum of H_0 is absolutely continuous, and contains a discrete set of thresholds. We perturb H_0 by an electric potential V which decays in a suitable sense at infinity, and study the spectral properties of the perturbed operator $H = H_0 + V$. First, we establish a Mourre estimate, and as a corollary prove that the singular continuous spectrum of H is empty, and any compact subset of the complement of the threshold set may contain at most a finite set of eigenvalues of H, each of them having a finite multiplicity. Next, we introduce the Krein spectral shift function (SSF) for the operator pair (H, H_0). We show that this SSF is bounded on any compact subset of the complement of the threshold set, and is continuous away from the threshold set and the eigenvalues of H. The main results of the article concern the asymptotic behaviour of the SSF at the thresholds, which is described in terms of the SSF for a pair of effective Hamiltonians.

AMS 2000 Mathematics Subject Classification: 35J10, 81Q10, 35P20

Keywords: Schrödinger operators, magnetic field, Mourre estimates, spectral shift function, effective Hamiltonians

1 Introduction

In the present article we consider a 2D Schrödinger operator H_0 with constant magnetic field $b > 0$ defined on a strip $\mathcal{S}_L$ of width $2L$. The spectrum of H_0 is absolutely continuous, equals the interval $[\mathcal{E}_1, \infty)$ with $\mathcal{E}_1 > 0$, and contains a countable set of thresholds $\mathcal{Z}$. This model is related to some aspects of the quantum Hall effect (see e.g. [2], [10]). We perturb H_0 by an electric potential V which decays in a suitable sense at infinity, and study some basic spectral properties of the perturbed operator H. First we establish a Mourre estimate (see [20]) with an appropriate conjugate operator, and as a consequence we show that the singular continuous spectrum of H is empty, and any compact subset of $\mathbb{R} \setminus \mathcal{Z}$ may contain at most a finite number of eigenvalues of H,

each of them having a finite multiplicity. Similar Mourre estimates for other magnetic Hamiltonians have been obtained in [7] and [12, Chapter 3].
Further, we introduce the Krein spectral shift function (SSF) for the operator pair (H, H_0) and prove that it is bounded on every compact subset of $\mathbb{R}\setminus\mathcal{Z}$, and is continuous on $\mathbb{R} \setminus (\mathcal{Z} \cup \sigma_p(H))$ where $\sigma_p(H)$ is the set of the eigenvalues of H. The main results of the article concern the asymptotic behaviour of the SSF near the thresholds of the spectrum of H_0. We show that this asymptotic behaviour is similar to the asymptotics near the origin of the SSF for a pair of effective Hamiltonians which are 1D Schrödinger operators. As a corollary we show that if the decay rate α of V is on the interval $(1, 2)$, then the SSF has a singularity at each threshold, and describe explicitly the leading term of this singularity; if $\alpha > 2$, then the SSF remains bounded at the thresholds. The threshold behaviour of the SSF for a pair of 3D Schrödinger operators with constant magnetic fields has been investigated in [9] (see also [23]). In that case the thresholds coincide with the Landau levels, and the threshold singularities of the SSF have different nature, related to the spectral properties of compact Berezin-Toeplitz operators.
The paper is organized as follows. In Section 2 we introduce some basic notations, describe the operators H_0 and H, formulate our main results, and briefly comment on them. Section 3 contains the proof of our results related to the Mourre estimates, while the proofs of the results concerning the SSF can be found in Section 4.

2 Main Results

2.1. In this subsection we introduce some basic notations used throughout the section. Let X_1, X_2 be two Hilbert spaces[1] We denote by $\mathcal{B}(X_1, X_2)$ (resp., by $S_\infty(X_1, X_2)$) the class of bounded (resp., compact) operators $T : X_1 \to X_2$. Further, we denote by $S_p(X_1, X_2)$, $p \in [1, \infty)$, the Schatten-von Neumann class of compact operators $T : X_1 \to X_2$ for which the norm $\|T\|_p := (\mathrm{Tr}\,|T|^p)^{1/p}$ is finite (see e.g. [25]). In this paper we will use only the trace class S_1 and the Hilbert-Schmidt class S_2. If $X_1 = X_2 = X$ we write $\mathcal{B}(X)$ or $S_p(X)$ instead of $\mathcal{B}(X, X)$ or $S_p(X, X)$, $p \in [1, \infty]$. Also, if the indication of the Hilbert space(s) where the corresponding operators act is irrelevant, we omit it in the notations of the classes $\mathcal{B}$ and S_p, $p \in [1, \infty]$.
Let $T = T^*$. We denote by $\mathbb{P}_{\mathcal{O}}(T)$ the spectral projection of T associated with the Borel set $\mathcal{O} \subset \mathbb{R}$.
Finally, if $T \in \mathcal{B}(X)$, we define the self-adjoint operators $\mathrm{Re}\ T := \frac{1}{2}(T + T^*)$ and $\mathrm{Im}\, T := \frac{1}{2i}(T - T^*)$.

2.2. In this subsection we introduce the operators H_0 and H, and summarize some of their spectral properties which will play a crucial role in the sequel.

[1] All the Hilbert spaces considered in the article are supposed to be separable.

For $L > 0$ put $I_L = (-L, L)$, $\mathcal{S} = I_L \times \mathbb{R}$. Let

$$H_0 := -\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} - bx\right)^2$$

be the 2D Schrödinger operator with constant scalar magnetic field $b > 0$, defined on $\{u \in \mathrm{H}^2(\mathcal{S}_L) \mid u_{|\partial\mathcal{S}_L} = 0\}$ where $\mathrm{H}^2(\mathcal{S}_L)$ denotes the second-order Sobolev space on $\mathcal{S}_L$. Then we have

$$\mathcal{F}H_0\mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} \hat{H}(k)dk,$$

where $\mathcal{F}$ is the partial Fourier transform with respect to y, i.e.

$$(\mathcal{F}u)(x,k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iyk}u(x,y)dy, \quad (x,k) \in \mathcal{S}_L,$$

and

$$\hat{H}(k) := -\frac{d^2}{dx^2} + (bx - k)^2, \quad k \in \mathbb{R},$$

is the operator defined on $D(\hat{H}) := \{w \in \mathrm{H}^2(I_L) | w(-L) = w(L) = 0\}$. In what follows, we will consider $D(\hat{H})$ as a Hilbert space equipped with the standard scalar product of $\mathrm{H}^2(I_L)$.

The spectrum $\sigma(\hat{H}(k))$ of the operator $\hat{H}(k)$, $k \in \mathbb{R}$, is discrete and simple. Let $\{E_j(k)\}_{j=1}^{\infty}$ be the increasing sequence of the eigenvalues of $\hat{H}(k)$, which are even real analytic functions of $k \in \mathbb{R}$ (see [15]). Further, the minimax principle easily implies

$$E_j(k) = k^2(1 + o(1)), \quad k \to \pm\infty. \tag{2.1}$$

Finally, by [10, Theorem 2] we have

$$kE_j'(k) > 0, \quad k \neq 0, \tag{2.2}$$

$$E_j(k) = \mathcal{E}_j + \mu_j k^2 + O(k^4), \quad k \to 0, \tag{2.3}$$

with

$$\mathcal{E}_j := E_j(0) > (2j-1)b, \quad \mu_j := \frac{1}{2}E_j''(0) > 0. \tag{2.4}$$

Thus $\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [\mathcal{E}_1, \infty)$, and $\mathcal{E}_j$, $j \in \mathbb{N} := \{1, 2, \ldots\}$, are thresholds in $\sigma(H_0)$. Set $\mathcal{Z} := \bigcup_{j\in\mathbb{N}} \{\mathcal{E}_j\}$.

Let $V : S_L \to \mathbb{R}$ be an electric potential such that the operator $|V|^{1/2}H_0^{-1/2}$ is compact. We define the perturbed operator $H := H_0 + V$ as a sum in the sense of the quadratic forms. Then we have $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0) = \sigma(H_0) = [\mathcal{E}_1, \infty)$.

2.3. In this subsection we formulate our result concerning the absence of singular continuous spectrum of H, and some generic properties of its eigenvalues.

Theorem 2.1. *(i) Assume*

$$VH_0^{-1} \in S_\infty, \tag{2.5}$$

$$H_0^{-1} y \frac{\partial V}{\partial y} H_0^{-1} \in S_\infty. \tag{2.6}$$

Then any compact subinterval of $\mathbb{R} \setminus \mathcal{Z}$ *may contain at most a finite number of eigenvalues, each of them having a finite multiplicity.*
(ii) Suppose moreover

$$H_0^{-1/2} y \frac{\partial V}{\partial y} H_0^{-1} \in \mathcal{B}, \tag{2.7}$$

$$H_0^{-1} y^2 \frac{\partial^2 V}{\partial y^2} H_0^{-1} \in \mathcal{B}. \tag{2.8}$$

Then $\sigma_{\rm sc}(H) = \emptyset$.

The proof of Theorem 2.1 is contained in Section 3.
Remark: Let $U : \mathcal{S}_L \to [0, \infty)$, and let Δ_D be the Dirichlet Laplacian on $\mathcal{S}_L$. The Sobolev embedding theorems imply that the inclusion $U^{1/2}(-\Delta_D)^{-1/2} \in \mathcal{B}$ (resp., $U^{1/2}(-\Delta_D)^{-1/2} \in S_\infty$) is ensured by $U \in L^q(\mathcal{S}_L) + L^\infty(\mathcal{S}_L)$ (resp., $U \in L^q(\mathcal{S}_L) + L^\infty_\varepsilon(\mathcal{S}_L)$, i.e. for each $\varepsilon > 0$ we have $U = U_1 + U_2$ with $U_1 \in L^q(\mathcal{S}_L)$, $U_2 \in L^\infty(\mathcal{S}_L)$, $\|U_2\|_{L^\infty(\mathcal{S}_L)} \leq \varepsilon$), $q > 1$. Similarly, the condition $U\Delta_D^{-1} \in \mathcal{B}$ (resp., $U\Delta_D^{-1} \in S_\infty$) follows from $U \in L^2(\mathcal{S}_L) + L^\infty(\mathcal{S}_L)$ (resp., $U \in L^2(\mathcal{S}_L) + L^\infty_\varepsilon(\mathcal{S}_L)$). On the other hand, by the diamagnetic inequality (see e.g. [25, Chapter 2]), we have $\|U^\gamma H_0^{-\gamma}\| \leq \|U^\gamma(-\Delta_D)^{-\gamma}\|$, $\gamma > 0$, and, moreover, $U^\gamma(-\Delta_D)^{-\gamma} \in S_\infty$ entails $U^\gamma H_0^{-\gamma} \in S_\infty$. These facts could be used in order to deduce sufficient conditions which guarantee the validity of the hypotheses of Theorem 2.1.

2.4. This subsection contains our results on the threshold behaviour of the spectral shift function for the operator pair (H, H_0). Let us recall the abstract setting for the SSF. Let $\mathcal{H}_0$ and $\mathcal{H}$ be two lower-bounded self-adjoint operators acting in the same Hilbert space. Assume that for some $\gamma > 0$, and $E_0 < \inf \sigma(\mathcal{H}_0) \cup \sigma(\mathcal{H})$, we have

$$(\mathcal{H} - E_0)^{-\gamma} - (\mathcal{H}_0 - E_0)^{-\gamma} \in S_1. \tag{2.9}$$

Then there exists a unique $\xi(\cdot; \mathcal{H}, \mathcal{H}_0) \in L^1(\mathbb{R}; \langle E\rangle^{-\gamma-1}dE)$ which vanishes identically on $(-\infty, E_0)$ such that *the Lifshits-Krein formula*

$$\mathrm{Tr}(f(\mathcal{H}) - f(\mathcal{H}_0)) = \int_{\mathbb{R}} \xi(E; \mathcal{H}, \mathcal{H}_0) f'(E) dE \tag{2.10}$$

holds for each $f \in C_0^\infty(\mathbb{R})$ (see [18] and [17]). The function $\xi(.; \mathcal{H}, \mathcal{H}_0)$ is called the SSF for the pair of the operators $(\mathcal{H}, \mathcal{H}_0)$. If $E < \inf \sigma(\mathcal{H}_0)$, then the spectrum of $\mathcal{H}$ below E could be at most discrete, and for almost every $E < \inf \sigma(\mathcal{H}_0)$ we have

$$\xi(E; \mathcal{H}, \mathcal{H}_0) = -N(E; \mathcal{H}) \tag{2.11}$$

where $N(E;\mathcal{H}) := \operatorname{rank} \mathbb{P}_{(-\infty,E)}(\mathcal{H})$. On the other hand, for almost every $E \in \sigma_{\rm ac}(\mathcal{H}_0)$, the SSF $\xi(E;\mathcal{H},\mathcal{H}_0)$ is related to the scattering determinant $\det S(E;\mathcal{H},\mathcal{H}_0)$ for the pair $(\mathcal{H},\mathcal{H}_0)$ by *the Birman-Krein* formula

$$\det S(E;\mathcal{H},\mathcal{H}_0) = e^{-2\pi i \xi(E;\mathcal{H},\mathcal{H}_0)} \tag{2.12}$$

(see [4]).
Next, we define the SSF for the pair (H, H_0). We will say that V satisfies condition $\mathcal{D}_\alpha$, $\alpha \in \mathbb{R}$, if

$$|V(x,y)| \leq c\langle y\rangle^{-\alpha}, \; c > 0, \; (x,y) \in S_L,$$

where, as usual, $\langle y\rangle := (1+y^2)^{1/2}$. Assume that V satisfies condition $\mathcal{D}_\alpha$ with $\alpha > 1$. Then (2.9) holds for $\mathcal{H} = H$, $\mathcal{H}_0 = H_0$, and $\gamma = 1$, and hence the SSF $\xi(\cdot; H, H_0)$ is well defined as an element of $L^1(\mathbb{R};\langle E\rangle^{-2}dE)$. In the present article we will identify this SSF with a representative of the corresponding class of equivalence described explicitly in Section 4.3 below.

Proposition 2.1. *Assume that V satisfies $\mathcal{D}_\alpha$ with $\alpha > 1$. Then the SSF $\xi(\cdot; H, H_0)$ is bounded on every compact subset of $\mathbb{R} \setminus \mathcal{Z}$ and continuous on $\mathbb{R} \setminus (\mathcal{Z} \cup \sigma_p(H))$.*

The proof of Proposition 2.1 can be found in Subsection 4.6 below.
Set

$$J(x,y) = \operatorname{sign} V(x,y) := \begin{cases} 1 & \text{if} \quad V(x,y) \geq 0, \\ -1 & \text{if} \quad V(x,y) < 0, \end{cases}$$

Fix $j \in \mathbb{N}$. Let $\psi_j(\cdot;k) : I_L \to \mathbb{R}$, $k \in \mathbb{R}$, be the real-valued normalized in $L^2(I_L)$ eigenfunction of the operator $\hat{H}(k)$ corresponding to the eigenvalue $E_j(k)$. For $\varepsilon \in (-1,1)$ introduce *the effective potential*

$$w_{j,\varepsilon}(y) := \int_{I_L} |V(x,y)|(J(x,y)-\varepsilon)^{-1}\psi_j(x;0)^2 dx, \quad y \in \mathbb{R},$$

so that $w_{j,0}(y) = \int_{I_L} V(x,y)\psi_j(x;0)^2 dx$, and *the effective Hamiltonians*

$$h_{0,j} := -\mu_j \frac{d^2}{dy^2}, \quad h_j(\varepsilon) := h_{0,j} + w_{j,\varepsilon},$$

the number μ_j being defined in (2.4). Note if V satisfies $\mathcal{D}_\alpha$ with $\alpha > 1$, then (2.9) holds for $\mathcal{H} = h_j(\varepsilon)$, $\mathcal{H}_0 = h_{0,j}$, and $\gamma = 1$, and hence the SSFs $\xi(\cdot; h_j(\varepsilon), h_{0,j})$, $j \in \mathbb{N}$, $\varepsilon \in (-1,1)$, are well defined.
For $\lambda > 0$ set

$$\theta_\beta(\lambda) := \begin{cases} 1 & \text{if} \quad \beta > 1/2, \\ |\ln \lambda| & \text{if} \quad \beta = 1/2, \\ \lambda^{-\frac{1}{2}+\beta} & \text{if} \quad 0 < \beta < 1/2. \end{cases} \tag{2.13}$$

If $\lambda < 0$, then

$$\theta_\beta(\lambda) := 1 \tag{2.14}$$

for all $\beta > 0$.

Theorem 2.2. *Assume that V satisfies $\mathcal{D}_\alpha$ with $\alpha > 1$. Fix $q \in \mathbb{N}$. Then for each $\varepsilon \in (0,1)$ we have*

$$\xi(\lambda; h_q(-\varepsilon), h_{0,q}) + O(\theta_{2\gamma}(\lambda)) \leq \xi(\mathcal{E}_q + \lambda; H, H_0) \leq \xi(\lambda; h_q(\varepsilon), h_{0,q}) + O(\theta_{2\gamma}(\lambda)), \quad (2.15)$$

as $\lambda \to 0$, for any $\gamma \in (0, (\alpha-1)/2)$, $\gamma \leq 1$.

The proof of Theorem 2.2 can be found in Subsection 4.7.
Assume now that $\alpha \in (1,2)$. Then there exists $\gamma \in (0, (\alpha-1)/2)$, $\gamma \leq 1$, such that $\theta_{2\gamma}(\lambda) = o(|\lambda|^{\frac{1}{2}-\frac{1}{\alpha}})$ as $\lambda \to 0$. Hence, using well-known results concerning the asymptotic behaviour of the SSF $\xi(\lambda; h_j(\varepsilon), h_{0,j})$ as $\lambda \to 0$ (see e.g. [24, Theorem XIII.82] in the case $\lambda \uparrow 0$, and [26] in the case $\lambda \downarrow 0$), we obtain the following

Corollary 2.1. *Let V satisfy $\mathcal{D}_\alpha$ with $\alpha \in (1,2)$. Fix $q \in \mathbb{N}$. Suppose that for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and some $\varepsilon_0 \in (0,1)$ there exist real numbers $\omega_{q,\pm}(\varepsilon)$ such that*

$$\lim_{y \to \pm\infty} |y|^\alpha w_{q,\varepsilon}(y) = \omega_{q,\pm}(\varepsilon) \quad (2.16)$$

uniformly with respect to ε. Then we have

$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{\alpha}-\frac{1}{2}} \xi(\mathcal{E}_q - \lambda; H, H_0) = -\mu_q^{-1/2} \mathcal{C}_\alpha \Omega_q^-, \quad (2.17)$$

$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{\alpha}-\frac{1}{2}} \xi(\mathcal{E}_q + \lambda; H, H_0) = -\mu_q^{-1/2} \mathcal{C}_\alpha \left(\csc(\pi/\alpha)\Omega_q^- + \cot(\pi/\alpha)\Omega_q^+\right), \quad (2.18)$$

where $\mathcal{C}_\alpha := \frac{1}{\pi}\int_0^1 (t^{-\alpha}-1)^{1/2} dt$, and $\Omega_q^\pm := \sum_{\varsigma=+,-} \omega_{q,\varsigma}(0)_\pm^{1/\alpha}$, while $\omega_{q,\varsigma}(0)_+$ and $\omega_{q,\varsigma}(0)_-$ denote the positive and the negative part of $\omega_{q,\varsigma}(0)$ respectively.

For the sake of completeness we include a sketch of the proof of Corollary 2.1 in Subsection 4.8.
Remark: If $q = 1$ and $\lambda > 0$, we have $\xi(\mathcal{E}_1 - \lambda; H, H_0) = -N(\mathcal{E}_1 - \lambda; H)$ (cf. (2.11)). Note that the spectrum of H below $\mathcal{E}_1$ is discrete if V satisfies $\mathcal{D}_\alpha$ with any $\alpha > 0$, and as in (2.17) we have

$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{\alpha}-\frac{1}{2}} N(\mathcal{E}_1 - \lambda; H) = \mu_q^{-1/2} \mathcal{C}_\alpha \Omega_1^- \quad (2.19)$$

for all $\alpha \in (0,2)$.
Similarly, using well known results on the asymptotic behaviour as $\lambda \uparrow 0$ of the SSF $\xi(\lambda; h_q(\varepsilon), h_{0,q})$ in the case $\alpha = 2$ (see [16]), we obtain the following

Corollary 2.2. *Assume the hypotheses of Corollary 2.1 with $\alpha = 2$. Fix $q \in \mathbb{N}$. Then we have*

$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-1} \xi(\mathcal{E}_q - \lambda; H, H_0) = -\frac{1}{2\pi} \sum_{\varsigma=+,-} \left(\frac{\omega_{q,\varsigma}(0)}{\mu_q} + \frac{1}{4}\right)_-^{1/2}.$$

Moreover, if $\omega_{q,\pm}(0) > -\mu_q/4$, then $\xi(\mathcal{E}_q - \lambda; H, H_0) = O(1)$ as $\lambda \downarrow 0$.

Remark: In the case $\alpha = 2$, the analysis of the asymptotic behaviour of $\xi(\lambda; h_j(\varepsilon), h_{0,j})$ as $\lambda \downarrow 0$ requires some additional estimates similar to those obtained in [26]. In order to avoid the inadequate increase of the size of the article, we omit these results.
Finally, in Subsection 4.9 we prove

Corollary 2.3. *Let V satisfy $\mathcal{D}_\alpha$ with $\alpha > 2$. Then for each $q \in \mathbb{N}$ we have*

$$\xi(\mathcal{E}_q + \lambda; H, H_0) = O(1), \quad \lambda \to 0. \tag{2.20}$$

3 Mourre estimates

In this section we prove Theorem 2.1 using an appropriate Mourre estimate established in Proposition 3.1. Similar Mourre estimates have been obtained in [7] for a 2D magnetic Schrödinger operator defined on the half-plane, and in [12, Chapter 3] for a 3D one defined in the whole space.

Lemma 3.1. *Let $n \in \mathbb{N}$, $E \in (\mathcal{E}_n, \mathcal{E}_{n+1})$. Then there exists $\delta = \delta(E) \in (0, \operatorname{dist}(E, \mathcal{Z}))$ such that the interval $\Delta_E = [E - \delta, E + \delta]$ satisfies*

$$E_r^{-1}(\Delta_E) = \emptyset, \quad r \geq n + 1, \tag{3.1}$$

and, if $n \geq 2$,

$$E_r^{-1}(\Delta_E) \cap E_s^{-1}(\Delta_E) = \emptyset, \quad r \neq s, \quad r, s = 1, \ldots, n. \tag{3.2}$$

Proof. First, (3.1) follows trivially from $\Delta_E \cap [\mathcal{E}_{n+1}, \infty) = \emptyset$.
Set $B_r := E_r^{-1}(\Delta_E) \cap [0, \infty)$, $r = 1, \ldots, n$. Since E_r are even functions of k, it suffices to show that

$$B_r \cap B_s = \emptyset, \quad r \neq s, \quad r, s = 1, \ldots, n, \tag{3.3}$$

instead of (3.2). Denote by E_r^{-1}, $r \in \mathbb{N}$, the function inverse to $E_r : [0, \infty) \to \mathbb{R}$. Since $\Delta_E \subset (\mathcal{E}_n, \infty)$, this interval is in the domain of all the functions E_r^{-1}, $r = 1, \ldots, n$, and we have

$$B_r = [E_r^{-1}(E - \delta), E_r^{-1}(E + \delta)], \quad r = 1, \ldots, n.$$

Therefore, in order to prove that there exists $\delta \in (0, \operatorname{dist}(E, \mathcal{Z}))$ such that (3.3) holds true, it suffices to show that there exists $\delta \in (0, \operatorname{dist}(E, \mathcal{Z}))$ such that

$$E_{r+1}^{-1}(E + \delta) < E_r^{-1}(E - \delta), \quad r = 1, \ldots, n - 1,$$

which is evident since $E_{r+1}^{-1}(E) < E_r^{-1}(E)$, the functions E_r^{-1} are continuous, and $n - 1$ is finite. □

Lemma 3.2. *Assume* (2.5). *Let $\chi \in C_0^\infty(\mathbb{R})$. Then $\chi(H) - \chi(H_0) \in S_\infty$.*

Proof. By the Helffer-Sjöstrand formula, we have

$$\chi(H) - \chi(H_0) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial\tilde{\chi}}{\partial\bar{z}}(H-z)^{-1}V(H_0-z)^{-1}dxdy$$

where $z = x+iy$, $\bar{z} = x-iy$, $\tilde{\chi}$ is the quasi-analytic extension of χ, and the convergence of the the integral is understood in the operator-norm sense (see e.g. [8, Chapter 8]). Since the support of $\tilde{\chi}$ is compact in $\mathbb{R}^2$, and the operator $\frac{\partial\tilde{\chi}}{\partial\bar{z}}(H-z)^{-1}V(H_0-z)^{-1}$ is compact for every $(x,y) \in \mathbb{R}^2$ with $y \neq 0$, and is uniformly norm-bounded on $\mathbb{R}^2$, we have $\chi(H) - \chi(H_0) \in S_\infty$. □

Introduce the operator

$$A = A^* = -\frac{i}{2}\left(y\frac{\partial}{\partial y} + \frac{\partial}{\partial y}y\right)$$

defined originally on $C_0^\infty(\mathbb{R}_y; D(\hat{H}))$ and then closed in $L^2(\mathcal{S}_L)$. Note that

$$(e^{itA}f)(x,y) = e^{t/2}f(x, e^t y), \quad t \in \mathbb{R}, \quad f \in L^2(\mathcal{S}_L),$$

and the unitary group e^{itA} preserves $D(H_0)$. In what follows, we will consider $D(H_0^\gamma)$, $\gamma > 0$, as a Hilbert space equipped with the scalar product $\langle H_0^\gamma u, H_0^\gamma v\rangle_{L^2(\mathcal{S}_L)}$, $u, v \in D(H_0^\gamma)$. Denote by $D(H_0^\gamma)^*$, $\gamma > 0$, the completion of $L^2(\mathcal{S}_L)$ with respect to the norm $\|H_0^{-\gamma}u\|_{L^2(\mathcal{S}_L)}$, $u \in L^2(\mathcal{S}_L)$.
Note that $C_0^\infty(\mathbb{R}_y; D(\hat{H}))$ is dense in $D(H_0)$, and, hence, $D(A) \cap D(H_0)$ is dense in $D(H_0)$.

Proposition 3.1. *Assume* (2.5) – (2.6). *Let $n \in \mathbb{N}$, $E \in (\mathcal{E}_n, \mathcal{E}_{n+1})$. Assume that $\delta \in (0, \mathrm{dist}(E, \mathcal{Z}))$ is chosen to satisfy* (3.1) *and* (3.2) *according to Lemma 3.1. Let $\chi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\chi = [E-\delta, E+\delta]$. Then there exists $K \in S_\infty$ and a constant $C > 0$ such that*

$$\chi(H)[H, iA]\chi(H) \geq C\chi(H)^2 + K \tag{3.4}$$

where the commutator $[H, iA]$ is understood as a bounded operator from $D(H_0)$ into $D(H_0)^*$.

Proof. A straightforward calculation yields

$$[H, iA] = [H_0, iA] + [V, iA] \tag{3.5}$$

where

$$[H_0, iA] = -2\frac{\partial^2}{\partial y^2} + 2ibx\frac{\partial}{\partial y}, \tag{3.6}$$

and

$$[V, iA] = -y\frac{\partial V(x,y)}{\partial y}. \tag{3.7}$$

Evidently, $[H_0, iA]$ is a bounded operator from $D(H_0)$ into $L^2(\mathcal{S}_L)$, and, hence, is a bounded operator from $D(H_0)$ into $D(H_0)^*$. On the other hand, $[V, iA]$ is a compact operator from $D(H_0)$ into $D(H_0)^*$. Hence, $[H, iA]$ is a bounded operator from $D(H_0)$ into $D(H_0)^*$. Further, for $\chi \in C_0^\infty(\mathbb{R})$ we have

$$\chi(H_0)[H_0, iA]\chi(H_0) = \mathcal{F}^* \left(2 \sum_{r,s=1}^{\infty} \int_{\mathbb{R}}^{\oplus} \chi(E_r(k))\chi(E_s(k))kp_r(k)(k - bx)p_s(k)dk \right) \mathcal{F}, \tag{3.8}$$

where

$$p_r(k) := \langle \cdot, \psi_r(\cdot; k)\rangle \psi_r(\cdot; k), \quad k \in \mathbb{R}, \quad r \in \mathbb{N}, \tag{3.9}$$

$\psi_r(\cdot; k)$ being the eigenfunction defined in Subsection 2.4 before the formulation of Theorem 2.2. Using (3.1) and (3.2), we find that (3.8) reduces to

$$\chi(H_0)[H_0, iA]\chi(H_0) = 2\mathcal{F}^* \left(\sum_{r=1}^{n} \int_{\mathbb{R}}^{\oplus} \chi(E_r(k))^2 k \langle (k - bx)\psi_r(k), \psi_r(k)\rangle p_r(k)dk \right) \mathcal{F}. \tag{3.10}$$

This, combined with the Feynman-Hellmann formula

$$E_r'(k) = 2\langle (k - bx)\psi_r(k), \psi_r(k)\rangle, \tag{3.11}$$

yields

$$\chi(H_0)[H_0, iA]\chi(H_0) = \mathcal{F}^* \left(\sum_{r=1}^{n} \int_{\mathbb{R}}^{\oplus} kE_r'(k)\chi(E_r(k))^2 p_r(k)dk \right) \mathcal{F}. \tag{3.12}$$

Moreover, by (2.2), we have

$$kE_r'(k)\chi(E_r(k))^2 \geq C_r \chi(E_r(k))^2,$$

with $C_r = \min_{k\in[E-\delta,E+\delta]} kE_r'(k) > 0$, $r = 1, \ldots, n$. Therefore,

$$\chi(H_0)[H_0, iA]\chi(H_0) \geq C\mathcal{F}^* \left(\sum_{r=1}^{n} \int_{\mathbb{R}}^{\oplus} \chi(E_r(k))^2 p_r(k) \right) \mathcal{F} = C\chi(H_0)^2, \tag{3.13}$$

where $C := \min_{r=1,\ldots,n} C_r > 0$. By (3.5),

$$\chi(H)[H, iA]\chi(H) = \chi(H_0)[H_0, iA]\chi(H_0) + K_0, \tag{3.14}$$

where

$$K_0 = \chi(H_0)[H_0, iA] (\chi(H) - \chi(H_0)) + (\chi(H) - \chi(H_0)) [H_0, iA]\chi(H) - \chi(H)y\frac{\partial V}{\partial y}\chi(H) :=$$

$$K_1 + K_2 + K_3.$$

We have

$$K_1 = \chi(H_0)H_0H_0^{-1}[H_0, iA]\left(\chi(H) - \chi(H_0)\right),$$

and the operators $\chi(H_0)H_0$ and $H_0^{-1}[H_0, iA]$ extend to bounded operators in $L^2(\mathcal{S}_L)$ (see (3.6)). Since the operator $\chi(H) - \chi(H_0)$ is compact by Lemma 3.2, we conclude that $K_1 \in S_\infty(L^2(\mathcal{S}_L))$. Similarly, taking into account that $\chi(H) - \chi(H_0)$ is compact, and the operators $[H_0, iA]H_0^{-1}$ and $H_0\chi(H) = H\chi(H) - V\chi(H)$ are bounded, we get

$$K_2 = \left(\chi(H) - \chi(H_0)\right)[H_0, iA]H_0^{-1}H_0\chi(H) \in S_\infty(L^2(\mathcal{S}_L)).$$

Finally, the operator

$$K_3 = \chi(H)y\frac{\partial V}{\partial y}\chi(H) = \chi(H)H_0H_0^{-1}y\frac{\partial V}{\partial y}H_0^{-1}H_0\chi(H)$$

is compact in $L^2(\mathcal{S}_L)$ since $H_0^{-1}y\frac{\partial V}{\partial y}H_0^{-1}$ is compact by (2.6), and $\chi(H)H_0 = (H_0\chi(H))^*$ is bounded in $L^2(\mathcal{S}_L)$. Therefore, $K_0 = K_1 + K_2 + K_3 \in S_\infty$. Combining (3.13) and (3.14), we get

$$\chi(H)[H, iA]\chi(H) \geq C\chi(H_0)^2 + K_0 = C\chi(H)^2 + K_0 + K_4, \tag{3.15}$$

where $K_4 := C\left(\chi(H_0)^2 - \chi(H)^2\right) \in S_\infty$ by Lemma 3.2. Hence (3.15) implies (3.4) with $K = K_0 + K_4$. □

For $E \in \mathbb{R}$ and $\delta > 0$ set $\Delta_E(\delta) := (E - \delta/2, E + \delta/2)$.

Corollary 3.1. *Assume* (2.5) – (2.6). *Fix* $E \in (\mathcal{E}_n, \mathcal{E}_{n+1})$, $n \in \mathbb{N}$. *Let* $\delta \in (0, \operatorname{dist}(E, \mathcal{Z}))$ *be chosen as in Proposition 3.1.*
(i) We have

$$\mathbb{P}_{\Delta_E(\delta)}(H)[H, iA]\mathbb{P}_{\Delta_E(\delta)}(H) \geq C\mathbb{P}_{\Delta_E(\delta)}(H) + \tilde{K} \tag{3.16}$$

where $\tilde{K} := \mathbb{P}_{\Delta_E(\delta)}(H)K\mathbb{P}_{\Delta_E(\delta)}(H) \in S_\infty$, C *and* K *being the same as in* (3.4).
(ii) Suppose moreover that $E \notin \overline{\sigma_p(H)}$. *Then for* $\delta' \in (0, \delta)$ *small enough we have*

$$\mathbb{P}_{\Delta_E(\delta')}(H)[H, iA]\mathbb{P}_{\Delta_E(\delta')}(H) \geq \frac{1}{2}C\mathbb{P}_{\Delta_E(\delta')}(H). \tag{3.17}$$

Proof. Choose χ in (3.4) to be equal to one on $\Delta_E(\delta)$, and multiply (3.4) from the left and the right by $\mathbb{P}_{\Delta_E(\delta)}(H)$. Thus we get (3.16). In order to obtain (3.17), we repeat the argument of the proof of [6, Lemma 4.8]. Pick $\delta' \in (0, \delta)$ and multiply (3.16) from the right and the left by $\mathbb{P}_{\Delta_E(\delta')}(H)$. We get

$$\mathbb{P}_{\Delta_E(\delta')}(H)[H, iA]\mathbb{P}_{\Delta_E(\delta')}(H) \geq C\mathbb{P}_{\Delta_E(\delta')}(H) + \mathbb{P}_{\Delta_E(\delta')}(H)\tilde{K}\mathbb{P}_{\Delta_E(\delta')}(H). \tag{3.18}$$

Since $E \notin \overline{\sigma_p(H)}$ and, hence, $\mathrm{s}-\lim_{\delta' \downarrow 0} \mathbb{P}_{\Delta_E(\delta')}(H) = 0$, while $\tilde{K}$ is compact, we have $\mathrm{n}-\lim_{\delta' \downarrow 0} \mathbb{P}_{\Delta_E(\delta')}(H) \tilde{K} \mathbb{P}_{\Delta_E(\delta')}(H) = 0$. Choose $\delta' \in (0, \delta)$ so small that

$$\|\mathbb{P}_{\Delta_E(\delta')}(H) \tilde{K} \mathbb{P}_{\Delta_E(\delta')}(H)\| \leq C/2$$

which implies

$$\mathbb{P}_{\Delta_E(\delta')}(H) \tilde{K} \mathbb{P}_{\Delta_E(\delta')}(H) \geq -\frac{1}{2} C \mathbb{P}_{\Delta_E(\delta')}(H). \tag{3.19}$$

Combining (3.18) with (3.19), we obtain (3.17). □

Since the unitary group e^{itA} preserves $D(H_0)$, and $[H, iA] : D(H_0) \to D(H_0)^*$ is a bounded operator, the Mourre estimate (3.16) entails the following

Corollary 3.2. [20], [6, Theorem 4.7], [11] *Assume* (2.5) – (2.6). *Let E, δ, and $\Delta_E(\delta)$ be as in Corollary 3.1. Then $\Delta_E(\delta)$ contains at most finitely many eigenvalues of H, each of them having a finite multiplicity.*

Now we are in position to prove Theorem 2.1. Let $\Delta \subset \mathbb{R} \setminus \mathcal{Z}$ be a compact interval. If $\Delta \subset (-\infty, \mathcal{E}_1)$, then $\Delta \cap \sigma_{\mathrm{ess}}(H) = \emptyset$ and Δ may contain at most a finite number of eigenvalues, each having a finite multiplicity. Assume $\Delta \subset (\mathcal{E}_n, \mathcal{E}_{n+1})$, $n \in \mathbb{N}$. For each $E \in \Delta$ choose $\delta = \delta(E)$ as in Proposition 3.1. Then we have $\Delta \subset \cup_{E \in \Delta} \Delta_E(\delta)$. Since Δ is compact, there exists a finite set $\{E_j\}_{j=1}^N$ of energies $E_j \in \Delta$ such that

$$\Delta \subset \cup_{j=1}^N \Delta_{E_j}(\delta). \tag{3.20}$$

Assume (2.5) – (2.6). Then (3.20) and Corollary 3.2 imply that Δ may contain at most a finite number of eigenvalues, each having a finite multiplicity. Hence, the first part of Theorem 2.1 is proved.
Assume moreover (2.7) – (2.8). It follows from (2.7) that $[H, iA]$ extends to a bounded operator from $D(H_0)$ to $D(H_0^{1/2})^*$, while (2.8) combined with (2.6), implies that the second commutator $[[H, iA], iA]$ extends to a bounded operator from $D(H_0)$ to $D(H_0)^*$. Then Corollary 3.1 ii) together with the results of [6, Corollary 4.10] and [11] (see also [20]) imply that $\sigma_{\mathrm{sc}}(H) \cap \left((\mathcal{E}_n, \mathcal{E}_{n+1}) \setminus \overline{\sigma_p(H)} \right) = \emptyset$, $n \in \mathbb{N}$. Since the set $(\mathcal{E}_n, \mathcal{E}_{n+1}) \cap \overline{\sigma_p(H)}$ is at most discrete, we get $\sigma_{\mathrm{sc}}(H) \cap (\mathcal{E}_n, \mathcal{E}_{n+1}) = \emptyset$, $n \in \mathbb{N}$. Finally, since $\mathcal{E}_1 = \inf \sigma_{\mathrm{ess}}(H)$ we have $\sigma_{\mathrm{sc}}(H) \cap (-\infty, \mathcal{E}_1) = \emptyset$. Therefore, $\sigma_{\mathrm{sc}}(H) \cap (\mathbb{R} \setminus \mathcal{Z}) = \emptyset$. Since $\mathcal{Z}$ is discrete, $\sigma_{\mathrm{sc}}(H) = \emptyset$. The second part of Theorem 2.1 is now proved too.
Remark: Mourre estimates and their corollaries concerning the spectrum of H could be also deduced from the general scheme for analytically fibered operators developed in [13]. The advantage of our approach is that it relies on an explicit and simple conjugate operator A, and offers an explicit description of the "exceptional set" $\mathcal{Z}$.

4 Analysis of the Spectral Shift Function

4.1. In this subsection we summarize some simple properties of compact operators which will be systematically used in the sequel. For $s > 0$ and $T^* = T \in S_\infty$ set

$$n_\pm(s;T) := \mathrm{rank}\, \mathbb{P}_{(s,\infty)}(\pm T).$$

For an arbitrary (not necessarily self-adjoint) operator $T \in S_\infty$ put

$$n_*(s;T) := n_+(s^2;T^*T), \quad s > 0. \tag{4.1}$$

If $T = T^*$, then evidently

$$n_*(s;T) = n_+(s,T) + n_-(s;T), \quad s > 0. \tag{4.2}$$

If $T_1, T_2 \in S_\infty$, and $s_1 > 0$, $s_2 > 0$, then the well known Weyl – Ky Fan inequalities

$$n_*(s_1 + s_2; T_1 + T_2) \le n_*(s_1;T_1) + n_*(s_2;T_2) \tag{4.3}$$

hold true. Moreover, if $T_j = T_j^*$, $T_1 \in S_\infty$, and $\mathrm{rank}\, T_2 < \infty$, we have

$$n_\pm(s;T_1) - \mathrm{rank}\, T_2 \le n_\pm(s;T_1 + T_2) \le n_\pm(s;T_1) + \mathrm{rank}\, T_2, \quad s > 0. \tag{4.4}$$

If $T \in S_p$, $p \in [1,\infty)$, then the following elementary Chebyshev-type inequality

$$n_*(s;T) \le s^{-p} \|T\|_p^p \tag{4.5}$$

holds for every $s > 0$.

4.2. In this subsection we introduce the concepts of index of a Fredholm pair of orthogonal projections, and index for a pair of selfadjoint operators, and discuss some of their properties. More details can be found in [1] and [5].
A pair of orthogonal projections (P,Q) is said to be Fredholm if

$$\{-1,1\} \cap \sigma_{\mathrm{ess}}(P - Q) = \emptyset.$$

In particular, if $P - Q \in S_\infty$, then the pair (P,Q) is Fredholm.
Assume that the pair of orthogonal projections (P,Q) is Fredholm. Set

$$\mathrm{index}(P,Q) := \dim \mathrm{Ker}\, (P - Q - I) - \dim \mathrm{Ker}\, (P - Q + I).$$

Let $\tilde{M}$, M, be bounded self-adjoint operators. If the spectral projections $\mathbb{P}_{(-\infty,0)}(\tilde{M})$ and $\mathbb{P}_{(-\infty,0)}(M)$ form a Fredholm pair, we will use the notation

$$\mathrm{ind}(\tilde{M}, M) := \mathrm{index}(\mathbb{P}_{(-\infty,0)}(\tilde{M}), \mathbb{P}_{(-\infty,0)}(M)).$$

A sufficient condition that the pair $\mathbb{P}_{(-\infty,0)}(\tilde{M}), \mathbb{P}_{(-\infty,0)}(M)$ be Fredholm, is $\tilde{M} = M + A$ where M is a bounded self-adjoint operator such that $0 \notin \sigma_{\mathrm{ess}}(M)$, and $A = A^* \in S_\infty$.

Lemma 4.1. [5, Subsection 3.2] *Let M be a bounded self-adjoint operator such that $0 \notin \sigma(M)$. Let A and B be compact self-adjoint operators. Then for $s \in (0, \infty)$ such that $[-s, s] \cap \sigma(M) = \emptyset$ we have*

$$\operatorname{ind}(M+s+B, M+s)-n_+(s;A) \le \operatorname{ind}(M+A+B, M) \le \operatorname{ind}(M-s+B, M-s)+n_-(s;A). \tag{4.6}$$

Assume, moreover, that the rank of A is finite. Then we have

$$\operatorname{ind}(M+B, M) - \operatorname{rank} A \le \operatorname{ind}(M+A+B, M) \le \operatorname{ind}(M+B, M) + \operatorname{rank} A. \tag{4.7}$$

Remark: Note that in the case $B = 0$, estimates (4.6) imply

$$|\operatorname{ind}(M+A, M)| \le n_*(s; A) \tag{4.8}$$

for any $s > 0$ such that $[-s, s] \cap \sigma(M) = \emptyset$.

Lemma 4.2. [21, Lemma 2.1], [5, Subsection 3.3] *Let M be a bounded self-adjoint operator such that $0 \notin \sigma(M)$. Let $T_1 = T_1^* \in S_\infty$ and $T_2 = T_2^* \in S_1$. Then for each $s_1 > 0$, $s_2 > 0$ such that $[-s, s] \cap \sigma(M) = \emptyset$ with $s = s_1 + s_2$, we have*

$$\int_{\mathbb{R}} |\operatorname{ind}(M + T_1 + t\,T_2, M)|\; d\mu(t) \le n_*(s_1; T_1) + \frac{1}{\pi s_2}\|T_2\|_1 \tag{4.9}$$

where $d\mu(t) := \frac{1}{\pi}\frac{dt}{1+t^2}$.

4.3. In this subsection we describe a representation of the SSF $\xi(E; \mathcal{H}, \mathcal{H}_0)$ which is a special case of the general representation of the SSF due to F. Gesztesy, K. Makarov, and A. Pushnitski (see [21], [14], [22]).
Let X_1 and X_2 be two Hilbert spaces. Let $\mathcal{H}$ and $\mathcal{H}_0$ be two lower bounded self-adjoint operators acting in X_1. Assume that (2.9) holds for some $\gamma > 0$. Next suppose that

$$\mathcal{V} := \mathcal{H} - \mathcal{H}_0 = \mathcal{K}^* \mathcal{J} \mathcal{K} \tag{4.10}$$

where $\mathcal{K} \in \mathcal{B}(X_1, X_2)$, $\mathcal{J} = \mathcal{J}^* \in \mathcal{B}(X_2)$, and $0 \notin \sigma(\mathcal{J})$. Finally, assume that

$$\mathcal{K}(\mathcal{H}_0 - E_0)^{-1/2} \in S_\infty(X_1, X_2), \tag{4.11}$$

$$\mathcal{K}(\mathcal{H}_0 - E_0)^{-\gamma'} \in S_2(X_1, X_2), \tag{4.12}$$

for some $E_0 < \inf \sigma(\mathcal{H}) \cup \sigma(\mathcal{H}_0)$ and $\gamma' > 0$. For $z \in \mathbb{C}_+ := \{\zeta \in \mathbb{C} \,|\, \operatorname{Im} \zeta > 0\}$ set

$$\mathcal{T}(z) := \mathcal{K}(\mathcal{H}_0 - z)^{-1}\mathcal{K}^*.$$

Evidently, $\mathcal{T}(z) \in S_\infty(X_2)$.

Lemma 4.3. [3] *Let (4.10) – (4.12) hold true. Then for almost every $E \in \mathbb{R}$ the operator-norm limit $\mathcal{T}(E) := \mathrm{n}-\lim_{\delta\downarrow 0} \mathcal{T}(E + i\delta)$ exists and by (4.12) we have $\mathcal{T}(E) \in S_\infty(X_2)$. Moreover, $0 \le \operatorname{Im} \mathcal{T}(E) \in S_1(X_2)$.*

Theorem 4.1. [21], [14], [22] *Let* (2.9) *and* (4.10) – (4.12) *hold true. Then for almost every* $E \in \mathbb{R}$ *we have*

$$\xi(E; \mathcal{H}, \mathcal{H}_0) = \int_{\mathbb{R}} \mathrm{ind}(\mathcal{J}^{-1} + \mathrm{Re}\, \mathcal{T}(E) + t\, \mathrm{Im}\, \mathcal{T}(E), \mathcal{J}^{-1})\, d\mu(t). \tag{4.13}$$

Note that the convergence of the integral in (4.13) is guaranteed by Lemma 4.2.
Now suppose that the electric potential V satisfies $\mathcal{D}_\alpha$ with $\alpha > 1$. Then relations (2.9) and (4.10) – (4.12) hold true with $X_1 = X_2 = L^2(\mathcal{S}_L)$, $\mathcal{H}_0 = H_0$, $\mathcal{H} = H$, $\mathcal{V} = V$, $\mathcal{K} = |V|^{1/2}$, $\mathcal{J} = J = \mathrm{sign}\, V$, and $\gamma = \gamma' = 1$. For $z \in \mathbb{C}_+$ set

$$T(z) := |V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}.$$

By Lemma 4.3 for almost every $E \in \mathbb{R}$ the operator-norm limit

$$T(E) := \mathrm{n} - \lim_{\delta \downarrow 0} T(E + i\delta) \tag{4.14}$$

exists, and

$$0 \leq \mathrm{Im}\, T(E) \in S_1. \tag{4.15}$$

In Corollary 4.1 below we will show that the limit (4.14) exists, and relation (4.15) holds true for *every* $E \in \mathbb{R} \setminus \mathcal{Z}$. Then Theorem 4.1 implies that for almost every $E \in \mathbb{R}$ we have

$$\xi(E; H, H_0) = \int_{\mathbb{R}} \mathrm{ind}(J + \mathrm{Re}\, T(E) + t\, \mathrm{Im}\, T(E), J)\, d\mu(t), \tag{4.16}$$

the right-hand-side being well defined for every $E \in \mathbb{R} \setminus \mathcal{Z}$. In this article we identify the SSF $\xi(E; H, H_0)$ for energies $E \notin \mathcal{Z}$ with the r.h.s. of (4.16).

4.4. Fix $j \in \mathbb{N}$. Denote by $\varphi_j : [0, \infty) \to [0, \infty)$ the function inverse to $E_j - \mathcal{E}_j$. In the following lemma we describe some properties of φ_j which will be used in the sequel.
Let β and η be two functions with values in $[0, \infty)$, and $\mathcal{O} \subseteq D(\beta) \cap D(\eta)$. We will write $\beta(s) \asymp \eta(s)$, $s \in \mathcal{O}$, if there exist two constants $c_\pm > 0$ such that for each $s \in \mathcal{O}$ we have $c_- \eta(s) \leq \beta(s) \leq c_+ \eta(s)$.

Lemma 4.4. *Let* $j \in \mathbb{N}$. *We have*

$$\varphi_j(s) \asymp s^{1/2}, \quad s \in [0, \infty), \tag{4.17}$$

$$\varphi_j'(s) \asymp s^{-1/2}, \quad s \in (0, \infty). \tag{4.18}$$

Moreover,

$$\varphi_j(s) = \sqrt{s}\, \Phi(s), \quad s \in [0, \infty), \tag{4.19}$$

where $\Phi \in C^\infty([0, \infty))$, *and*

$$\Phi(0) = \mu_j^{-1/2}, \tag{4.20}$$

the number μ_j *being defined in* (2.4). *In particular, we have*

$$|\varphi_j''(s)| = O(s^{-3/2}), \quad s \in (0, s_0), \quad s_0 \in (0, \infty). \tag{4.21}$$

Proof. By (2.1) and (2.3) we have

$$E_j(k) - \mathcal{E}_j \asymp k^2, \quad k \in \mathbb{R}, \tag{4.22}$$

which implies immediately (4.17). On the other hand, (3.11) and (2.2) easily yield

$$E_j'(k) \asymp k, \quad k \in [0, \infty). \tag{4.23}$$

Bearing in mind the formula for the derivative of an inverse function, we find that (4.17) and (4.23) imply (4.18).
Further, for $t \geq 0$ introduce the function $E_j(\sqrt{t}) - \mathcal{E}_j$, and denote by $\Psi = \Psi_j : [0, \infty) \to [0, \infty)$ its inverse. By (4.18) we have $\Psi'(s) \asymp 1$, $s \in [0, \infty)$. Since E_j is analytic, we find that $\Psi \in C^\infty([0, \infty))$. Moreover, $\Psi(0) = 0$ and $\Psi'(0) = \mu_j^{-1}$. Since $\varphi(s) = \sqrt{\Psi(s)}$, we get (4.19) with $\Phi(s) = \sqrt{\Psi(s)/s}$, which on its turn implies (4.20). □

For $j \in \mathbb{N}$ set

$$P_j := \mathcal{F}^* \int_{\mathbb{R}}^{\oplus} p_j(k) dk \mathcal{F},$$

the orthogonal projections $p_j(k)$, $k \in \mathbb{R}$, being defined in (3.9). For $z \in \mathbb{C}_+$ and $j \in \mathbb{N}$ put

$$T_j(z) := |V|^{1/2} P_j (H_0 - z)^{-1} |V|^{1/2}.$$

Lemma 4.5. *Assume that V satisfies $\mathcal{D}_\alpha$ with $\alpha > 1$. Fix $j \in \mathbb{N}$. Then for each $z \in \mathbb{C}_+$ we have $T_j(z) \in S_1$ and the operator-valued function $T_j : \mathbb{C}_+ \to S_1$ is analytic. Moreover, for $E \in \mathbb{R} \setminus \{\mathcal{E}_j\}$ the limit*

$$T_j(E) = \lim_{\delta \downarrow 0} T_j(E + i\delta) \tag{4.24}$$

exists in S_1, and $T_j : \mathbb{R} \setminus \{\mathcal{E}_j\} \to S_1$ is continuous. Next, if $E - \mathcal{E}_j < 0$, then the operator $T_j(E)$ is self-adjoint, and if $E - \mathcal{E}_j > 0$, we have

$$0 \leq \operatorname{Im} T_j(E), \quad \operatorname{rank} \operatorname{Im} T_j(E) \leq 2. \tag{4.25}$$

Finally, for each $\lambda_0 > 0$ there exists $C_j = C_j(\lambda_0)$ such that for $0 < |E - \mathcal{E}_j| < \lambda_0$ we have

$$\|T_j(E)\|_1 \leq C_j |E - \mathcal{E}_j|^{-1/2}; \tag{4.26}$$

if $E - \mathcal{E}_j < 0$, then C_j could be chosen independent of λ_0.

Proof. Let $G = G_j : \mathbb{R} \to S_2(L^2(\mathcal{S}_L), \mathbb{C})$ be the operator-valued function given for $k \in \mathbb{R}$ by

$$G(k)u := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{I_L} e^{-iky} |V(x,y)|^{1/2} \psi_j(x;k) u(x,y) dx dy, \quad u \in L^2(\mathcal{S}_L).$$

Evidently,

$$\|G(k)^*G(k)\|_1 = \|G(k)\|_2^2 \leq c_1 := \frac{1}{2\pi} \sup_{x \in I_L} \int_{\mathbb{R}} |V(x,y)| dy \tag{4.27}$$

for any $k \in \mathbb{R}$. Next,

$$\|G(k_1) - G(k_2)\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{I_L} |V(x,y)| |e^{-ik_1 y} \psi_j(x;k_1) - e^{-ik_2 y} \psi_j(x;k_2)|^2 dx dy \leq$$

$$\frac{2^{2(1-\gamma)}}{\pi} \sup_{x \in I_L} \int_{\mathbb{R}} |V(x,y)| |y|^{2\gamma} dy |k_1 - k_2|^{2\gamma} + 2c_1 \int_{I_L} |\psi(x;k_1) - \psi(x;k_2)|^2 dx$$

for $k_1, k_2 \in \mathbb{R}$, and $\gamma \in (0, (\alpha-1)/2)$, $\gamma \leq 1$. Since $\psi \in C^\infty(\mathbb{R}_k; L^2(I_L))$, we have

$$\int_{I_L} |\psi(x;k_1) - \psi(x;k_2)|^2 dx = O(|k_1 - k_2|^2)$$

for $k_1, k_2 \in (-k_0, k_0)$ with $k_0 \in (0, \infty)$. Therefore,

$$\|G(k_1) - G(k_2)\|_2 = O(|k_1 - k_2|^\gamma) \tag{4.28}$$

for $k_1, k_2 \in (-k_0, k_0)$, $k_0 \in (0, \infty)$, and $\gamma \in (0, (\alpha-1)/2)$, $\gamma \leq 1$. Taking into account (4.27) and (2.1), we find that if $z \in \mathbb{C}_+$, then

$$\|G_j^* G_j (E_j - z)^{-1}\|_1 \in L^1(\mathbb{R}). \tag{4.29}$$

Then the spectral theorem implies

$$T_j(z) = \int_{\mathbb{R}} \frac{G_j(k)^* G_j(k)}{E_j(k) - z} dk, \quad z \in \mathbb{C}_+, \tag{4.30}$$

where, due to (4.29) and the continuity of the functions $G_j : \mathbb{R} \to S_2(L^2(\mathcal{S}_L), \mathbb{C})$ and $E_j : \mathbb{R} \to \mathbb{R}$, the integral admits an interpretation as a Bochner integral in the Banach space S_1 (see e.g. [19]), and it is easy to see that $T_j : \mathbb{C}_+ \to S_1$ is analytic.
Let $F = F_j : (0, \infty) \to S_2(L^2(\mathcal{S}_L), \mathbb{C}^2)$ be the operator-valued function defined for $s \in (0, \infty)$ by

$$F(s)u := \sqrt{\varphi'(s)} (G(\varphi(s))u, G(-\varphi(s))u), \quad u \in L^2(\mathcal{S}_L),$$

where, as above, $\varphi = \varphi_j$ denotes the function inverse to $E_j - \mathcal{E}_j$. Then we have

$$T_j(z) = \int_0^\infty \frac{F_j(s)^* F_j(s)}{s - \lambda - i\delta} ds, \quad z = \mathcal{E}_j + \lambda + i\delta \in \mathbb{C}_+.$$

Further, if $\lambda := E - \mathcal{E}_j < 0$, set

$$T_j(E) = \int_0^\infty \frac{F_j(s)^* F_j(s)}{s - \lambda} ds. \tag{4.31}$$

Evidently, the operator $T_j(E)$ is self-adjoint. Also, it is easy to check that (4.24) holds true, and the function $T_j : (-\infty, \mathcal{E}_j) \to S_1$ is continuous. By (4.27) and (4.18),

$$\|T_j(E)\|_1 \leq 2c_1 \int_0^\infty \frac{\varphi'(s)}{s+|\lambda|} ds = O\left(\int_0^\infty \frac{ds}{s^{1/2}(s+|\lambda|)}\right) = O(|\lambda|^{-1/2}), \quad \lambda < 0,$$

so that (4.26) holds in this case as well.
Let now $\lambda = E - \mathcal{E}_j > 0$. For $E = \mathcal{E}_j + \lambda$ put

$$\mathrm{Re}\, T_j(E) := \text{v.p.} \int_0^\infty \frac{F_j(s)^* F_j(s)}{s - \lambda} ds, \tag{4.32}$$

$$\mathrm{Im}\, T_j(E) := \pi F_j(\lambda)^* F_j(\lambda), \tag{4.33}$$

$$T_j(E) := \mathrm{Re}\, T_j(E) + i \mathrm{Im}\, T_j(E).$$

Note that (4.33) immediately implies (4.25). Moreover,

$$\text{v.p.} \int_0^\infty \frac{F(s)^* F(s)}{s-\lambda} ds = \int_0^{\lambda/2} \frac{F(s)^* F(s)}{s-\lambda} ds + \int_{3\lambda/2}^\infty \frac{F(s)^* F(s)}{s-\lambda} ds +$$

$$\int_0^{\lambda/2} \left(F(\lambda+\nu)^* F(\lambda+\nu) - F(\lambda-\nu)^* F(\lambda-\nu)\right) \frac{d\nu}{\nu}. \tag{4.34}$$

By (4.18) and (4.21),

$$|\varphi(\lambda+\nu) - \varphi(\lambda-\nu)| = O\left((\lambda+\nu)^{1/2} - (\lambda-\nu)^{1/2}\right), \tag{4.35}$$

$$|\varphi'(\lambda+\nu) - \varphi'(\lambda-\nu)| = O\left((\lambda-\nu)^{-1/2} - (\lambda+\nu)^{-1/2}\right), \tag{4.36}$$

for $\nu \in (0, \lambda/2)$, $\lambda \in (0, \lambda_0)$.
Taking into account (4.27) - (4.28), (4.18), and (4.32) - (4.36) we find that the operator $T_j(E)$ is well defined, that (4.24) holds true again, and

$$\|F(\lambda)^* F(\lambda)\|_1 = O(\lambda^{-1/2}), \quad \lambda > 0,$$

$$\left\| \int_0^{\lambda/2} \frac{F(s)^* F(s)}{s-\lambda} ds \right\|_1 = O(\lambda^{-1/2}), \quad \left\| \int_{3\lambda/2}^\infty \frac{F(s)^* F(s)}{s-\lambda} ds \right\|_1 = O(\lambda^{-1/2}), \quad \lambda > 0,$$

$$\left\| \int_0^{\lambda/2} (F(\lambda+\nu)^* F(\lambda+\nu) - F(\lambda-\nu)^* F(\lambda-\nu)) \frac{d\nu}{\nu} \right\|_1 = O(\lambda^{-1/2}), \quad \lambda \in (0, \lambda_0),$$

which yields again (4.26). □

4.5. Let $j \in \mathbb{N}$. Set $P_j^+ := \sum_{m=j}^\infty P_m$ where the convergence of the infinite sum is understood in the strong sense. For $z \in \mathbb{C}_+$, $\mathrm{Re}\, z < \mathcal{E}_j$, put

$$T_j^+(z) := |V|^{1/2} P_j^+ (H_0 - z)^{-1} |V|^{1/2}.$$

Lemma 4.6. *Fix $j \in \mathbb{N}$. Let $E \in (-\infty, \mathcal{E}_j)$. Then the limit*

$$T_j^+(E) = T_j^+(E)^* = \mathrm{n} - \lim_{\delta \downarrow 0} T_j^+(E + i\delta) \tag{4.37}$$

exists. Moreover, for any $z \in \overline{\mathbb{C}_+} \setminus [\mathcal{E}_j, \infty)$ we have $T_j^+(z) \in S_2$, and the operator-valued function $T_j^+ : \overline{\mathbb{C}_+} \setminus [\mathcal{E}_j, \infty) \to S_2$ is continuous. Finally, there exists a constant C_+ which depends on V, but is independent of E and j, such that

$$\|T_j^+(E)\|_2 \leq C_+ \mathcal{E}_j (\mathcal{E}_j - E)^{-1}, \quad E \in (-\infty, \mathcal{E}_j). \tag{4.38}$$

Proof. We have

$$P_j^+ (H_0 - z)^{-1} = P_j^+ \mathbb{P}_{[\mathcal{E}_j, \infty)}(H_0)(H_0 - z)^{-1} \tag{4.39}$$

and the operator valued function $\mathbb{P}_{[\mathcal{E}_j,\infty)}(H_0)(H_0 - z)^{-1}$ is analytic even on $\mathbb{C} \setminus [\mathcal{E}_j, \infty)$. Since P_j^+ and $|V|^{1/2}$ are bounded operators, this analyticity implies, in particular, the existence of the limit in (4.37) and the continuity of $T_j^+ : \overline{\mathbb{C}_+} \setminus [\mathcal{E}_j, \infty) \to \mathcal{B}$. Further,

$$\||V|^{1/2} P_j^+ (H_0 - E)^{-1} |V|^{1/2}\|_2 \leq \sup_{(x,y) \in \mathcal{S}_L} |V(x,y)|^{1/2} \|P_j^+ (H_0 - E)^{-1} H_0\| \, \|H_0^{-1} |V|^{1/2}\|_2. \tag{4.40}$$

By (4.39),

$$\|P_j^+ (H_0 - E)^{-1} H_0\| \leq \|\mathbb{P}_{[\mathcal{E}_j,\infty)}(H_0)(H_0 - E)^{-1} H_0\| \leq \sup_{\lambda \in [\mathcal{E}_j, \infty)} \lambda (\lambda - E)^{-1} = \mathcal{E}_j (\mathcal{E}_j - E)^{-1}. \tag{4.41}$$

On the other hand, the diamagnetic inequality for Hilbert-Schmidt operators (see e.g. [25, Theorem 2.13]) implies

$$\|H_0^{-1} |V|^{1/2}\|_2 \leq \|\Delta_D^{-1} |V|^{1/2}\|_2 \tag{4.42}$$

where, as above, Δ_D is the Dirichlet Laplacian defined on $\mathcal{S}_L$. The integral kernel of Δ_D is explicitly known, and we easily find

$$\|\Delta_D^{-1} |V|^{1/2}\|_2^2 \leq 16 c_1 \frac{L^3}{\pi^3} \sum_{n=1}^{\infty} n^{-3} \int_0^{\infty} \frac{d\xi}{(\xi^2 + 1)^2}. \tag{4.43}$$

Putting together (4.40) - (4.43), we obtain (4.38).
Finally, an estimate similar to (4.40) of the Hilbert-Schmidt norm of the difference $|V|^{1/2} P_j^+ (H_0 - z_1)^{-1} |V|^{1/2} - |V|^{1/2} P_j^+ (H_0 - z_2)^{-1} |V|^{1/2}$, $z_1, z_2 \in \overline{\mathbb{C}_+} \setminus [\mathcal{E}_j, \infty)$ easily implies the continuity of $T_j^+ : \overline{\mathbb{C}_+} \setminus [\mathcal{E}_j, \infty) \to S_2$. □

4.6. In this subsection we prove (4.14) - (4.15) as well as Proposition 2.1.
Let $E \in \mathbb{R} \setminus \mathcal{Z}$. If E has one nearest element from $\mathcal{Z}$, let $q = q(E)$ be the number of

this neighbour; if E has two nearest elements from $\mathcal{Z}$, for definiteness let $q(E)$ be the number of the greater of these elements. Set

$$T(E) := \sum_{j=1}^{q(E)} T_j(E) + T^+_{q(E)+1}(E). \tag{4.44}$$

Corollary 4.1. *Let V satisfy $\mathcal{D}_\alpha$ with $\alpha > 1$, and let $E \in \mathbb{R} \setminus \mathcal{Z}$. Then* (4.14) - (4.15) *hold true, the limiting operator $T(E)$ being defined in* (4.44)*. Moreover,*

$$\operatorname{rank} \operatorname{Im} T(E) \le 2q(E). \tag{4.45}$$

Proof. In order to prove the existence of the limit (4.14), we just have to write

$$T(E+i\delta) := \sum_{j=1}^{q(E)} T_j(E+i\delta) + T^+_{q(E)+1}(E+i\delta), \quad \delta > 0,$$

and to apply (4.24) and (4.37). In order to prove (4.15) and (4.45), it suffices to apply (4.25), bearing in mind that $\operatorname{Im} T(E) = \sum_{j=1}^{q(E)} \operatorname{Im} T_j(E)$. □

Next we prove Proposition 2.1. The proof of the continuity of the SSF repeats word by word the proof of the continuity part of [5, Proposition 2.5]. Let us show that the SSF is locally bounded, i.e. that it is bounded on every compact subset of $\mathbb{R} \setminus \mathcal{Z}$.
Let $E \in \mathbb{R} \setminus \mathcal{Z}$. Applying (4.16), (4.8), and (4.7), we get

$$|\xi(E; H, H_0)| \le n_*(s; \operatorname{Re} T(E)) + \operatorname{rank} \operatorname{Im} T(E), \quad s \in (0,1). \tag{4.46}$$

By (4.3),

$$n_*(s; \operatorname{Re} T(E)) \le n_*(s/2; \sum_{j=1}^{q(E)} \operatorname{Re} T_j(E)) + n_*(s/2; T^+_{q(E)+1}(E)). \tag{4.47}$$

Using (4.5) with $p = 1$ and $p = 2$, as well as (4.26) and (4.25), we get

$$n_*(s/2; \sum_{j=1}^{q(E)} \operatorname{Re} T_j(E)) \le \frac{2}{s} \sum_{j=1}^{q(E)} \|T_j(E)\|_1 \le \frac{2}{s} \sum_{j=1}^{q(E)} C_j |E - \mathcal{E}_j|^{-1/2}, \tag{4.48}$$

$$n_*(s/2; T^+_{q(E)+1}(E)) \le \frac{4}{s^2} \|T^+_{q(E)+1}(E)\|_2^2 \le \frac{4}{s^2} C_+^2 \mathcal{E}^2_{q(E)+1} (\mathcal{E}_{q(E)+1} - E)^{-2}. \tag{4.49}$$

Now the combination of (4.46), (4.25), and (4.47) - (4.49) implies the local boundedness of the SSF.

4.7. In this subsection we prove Theorem 2.2.

Proposition 4.1. *Assume that V satisfies $\mathcal{D}_\alpha$ with $\alpha > 1$. Pick $q \in \mathbb{N}$ and $\lambda \neq 0$ such that $E := \mathcal{E}_q + \lambda \notin \mathcal{Z}$. Then we have*

$$\mathrm{ind}\,(J+\varepsilon+\mathrm{Re}\,T_q(E), J+\varepsilon)+O(1) \leq \xi(E; H, H_0) \leq \mathrm{ind}\,(J-\varepsilon+\mathrm{Re}\,T_q(E), J-\varepsilon)+O(1) \tag{4.50}$$

as $\lambda \to 0$ for each $\varepsilon \in (0, 1)$.

Proof. Applying (4.16), (4.7), and (4.45), we get

$$|\xi(E; H, H_0) - \mathrm{ind}(J + \mathrm{Re}\,T(E), J)| \leq 2q(E). \tag{4.51}$$

Write $\mathrm{Re}\,T(E) = \mathrm{Re}\,T_q(E) + \tilde{T}(E)$ where $\tilde{T}(E) := \sum_{j<q} \mathrm{Re}\,T_j(E) + T^+_{q+1}(E)$. By (4.6),

$$\mathrm{ind}\,(J + \varepsilon + \mathrm{Re}\,T_q(E), J + \varepsilon) - n_*(\varepsilon; \tilde{T}(E)) \leq \mathrm{ind}(J + \mathrm{Re}\,T(E), J) \leq$$

$$\mathrm{ind}\,(J - \varepsilon + \mathrm{Re}\,T_q(E), J - \varepsilon) + n_*(\varepsilon; \tilde{T}(E)). \tag{4.52}$$

Using (4.3) and arguing as in the derivation of (4.48), (4.49), we get

$$n_*(\varepsilon; \tilde{T}(E)) \leq \frac{2}{\varepsilon} \sum_{j:j<q} C_j |\mathcal{E}_q - \mathcal{E}_j + \lambda|^{-1/2} + \frac{4}{\varepsilon^2} C_+^2 \mathcal{E}_{q+1}^2 (\mathcal{E}_{q+1} - \mathcal{E}_q - \lambda)^{-2} = O(1), \quad \lambda \to 0. \tag{4.53}$$

Now the combination of (4.51) – (4.53) yields (4.50). □

Fix $j \in \mathbb{N}$. Let $g = g_j : \mathbb{R} \to S_2(L^2(\mathcal{S}_L), \mathbb{C})$ be the operator-valued function given for $k \in \mathbb{R}$ by

$$g_j(k)u = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{I_L} e^{-iky} |V(x, y)|^{1/2} \psi_j(x; 0) u(x, y) dxdy, \quad u \in L^2(\mathcal{S}_L).$$

Similarly to (4.27) and (4.28) we have

$$\|g(k)\|_2^2 \leq c_1, \quad k \in \mathbb{R}, \tag{4.54}$$

$$\|g(k_1) - g(k_2)\|_2 = O(|k_1 - k_2|^\gamma), \quad k_1, k_2 \in \mathbb{R}, \tag{4.55}$$

for any $\gamma \in (0, (\alpha - 1)/2)$ such that $\gamma \leq 1$. By analogy with (4.30) set

$$\tilde{\tau}_j(z) := \int_{\mathbb{R}} \frac{g_j(k)^* g_j(k)}{E_j(k) - z} dk, \quad z \in \mathbb{C}_+. \tag{4.56}$$

As in the case of the operator $T_j(z)$ (see Lemma 4.5) we can show that in S_1 there exists a limit

$$\tilde{\tau}_j(E) = \lim_{\delta \downarrow 0} \tilde{\tau}_j(E + i\delta), \quad E \in \mathbb{R} \setminus \{\mathcal{E}_j\}.$$

Proposition 4.2. *Let V satisfy $\mathcal{D}_\alpha$ with $\alpha > 1$. Fix $q \in \mathbb{N}$, and let $E = \mathcal{E}_q + \lambda \notin \mathcal{Z}$. Then for each $\varepsilon \in (0, 1/2)$ we have*

$$\operatorname{ind}(J + 2\varepsilon + \operatorname{Re}\tilde{\tau}_q(E), J + 2\varepsilon) + O(1) \leq \operatorname{ind}(J + \varepsilon + \operatorname{Re} T_q(E), J + \varepsilon), \tag{4.57}$$

$$\operatorname{ind}(J - 2\varepsilon + \operatorname{Re}\tilde{\tau}_q(E), J - 2\varepsilon) + O(1) \geq \operatorname{ind}(J - \varepsilon + \operatorname{Re} T_q(E), J - \varepsilon), \tag{4.58}$$

as $\lambda \downarrow 0$.

Proof. Using (4.6) and (4.8), we obtain

$$\operatorname{ind}(J+2\varepsilon+\operatorname{Re}\tilde{\tau}_q(E), J+2\varepsilon) - n_*(\varepsilon; \operatorname{Re} T_q(E) - \operatorname{Re}\tilde{\tau}_q(E)) \leq \operatorname{ind}(J+\varepsilon+\operatorname{Re} T_q(E), J+\varepsilon),$$

$$\operatorname{ind}(J-2\varepsilon+\operatorname{Re}\tilde{\tau}_q(E), J-2\varepsilon) + n_*(\varepsilon; \operatorname{Re} T_q(E) - \operatorname{Re}\tilde{\tau}_q(E)) \geq \operatorname{ind}(J-\varepsilon+\operatorname{Re} T_q(E), J-\varepsilon).$$

Hence, in order to prove (4.57) – (4.58), it suffices to show that for each $\varepsilon > 0$ we have

$$n_*(\varepsilon; \operatorname{Re} T_q(E) - \operatorname{Re}\tilde{\tau}_q(E)) = O(1), \quad \lambda \to 0. \tag{4.59}$$

Let again $\varphi = \varphi_q$ be the function inverse to $E_q - \mathcal{E}_q$. Denote by $f = f_q : (0, \infty) \to S_2(L^2(\mathcal{S}_L), \mathbb{C}^2)$ the operator-valued function defined for $s \in (0, \infty)$ by

$$f(s)u := \sqrt{\varphi'(s)}(g(\varphi(s))u, g(-\varphi(s))u), \quad u \in L^2(\mathcal{S}_L).$$

Then similarly to (4.31), (4.32), and (4.34), we have

$$\tilde{\tau}_q(\mathcal{E}_q + \lambda) = \tilde{\tau}_q(\mathcal{E}_q + \lambda)^* = \int_0^\infty \frac{f_q(s)^* f_q(s)}{s - \lambda} ds$$

if $\lambda < 0$, and

$$\operatorname{Re}\tilde{\tau}_q(\mathcal{E}_q + \lambda) = \int_0^{\lambda/2} \frac{f_q(s)^* f_q(s)}{s - \lambda} ds + \int_{3\lambda/2}^\infty \frac{f_q(s)^* f_q(s)}{s - \lambda} ds +$$

$$\int_0^{\lambda/2} (f_q(\lambda+\nu)^* f_q(\lambda+\nu) - f_q(\lambda-\nu)^* f_q(\lambda-\nu)) \frac{d\nu}{\nu}$$

if $\lambda > 0$. Further, we have

$$G(k) = g(k) + k\varrho(k)$$

where $\varrho : \mathbb{R} \to L^2(L^2(\mathcal{S}_L), \mathbb{C})$ is the operator-valued function given for $k \in \mathbb{R}$ by

$$\varrho(k)u = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{I_L} e^{-iky} |V(x,y)|^{1/2} \tilde{\psi}(x;k) u(x,y) dx dy, \quad u \in L^2(\mathcal{S}_L),$$

where $\tilde{\psi}(x;k) := \frac{\psi_q(x;k) - \psi_q(x;0)}{k}$. Evidently,

$$\|\varrho(k)\|_2^2 \leq c_1 \int_{I_L} \tilde{\psi}(x;k)^2 dx, \quad k \in \mathbb{R}, \tag{4.60}$$

$$\|\varrho(k_1) - \varrho(k_2)\|_2 = O(|k_1 - k_2|^\gamma) \tag{4.61}$$

for $k_1, k_2 \in (-k_0, k_0)$ with $k_0 \in (0, \infty)$, and $\gamma \in (0, (\alpha - 1)/2)$, $\gamma \leq 1$. Next, we have

$$F(s) = f(s) + \varphi(s) r(s)$$

where $r : (0, \infty) \to S_2(L^2(\mathcal{S}_L), \mathbb{C}^2)$ is the operator-valued function defined for $s \in (0, \infty)$ by

$$r(s)u := \sqrt{\varphi'(s)}(\varrho(\varphi(s))u, -\varrho(-\varphi(s))u), \quad u \in L^2(\mathcal{S}_L).$$

Therefore,

$$F(s)^* F(s) = f(s)^* f(s) + 2\varphi(s) \mathrm{Re}\, f(s)^* r(s) + \varphi(s)^2 r(s)^* r(s). \tag{4.62}$$

Note that

$$\varphi(s) f(s)^* r(s) = \varphi(s)\varphi'(s) \left(g(\varphi(s))^* \varrho(\varphi(s)) - g(-\varphi(s))^* \varrho(-\varphi(s)) \right) =$$

$$\varphi(s)\varphi'(s) \left(g(\varphi(s))^* (\varrho(\varphi(s)) - \varrho(-\varphi(s))) + (g(\varphi(s))^* - g(-\varphi(s))^*) \varrho(-\varphi(s)) \right).$$

Hence, by (4.17) – (4.18), (4.54) – (4.55), and (4.60) – (4.61), we have

$$\varphi(s) \|f(s)^* r(s)\|_1 = O(s^{\gamma/2}), \quad \gamma \in (0, (\alpha - 1)/2), \quad \gamma \leq 1, \tag{4.63}$$

$$\varphi(s)^2 \|r(s)^* r(s)\|_1 = O(s^{1/2}) \tag{4.64}$$

for $s \in (0, s_0)$ and $s_0 \in (0, \infty)$. By (4.62), for a fixed $s_0 > 0$ we have

$$\mathrm{Re}\, T_q(\mathcal{E}_q + \lambda) - \mathrm{Re}\, \tilde{\tau}_q(\mathcal{E}_q + \lambda) = T_q(\mathcal{E}_q + \lambda) - \tilde{\tau}_q(\mathcal{E}_q + \lambda) =$$

$$\int_{s_0}^\infty \frac{F(s)^* F(s)}{s - \lambda} ds - \int_{s_0}^\infty \frac{f(s)^* f(s)}{s - \lambda} ds + \int_0^{s_0} \frac{2\varphi(s) \mathrm{Re}\, f(s)^* r(s) + \varphi(s)^2 r(s)^* r(s)}{s - \lambda} ds$$

if $\lambda < 0$, and

$$\mathrm{Re}\, T_q(\mathcal{E}_q + \lambda) - \mathrm{Re}\, \tilde{\tau}_q(\mathcal{E}_q + \lambda) = \int_{s_0}^\infty \frac{F(s)^* F(s)}{s - \lambda} ds - \int_{s_0}^\infty \frac{f(s)^* f(s)}{s - \lambda} ds +$$

$$\int_{3\lambda/2}^{s_0} \frac{2\varphi(s) \mathrm{Re}\, f(s)^* r(s) + \varphi(s)^2 r(s)^* r(s)}{s - \lambda} ds + \int_0^{\lambda/2} \frac{2\varphi(s) \mathrm{Re}\, f(s)^* r(s) + \varphi(s)^2 r(s)^* r(s)}{s - \lambda} ds +$$

$$2 \int_0^{\lambda/2} (\varphi(\lambda + \nu) \mathrm{Re}\, f(\lambda + \nu)^* r(\lambda + \nu) - \varphi(\lambda - \nu) \mathrm{Re}\, f(\lambda - \nu)^* r(\lambda - \nu)) \frac{d\nu}{\nu} +$$

$$\int_0^{\lambda/2} (\varphi(\lambda + \nu)^2 r(\lambda + \nu)^* r(\lambda + \nu) - \varphi(\lambda - \nu)^2 r(\lambda - \nu)^* r(\lambda - \nu)) \frac{d\nu}{\nu}$$

if λ is positive and small enough (say, $\lambda \in (0, s_0/2)$). Using estimates (4.35) - (4.36) as well as (4.54) - (4.55), (4.60) - (4.61), and (4.63) - (4.64), we obtain

$$\|\mathrm{Re}\, T_q(\mathcal{E}_q + \lambda) - \mathrm{Re}\, \tilde{\tau}_q(\mathcal{E}_q + \lambda)\|_1 = O(1), \quad \lambda \to 0,$$

which combined with (4.5) for $p = 1$ yields (4.59), and hence (4.57) - (4.58). $\square$

Fix $j \in \mathbb{N}$. By analogy with (4.30) and (4.56) set

$$\tau_j(z) := \int_{\mathbb{R}} \frac{g(k)^* g(k)}{\mu_j k^2 - z} dk, \quad z \in \mathbb{C}_+.$$

As in the case of the operators $T_j(z)$ and $\tilde{\tau}(z)$, in S_1 there exists a limit

$$\tau_j(E) = \lim_{\delta \downarrow 0} \tau_j(E + i\delta), \quad E \in \mathbb{R} \setminus \{0\}.$$

Proposition 4.3. *Let V satisfy $\mathcal{D}_\alpha$ with $\alpha > 1$. Fix $q \in \mathbb{N}$. Then for each $\varepsilon \in (0, 1/2)$ and $\gamma \in (0, (\alpha - 1)/2)$, $\gamma \leq 1$, we have*

$$\text{ind}\,(J + 2\varepsilon + \text{Re}\tau_q(\lambda), J + 2\varepsilon) + O(\theta_{2\gamma}(\lambda)) \leq \text{ind}\,(J + \varepsilon + \text{Re}\tilde{\tau}_q(\mathcal{E}_q + \lambda), J + \varepsilon), \quad (4.65)$$

$$\text{ind}\,(J - 2\varepsilon + \text{Re}\tau_q(\lambda), J - 2\varepsilon) + O(\theta_{2\gamma}(\lambda)) \geq \text{ind}\,(J - \varepsilon + \text{Re}\tilde{\tau}_q(\mathcal{E}_q + \lambda), J - \varepsilon), \quad (4.66)$$

as $\lambda \to 0$, the functions θ_β being defined in (2.13) – (2.14).

Proof. Similarly to the proof of Proposition 4.2 (see (4.59)), it suffices to show that for each $\varepsilon > 0$ we have

$$n_*(\varepsilon; \text{Re}\,\tilde{\tau}_q(\mathcal{E}_q + \lambda) - \text{Re}\,\tau_q(\lambda)) = O(\theta_{2\gamma}(\lambda)), \quad \lambda \to 0. \quad (4.67)$$

Let at first $\lambda < 0$. In this case we have

$$\text{Re}\,\tilde{\tau}_q(\mathcal{E}_q + \lambda) - \text{Re}\,\tau_q(\lambda) = \tilde{\tau}_q(\mathcal{E}_q + \lambda) - \tau_q(\lambda) =$$

$$\int_{\mathbb{R}} g_q(k)^* g_q(k) \frac{\mu_q k^2 - E_q(k) + \mathcal{E}_q}{(E_q(k) - \mathcal{E}_q - \lambda)(\mu_q k^2 - \lambda)} dk$$

and

$$\|\tilde{\tau}(\mathcal{E}_q + \lambda) - \tau_q(\lambda)\|_1 \leq \frac{c_1}{\mu_q} \int_{\mathbb{R}} \frac{|E_q(k) - \mathcal{E}_q - \mu_q k^2|}{k^2(E_q(k) - \mathcal{E}_q)} dk = O(1), \quad \lambda \uparrow 0,$$

which combined with (4.5) for $p = 1$ yields (4.67) in the case $\lambda < 0$.
Let now $\lambda > 0$. As above, let $\varphi = \varphi_q$ be the function inverse to $E_q - \mathcal{E}_q$. Set

$$\phi(s) = \phi_q(s) := \mu_q^{-1/2} s^{1/2}, \quad s > 0.$$

By (4.19) - (4.20),

$$\varphi(s) - \phi(s) = O(s^{3/2}), \quad (4.68)$$

$$\varphi'(s) - \phi'(s) = O(s^{1/2}), \quad (4.69)$$

for $s \in (0, s_0)$ and $s_0 \in (0, \infty)$. Fix $s_0 \in (0, \infty)$ and assume $\lambda < s_0/2$. For $\eta = \varphi$ or $\eta = \phi$ define the operator-valued function $\Gamma_\eta : (0, \infty) \to S_2(L^2(\mathcal{S}_L), \mathbb{C}^2)$ by

$$\Gamma_\eta(s)u := (g(\eta(s))u, g(-\eta(s))u), \quad s > 0, \quad u \in L^2(\mathcal{S}_L),$$

and

$$M_{\eta,1}(\lambda) := \int_{s_0}^{\infty} \eta'(s) \frac{\Gamma_\eta(s)^* \Gamma_\eta(s)}{s-\lambda} ds,$$

$$M_{\eta,2}(\lambda) := \text{v.p.} \int_0^{s_0} \eta'(s) \frac{2\text{Re}\,\Gamma_\eta(s)^*\Gamma_\eta(\lambda) - \Gamma_\eta(\lambda)^*\Gamma_\eta(\lambda)}{s-\lambda} ds,$$

$$M_{\eta,3}(\lambda) := \int_0^{s_0} \eta'(s) \frac{(\Gamma_\eta(s) - \Gamma_\eta(\lambda))^*(\Gamma_\eta(s) - \Gamma_\eta(\lambda))}{s-\lambda} ds.$$

Then we have

$$\tilde{\tau}_q(\mathcal{E}_q + \lambda) = \sum_{l=1,2,3} M_{\varphi,l}(\lambda), \quad \tau_q(\lambda) = \sum_{l=1,2,3} M_{\phi,l}(\lambda).$$

It is easy to see that

$$\|M_{\eta,1}(\lambda)\|_1 = O(1), \quad \lambda \downarrow 0, \quad \eta = \varphi, \phi, \tag{4.70}$$

$$\text{rank}\, M_{\eta,2}(\lambda) \le 6, \quad \lambda > 0, \quad \eta = \varphi, \phi, \tag{4.71}$$

$$\|M_{\eta,3}(\lambda)\|_1 = O(\theta_\gamma(\lambda)), \quad \lambda \downarrow 0, \quad \eta = \varphi, \phi. \tag{4.72}$$

Let us show that

$$\|M_{\varphi,3}(\lambda) - M_{\phi,3}(\lambda)\|_1 = O(\theta_{2\gamma}(\lambda)), \quad \lambda \downarrow 0. \tag{4.73}$$

We have

$$M_{\varphi,3}(\lambda) - M_{\phi,3}(\lambda) =$$
$$\int_0^{s_0} (\varphi'(s) - \phi'(s)) \frac{(\Gamma_\varphi(s) - \Gamma_\varphi(\lambda))^*(\Gamma_\varphi(s) - \Gamma_\varphi(\lambda))}{s-\lambda} ds +$$
$$\int_0^{s_0} \phi'(s) \frac{(\Gamma_\varphi(s) - \Gamma_\phi(s) - \Gamma_\varphi(\lambda) + \Gamma_\phi(\lambda))^*(\Gamma_\varphi(s) - \Gamma_\varphi(\lambda))}{s-\lambda} ds +$$
$$\int_0^{s_0} \phi'(s) \frac{(\Gamma_\phi(s) - \Gamma_\phi(\lambda))^*(\Gamma_\varphi(s) - \Gamma_\phi(s) - \Gamma_\varphi(\lambda) + \Gamma_\phi(\lambda))}{s-\lambda} ds :=$$
$$I_1 + I_2 + I_3.$$

Using (4.69), (4.55), and (4.18) which implies $|\varphi(s) - \varphi(\lambda)| = O(|\sqrt{s} - \sqrt{\lambda}|)$, $s \in (0, s_0)$, we get

$$\|I_1\|_1 = O\left(\int_0^{s_0} s^{1/2} \frac{|\sqrt{s} - \sqrt{\lambda}|^{2\gamma}}{|s-\lambda|} ds\right) = O(1), \quad \lambda \downarrow 0. \tag{4.74}$$

Further, for $s, \lambda > 0$, and $\gamma \in (0, (\alpha-1)/2)$, $\gamma \le 1$, we have

$$\|\Gamma_\varphi(s) - \Gamma_\phi(s) - \Gamma_\varphi(\lambda) + \Gamma_\phi(\lambda)\|_2^2 \le$$

$$\frac{1}{\pi} \sup_{(x,y) \in \mathcal{S}_L} \langle y \rangle^\alpha |V(x,y)| \int_{\mathbb{R}} |e^{i\varphi(s)y} - e^{i\phi(s)y} - e^{i\varphi(\lambda)y} + e^{i\phi(\lambda)y}|^2 \langle y \rangle^{-\alpha} dy \le$$

$$\frac{2^{3-2\gamma}}{\pi}\sup_{(x,y)\in\mathcal{S}_L}\langle y\rangle^{\alpha}|V(x,y)|\int_{\mathbb{R}}|y|^{2\gamma}\langle y\rangle^{-\alpha}dy\left(|\varphi(s)-\phi(s)|^{2\gamma}+|\varphi(\lambda)-\phi(\lambda)|^{2\gamma}\right).$$

Using (4.68), we get

$$\|I_j\|_1 = O\left(\int_0^{s_0} s^{-1/2}\frac{(s^{3\gamma}+\lambda^{3\gamma})^{1/2}|\sqrt{s}-\sqrt{\lambda}|^{\gamma}}{|s-\lambda|}ds\right) = O(\theta_{2\gamma}(\lambda)), \quad \lambda\downarrow 0, \quad j=2,3. \tag{4.75}$$

Putting together (4.74) and (4.75), we obtain (4.73). Now the combination of (4.70) – (4.73) with (4.4) and (4.5) for $p=1$ yields (4.67) in the case $\lambda>0$. □

Next, we note that for each $\lambda>0$ and $q\in\mathbb{N}$ we have $\operatorname{rank}\operatorname{Im}\tau_q(\lambda)\leq 2$, while $\operatorname{Im}\tau_q(\lambda)=0$ if $\lambda<0$. Therefore,

$$\operatorname{ind}(J-\varepsilon+\operatorname{Re}\tau_q(\lambda), J-\varepsilon) = \int_{\mathbb{R}}\operatorname{ind}(J-\varepsilon+\operatorname{Re}\tau_q(\lambda)+t\operatorname{Im}\tau_q(\lambda), J-\varepsilon)d\mu(t)+O(1), \quad \lambda\to 0, \tag{4.76}$$

for each $\varepsilon\in(-1,1)$. On the other hand, we have

$$w_{q,\varepsilon} = \varkappa^*(J-\varepsilon)^{-1}\varkappa, \quad \varepsilon\in(-1,1),$$

$$\tau_q(z) = \varkappa(h_{0,q}-z)^{-1}\varkappa^*, \quad z\in\overline{\mathbb{C}_+}\setminus\{0\},$$

where $\varkappa: L^2(\mathbb{R})\to L^2(\mathcal{S}_L)$ is the operator defined by

$$(\varkappa u)(x,y) := \psi(x,0)|V(x,y)|^{1/2}u(y), \quad u\in L^2(\mathbb{R}).$$

By Theorem 4.1 we have

$$\int_{\mathbb{R}}\operatorname{ind}(J-\varepsilon+\operatorname{Re}\tau_q(\lambda)+t\operatorname{Im}\tau_q(\lambda), J-\varepsilon)d\mu(t) = \xi(\lambda; h_q(\varepsilon), h_{0,q}), \quad \lambda\neq 0. \tag{4.77}$$

Combining (4.50), (4.57) – (4.58), (4.65) – (4.66), (4.76), and (4.77), we obtain (2.15).

4.8. In this subsection we give a sketch of the proof of Corollary 2.1. Let $w=\bar{w}\in L^\infty(\mathbb{R})$. Set

$$h_0 := -\frac{d^2}{dy^2}, \quad D(h_0)=\mathrm{H}^2(\mathbb{R}), \quad h := h_0+w, \quad D(h)=D(h_0).$$

Assume that for some $\alpha>0$ there exist real numbers $\omega_\pm$ such that

$$\lim_{y\to\pm\infty}|y|^{\alpha}w(y) = \omega_\pm. \tag{4.78}$$

Set $\omega_\pm^{(-)} := \max\{0,-\omega_\pm\}$, $\omega_\pm^{(+)} := \max\{0,\omega_\pm\}$.

Lemma 4.7. [24, Theorem XIII.82] *Assume that* (4.78) *holds with* $\alpha \in (0,2)$. *Then we have*

$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{2}-\frac{1}{\alpha}} N(-\lambda; h) = \mathcal{C}_\alpha \left(\left(\omega_-^{(-)}\right)^{1/\alpha} + \left(\omega_+^{(-)}\right)^{1/\alpha} \right). \tag{4.79}$$

Remark: Under the hypotheses of Lemma 4.7, relation (4.79) is equivalent to the standard semiclassical formula

$$N(-\lambda; h) = (2\pi)^{-1} \left| \left\{ (y,\eta) \in T^*\mathbb{R} \,|\, \eta^2 + w(y) < -\lambda \right\} \right| (1 + o(1)), \quad \lambda \downarrow 0,$$

where $|\cdot|$ denotes the Lebesgue measure, provided that $\omega_-^{(-)} + \omega_+^{(-)} > 0$.
Recall now that $\xi(-\lambda; h_q(\varepsilon), h_{0,q}) = -N(-\lambda; h_q(\varepsilon))$, $\lambda > 0$. Since the operator $h_q(\varepsilon)$ is unitarily equivalent to the operator $h = h_0 + w$ with $w(y) = w_{q,\varepsilon}(\mu_q^{1/2} y)$, and the quantities $\omega_{q,\pm}(\varepsilon)$ are continuous at $\varepsilon = 0$, we find that (2.15) and (4.79) imply (2.17).

Lemma 4.8. *Assume that* (4.78) *holds with* $\alpha \in (1,2)$. *Then we have*

$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{2}-\frac{1}{\alpha}} \xi(\lambda; h, h_0) =$$

$$-\mathcal{C}_\alpha \left(\csc(\pi/\alpha) \left(\left(\omega_-^{(-)}\right)^{1/\alpha} + \left(\omega_+^{(-)}\right)^{1/\alpha} \right) + \cot(\pi/\alpha) \left(\left(\omega_-^{(+)}\right)^{1/\alpha} + \left(\omega_+^{(+)}\right)^{1/\alpha} \right) \right). \tag{4.80}$$

Proof. Set

$$h_0^{(+)} := -\frac{d^2}{dy^2}, \quad D(h_0) = \left\{ u \in \mathrm{H}^2(0,\infty) \,|\, u(0) = 0 \right\},$$

$$h^{(+)} := h_0^{(+)} + w_{|(0,\infty)}, \quad D(h^{(+)}) = D(h_0^{(+)}).$$

By the Birman-Krein formula (2.12), and [26, Section 7, Corollary],

$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{2}-\frac{1}{\alpha}} \xi(\lambda; h^{(+)}, h_0^{(+)}) = -\mathcal{C}_\alpha \left(\csc(\pi/\alpha) \left(\omega_+^{(-)}\right)^{1/\alpha} + \cot(\pi/\alpha) \left(\omega_+^{(+)}\right)^{1/\alpha} \right). \tag{4.81}$$

Now put

$$h_0^{(-)} := -\frac{d^2}{dy^2}, \quad D(h_0) = \left\{ u \in \mathrm{H}^2(-\infty,0) \,|\, u(0) = 0 \right\},$$

$$h^{(-)} := h_0^{(-)} + w_{|(-\infty,0)}, \quad D(h^{(-)}) = D(h_0^{(-)}).$$

Since the operator $h_0^{(-)}$ is unitarily equivalent to $h_0^{(+)}$, and the operator $h^{(-)}$ is unitarily equivalent to $h_0^{(+)} + \tilde{w}$ with $\tilde{w}(y) = w(-y)$, $y > 0$, we find that (4.81) entails

$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{2}-\frac{1}{\alpha}} \xi(\lambda; h^{(-)}, h_0^{(-)}) = -\mathcal{C}_\alpha \left(\csc(\pi/\alpha) \left(\omega_-^{(-)}\right)^{1/\alpha} + \cot(\pi/\alpha) \left(\omega_-^{(+)}\right)^{1/\alpha} \right). \tag{4.82}$$

Making use of the orthogonal decomposition $L^2(\mathbb{R}) = L^2(-\infty, 0) \oplus L^2(0, \infty)$, introduce the operators $h_0^{(-)} \oplus h_0^{(+)}$ and $h^{(-)} \oplus h^{(+)}$, self-adjoint in $L^2(\mathbb{R})$. Evidently,

$$\xi(\lambda; h^{(-)} \oplus h^{(+)}, h_0^{(-)} \oplus h_0^{(+)}) = \xi(\lambda; h^{(-)}, h_0^{(-)}) + \xi(\lambda; h^{(+)}, h_0^{(+)}). \tag{4.83}$$

On the other hand, since the ranks of the resolvent differences

$$(h_0 - E_0)^{-1} - (h_0^{(-)} \oplus h_0^{(+)} - E_0)^{-1}, \quad (h - E_0)^{-1} - (h^{(-)} \oplus h^{(+)} - E_0)^{-1},$$

with $E_0 < \inf \sigma(h)$ are finite, we have

$$\xi(\lambda; h, h_0) = \xi(\lambda; h^{(-)} \oplus h^{(+)}, h_0^{(-)} \oplus h_0^{(+)}) + O(1), \quad \lambda > 0. \tag{4.84}$$

Now (4.80) follows from the combination of (4.84), (4.83), and (4.81) – (4.82). $\square$

The combination of (2.15) and (4.80) easily yields (2.18).

4.9. Finally, we assume that $\alpha > 2$ and prove Corollary 2.3. If $\lambda < 0$, then (2.20) is an immediate consequence of Theorem 2.2 and the well-known fact that the 1D Schrödinger operator $-\frac{d^2}{dy^2} + w(y)$, $y \in \mathbb{R}$, has a most a finite number of negative eigenvalues if $w(y) = o(|y|^{-2})$ as $|y| \to \infty$ (see e.g. [24]). Assume $\lambda > 0$. Combining (4.8), (4.7), (4.4), and (4.5) with $p = 1$, we obtain

$$|\mathrm{ind}\,(J - \varepsilon + \mathrm{Re}\,\tau_q(\lambda), J - \varepsilon)| \leq n_*(1 - |\varepsilon|; \mathrm{Re}\,\tau_q(\lambda)) \leq$$

$$(1 - |\varepsilon|)^{-1} \|M_{\phi,1}(\lambda)\|_1 + \mathrm{rank}\, M_{\phi,2}(\lambda) + (1 - |\varepsilon|)^{-1} \|M_{\phi,3}(\lambda)\|_1, \quad \varepsilon \in (-1, 1). \tag{4.85}$$

Pick $\gamma < (\alpha - 1)/2$, $\gamma \leq 1$, $\gamma > 1/2$. Using (4.70) – (4.72), we find that the r.h.s. of (4.85) remains bounded as $\lambda \downarrow 0$.
Putting together (4.50), (4.57) – (4.58), (4.65) – (4.66), and (4.85), we obtain (2.20) in the case $\lambda > 0$.

Acknowledgements. The authors thank the referee whose valuable remarks contributed to the improvement of the article. Georgi Raikov gratefully acknowledges the financial support by the Chilean Scientific Foundation *Fondecyt* under Grant 1050716.

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