DETERMINING AN IWATSUKA HAMILTONIAN THROUGH QUANTUM VELOCITY MEASUREMENT

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ABSTRACT. We investigate the inverse problem of retrieving the magnetic potential of an Iwatsuka Hamiltonian by knowledge of the second component of the quantum velocity. We show that knowledge of the quantum currents carried by a suitable set of states with energy concentration within the first spectral band of the Schrödinger operator, uniquely determines the magnetic field.

1. INTRODUCTION: IWATSUKA HAMILTONIANS, EDGE CURRENTS AND MAGNETIC INVERSE PROBLEMS

Quantum Hall Hamiltonians describe the motion of a charged particle constrained to a bounded or unbounded subdomain of the plane, subject to a constant transverse magnetic field with strength $b \geq 0$. Confined quantum Hall systems, such as motion in a half-plane or a strip are particularly interesting as a current flowing along an edge is created. Confinement may be obtained by Dirichlet boundary conditions (hard edge) or an electrostatic potential barrier (soft edge), but in any case the edges of the confinement induce edge currents. These edge currents are carried by states with energy localized between any two Landau levels $(2n-1)b$, $n = 1, 2, \ldots$, see e.g. [7, 12, 14, 15].

In the present article we are interested in edge currents created by purely magnetic barriers. Namely, we consider a two-dimensional Schrödinger operator with a non-constant magnetic field $b(x, y) = b(x)$, $(x, y) \in \mathbb{R}^2$, depending only on x. When the real-valued function b is bounded and has different limits as x goes to $\pm\infty$, it was shown in [18] by Iwatsuka that the spectrum is absolutely continuous. Later on, the transport properties of these so-called Iwatsuka Hamiltonians were investigated in [24] by physicists Reijniers and Peeters. When $b(x)$ assumes constant value $b_\pm$ for $\pm x > 0$, $0 < b_- < b_+ < \infty$, they argued that this discontinuity in the magnetic field at $x = 0$ creates an effective edge and that currents flow along the edge. This is the magnetic analog of the barriers created by Dirichlet boundary conditions along $x = 0$ or a confining electrostatic potential filling the half-space $x < 0$ described in the paragraph above.

Besides, when $0 < b_- \leq b(x) \leq b_+$ for $x \in (-\epsilon, \epsilon)$, where $\epsilon \in [0, b_-^{-1/2})$, and $b(x) = b_\pm$ for $\pm x > \epsilon$, the existence of currents flowing in the y-direction was rigorously established in [16]. These currents are carried by states with energy concentration in the energy bands of the Iwatsuka Hamiltonian, and they are well-localized in x to a region of size $b_-^{-1/2}$, centered at $x = 0$. The study of the case of a jump in the magnetic field at $x = 0$ corresponding to $\epsilon = 0$ in [16], was extended to $b_+ = -b_- = b > 0$ in [10]. It shows that the magnetic field creates an effective barrier near $x = 0$ that causes edge currents to flow along it consistent with the heuristic approach conducted in [24]. Moreover, partially motivated by [24], Dombrowski, Germinet and Raikov studied the edge conductance for generalized Iwatsuka models in [9].

In this article we examine the inverse problem of determining the vector potential $A(x, y) = (0, a(x))$, where $a(x) = \int_0^x b(s)\mathrm{d}s$, $x \in \mathbb{R}$, of the Iwatsuka Hamiltonian defined on the dense domain $C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ by $(-i\nabla - A)^2$, from knowledge of the edge currents carried by quantum states with energy localized in the first band (b_-, b_+). More precisely, these edge currents are generated upon triggering the dynamic quantum system governed by the Iwatsuka Hamiltonian, by a suitable set of initial states with energy concentration within (b_-, b_+).

Inverse coefficient problems for the magnetic Schrödinger operator have attracted a great deal of attention over the last years. For instance, in [8], using the Bukhgeim-Klibanov method, see [4], the time-independent divergence-free magnetic potential was Lipschitz stably retrieved from a finite number of partial boundary observations (over the entire course of time) of the solution. The authors proceed by suitably changing the initial state of the corresponding dynamic Schrödinger equation and then measuring the solution on a sub-boundary fulfilling the geometric optics condition for the observability derived by Bardos, Lebeau and Rauch in [1]. The case of non-zero divergence magnetic vectors was treated with the same approach in [17].

In [11, 25] the magnetic field of the Schrödinger operator was identified by the Dirichlet-to-Neumann map. These results are based on a different approach using geometric optics solutions. The stability issue in the same problem was treated in [3] but for dynamic magnetic Schrödinger equation in a bounded domain, and in [2] for the same equation on a Riemannian manifold. As for the determination of the magnetic vector potential of the Schrödinger operator by spectral data, it was established in [20]. All the above mentioned inverse results were derived for magnetic Schrödinger systems in a bounded domain, and we refer the reader to [5] for the study of the inverse problem of retrieving the magnetic field of the Schrödinger equation in an unbounded cylindrical domain.

It is worth noticing that the analysis of the inverse problem under study in this manuscript is fundamentally different from the ones used for solving the magnetic inverse problems of [2, 3, 4, 5, 8, 11, 20, 25]. This is mostly due to the fact that our data are not naturally related to the Neumann boundary data or the spectral data used by [2, 3, 4, 5, 8, 11, 20, 25]. Nevertheless, due to its translational invariance in the y direction, the Iwatsuka Hamiltonian admits a fiber decomposition. The fibers are Sturm-Liouville operators with a zero-th order perturbation expressed in terms of the unknown function a. But despite of this, and since the fibers are defined on the real line here, it is still unclear whether the analysis conducted in [22, Chapter 2, Section 3] on the half-line could be adapted to the inverse problem under investigation in this manuscript. for inverse spectral problems for one-dimensional Schrödinger operators defined either on the real-line or on the half-line, we refer the reader to [13] and the references therein, where the continuous and bounded from below real-valued electric potential was identified by the Krein spectral shift function.

The remaining part of the text is structured as follows. Section 2 contains the main definitions and results of this article. In Section 3 we rigorously define the data used for solving the inverse problem under examination in this article, and we briefly comment on them. Finally, in Section 4, we give the proof of the main results stated below in Theorems 2.1 and 2.2, and in Corollary 2.3.

2. Definitions and results

2.1. **Iwatsuka Hamiltonians.** Let $b \in L^\infty(\mathbb{R})$ be a non-decreasing function satisfying

$$\lim_{x\to\pm\infty} b(x) = b_\pm, \tag{2.1}$$

where $b_\pm$ are two positive real numbers such that

$$0 < b_- < b_+ < 3b_-. \tag{2.2}$$

Notice for further use that (2.1)-(2.2) yields

$$b_- \leq b(x) \leq b_+,\ x \in \mathbb{R}. \tag{2.3}$$

Next, we put

$$a(x) := \int_0^x b(s)\mathrm{d}s,\ x \in \mathbb{R}, \tag{2.4}$$

and we introduce the magnetic potential $A(x) = (A_1(x), A_2(x))$ with $A_1(x) = 0$ and $A_2(x) = a(x)$. In the present article we consider the two-dimensional magnetic Hamiltonian $(-i\nabla - A)^2$, defined on $C_0^\infty(\mathbb{R}^2)$ by

$$H := -\partial_x^2 + (-i\partial_y - a)^2. \tag{2.5}$$

Since $A \in L^4_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $b = \partial_x A_2 - \partial_y A_1 \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$, the operator H is essentially self-adjoint on the dense domain $C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ by [21, Theorems 1 and 2], and we still denote by H its unique self-adjoint extension in $L^2(\mathbb{R}^2)$.

2.2. **Reduction to one-dimensional operators.** The Schrödinger operator defined in (2.5) being invariant with respect to translations in the y-direction, it decomposes into a family of parameterized Hamiltonians on $L^2(\mathbb{R})$. Let $\mathcal{F}$ denote the partial Fourier transform with respect to y, i.e.

$$(\mathcal{F}u)(x,\xi) = \hat{u}(x,\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\xi y} u(x,y) \mathrm{d}y,\ u \in L^2(\mathbb{R}^2),\ (x,\xi) \in \mathbb{R}^2.$$

The Hilbert space $L^2(\mathbb{R}^2)$ can be expressed as a constant fiber direct integral over $\mathbb{R}$ with fibers $L^2(\mathbb{R})$, i.e. $L^2(\mathbb{R}^2) = \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) \mathrm{d}\xi$, and the operator H admits a partial Fourier decomposition with respect to the y-variable, with

$$\mathcal{F} H \mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} h(\xi) \mathrm{d}\xi,$$

where each $h(\xi)$, $\xi \in \mathbb{R}$, is self-adjoint in $L^2(\mathbb{R})$ with domain $D(h(\xi))$ which is independent of ξ, i.e. $D(h(\xi)) = D(h(0))$, according to [18, Lemma 2.3]. Moreover, we have $h(\xi) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x,\xi)$ on $C_0^\infty(\mathbb{R})$, where $q(x,\xi) := v(x,\xi)^2/4$ and $v(x,\xi) := 2(\xi - a(x))$ denotes the quantum velocity at frequency ξ.

In light of (2.1), the potential $q(\cdot,\xi)$, $\xi \in \mathbb{R}$, is unbounded as $|x|$ goes to infinity, hence $h(\xi)$ has a compact resolvent. Let $\{\lambda_j(\xi),\ j \in \mathbb{N}\}$, be the non decreasing sequence of the eigenvalues of the operator $h(\xi)$, $\xi \in \mathbb{R}$. Since all the eigenvalues $\lambda_j(\xi)$ are simple (see [14, Proposition A2] or [18, Lemma 2.3]), we have for all $j \geq 3$,

$$\lambda_1(\xi) < \lambda_2(\xi) < \ldots < \lambda_j(\xi) < \lambda_{j+1}(\xi) < \ldots$$

and the functions $\xi \mapsto \lambda_j(\xi)$, $j \geq 1$, are real analytic by the Kato perturbation theory, see [19, Chap. VII]. Moreover we have

$$(2j-1)b_- \leq \lambda_j(\xi) \leq (2j-1)b_+,\ \xi \in \mathbb{R},\ j \in \mathbb{N}, \tag{2.6}$$

and

$$\lim_{\xi \to \pm\infty} \lambda_j(\xi) = (2j-1)b_\pm,\ j \in \mathbb{N}, \tag{2.7}$$

from [9, Proposition 3.1]. As a consequence the spectrum of H is purely absolutely continuous (see, e.g. [23, Theorem XIII.86]) and

$$\sigma(H) = \bigcup_{j=1}^{\infty} [(2j-1)b_-, (2j-1)b_+].$$

Therefore, $\sigma(H)$ has a band structure and it follows from (2.2) that the first band $[b_-, b_+]$ does not overlap with the remaining part of the spectrum $\cup_{j=2}^{\infty}[(2j-1)b_-, (2j-1)b_+]$.

2.3. **Quantum velocity.** In light of [18, Lemma 2.3], there exists a $L^2(\mathbb{R})$-orthonormal basis $\{\varphi_j(\cdot,\xi),\ j \in \mathbb{N}\}$ is of eigenfunctions of $h(\xi)$, $\xi \in \mathbb{R}$, such that

$$h(\xi)\varphi_j(\cdot,\xi) = \lambda_j(\xi)\varphi_j(\cdot,\xi),\ j \in \mathbb{N}.$$

Moreover, $\varphi_j(\cdot,\xi) \in D(h(0)) = \{u \in H^1(\mathbb{R}),\ -u'' + a^2 u \in L^2(\mathbb{R})\}$, $j \in \mathbb{N}$, depends analytically on $\xi \in \mathbb{R}$ with respect to the graph norm of $h(0)$, defined by

$$\|u\|_{D(h(0))} := \left(\|u\|_2^2 + \|h(0)u\|_2^2 \right)^{1/2},\ u \in D(h(0)),$$

where $\|\cdot\|_2$ denotes the usual norm in $L^2(\mathbb{R})$. In what follows, all the eigenfunctions $\varphi_j(\cdot,\xi)$, $j \in \mathbb{R}$, are chosen to be real-valued, and since $\varphi_1(\cdot,\xi)$ is non-degenerate, we will always assume that

$$\varphi_1(x,\xi) > 0,\ x \in \mathbb{R}.$$

This being said, we introduce the *current operator* ϑ as

$$\vartheta(\chi) := \int_{\mathbb{R}} \chi(\xi)^2 \langle v(\cdot,\xi)\varphi_1(\cdot,\xi), \varphi_1(\cdot,\xi)\rangle_2 \mathrm{d}\xi,\ \chi \in C_0^\infty(\mathbb{R}) := C_0^\infty(\mathbb{R},\mathbb{R}), \tag{2.8}$$

where $\langle\cdot,\cdot\rangle_2$ is the usual scalar product in $L^2(\mathbb{R})$. We shall see in Section 3.2, that $\vartheta(\chi)$ is the expectation of the second component of the *velocity operator* $2(i\partial_y + a)$ expressed in the quantum state $e^{-itH}u_{0,\chi}$, $t \in [0,+\infty)$, where

$$u_{0,\chi} := \mathcal{F}^* \psi_\chi,\ \psi_\chi(x,\xi) := \chi(\xi)\varphi_1(x,\xi),\ (x,\xi) \in \mathbb{R}^2, \tag{2.9}$$

that is to say the quantum current current carried by $e^{-itH}u_{0,\chi}$. Since it is time-independent according to (2.8), we rather call it *quantum current carried by* $u_{0,\chi}$ in the sequel.

In the present article we investigate the inverse problem to know whether knowledge of the transport properties of the Iwatsuka Hamiltonian H, expressed through its current operator ϑ, uniquely determine the magnetic potential a of H.

2.4. **Main results.** We denote by $\mathbb{B}$ the (convex) set of Iwatsuka magnetic fields, i.e. the set of non-decreasing functions $b \in L^\infty(\mathbb{R})$ satisfying the condition (2.1). Given b and $\tilde{b}$ in $\mathbb{B}$, we introduce

$$b_\varepsilon := (1-\varepsilon)b + \varepsilon\tilde{b},\ \varepsilon \in (0,1).$$

Evidently, $b_\varepsilon \in \mathbb{B}$ for all $\varepsilon \in (0,1)$ and we denote by ϑ_ε the current operator associated with b_ε. Otherwise stated, $\vartheta_\varepsilon(\chi)$, $\varepsilon \in (0,1)$ and $\chi \in C_0^\infty(\mathbb{R})$, is the quantum current carried by the state $u_{0,\varepsilon,\chi}$, characterized by

$$\hat{u}_{0,\varepsilon,\chi}(\cdot,\xi) = \chi(\xi)\varphi_{\varepsilon,1}(\cdot,\xi),\ \xi \in \mathbb{R}.$$

Here, $\{\varphi_{\varepsilon,j}(\cdot,\xi),\ j \in \mathbb{N}\}$ is a $L^2(\mathbb{R})$-orthonormal basis of eigenfunctions of the operator

$$h_\varepsilon(\xi) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_\varepsilon(\cdot,\xi),\ \xi \in \mathbb{R}, \tag{2.10}$$

where $q_\varepsilon(\cdot,\xi) := v_\varepsilon(\cdot,\xi)^2/4$, $v_\varepsilon(x,\xi) := 2(\xi - a_\varepsilon(x))$ and $a_\varepsilon(x) := \int_0^x b_\varepsilon(s)\mathrm{d}s$.

Our first identification result is as follows.

Theorem 2.1. *Let $b \in \mathbb{B}$ and $\tilde{b} \in \mathbb{B}$ be such that*

$$\exists r \in (0,r_0),\ \mathrm{supp}\,(\tilde{a}-a) \subset [-r,r],\ r_0 := \frac{(3b_- - b_+)^{1/2}}{2b_+}, \tag{2.11}$$

where $\tilde{a}(x) := \int_0^x \tilde{b}(s)\mathrm{d}s$ for all $x \in \mathbb{R}$.
Then, if 0 is an accumulation point of $\{\epsilon \in (0,1], \vartheta_\epsilon = \vartheta\}$, we have $\tilde{a} = a$.

In contrast to Theorem 2.1 where an infinite number of current operators ϑ_ϵ, $\epsilon \in (0,1]$, are supposed to be known, the following result assumes knowledge of one current operator only, but additional spectral data is needed for the identification of the Iwatsuka Hamiltonian. More precisely, Theorem 2.2, below, aims to recover the unknown magnetic field $\tilde{b} \in \mathbb{B}$ from combined knowledge of 1) the quantum current $\tilde{\vartheta}(\chi)$ carried by the state $\tilde{u}_{0,\chi}(x,y) = \mathcal{F}^*(\chi\tilde{\varphi}_1(x,\cdot))(y)$, $(x,y) \in \mathbb{R}^2$, for all $\chi \in C_0^\infty(\mathbb{R})$, where $\{\tilde{\varphi}_j(\cdot,\xi),\ j \in \mathbb{N}\}$ is a $L^2(\mathbb{R})$-orthonormal basis of eigenfunctions of the operator

$$\tilde{h}(\xi) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\xi - \tilde{a}(x))^2,\ \xi \in \mathbb{R},$$

and, 2) $\tilde{\varphi}_1(\cdot,\pm\xi_0)$ for some arbitrarily fixed $\xi_0 \in (0,\infty)$.

Theorem 2.2. *Let $b \in \mathbb{B}$ and $\tilde{b} \in \mathbb{B}$ be such that* $\mathrm{supp}\,(\tilde{a}-a)$ *is compact. Assume that*

$$\forall \chi \in C_0^\infty(\mathbb{R}),\ \vartheta(\chi) = \tilde{\vartheta}(\chi), \tag{2.12}$$

and that

$$\exists \xi_0 \in (0,\infty), \forall x \in \mathbb{R},\ \varphi_1(x,\pm\xi_0) = \tilde{\varphi}_1(x,\pm\xi_0). \tag{2.13}$$

Then, we have $\tilde{a} = a$.

The following statement is a byproduct of Theorem 2.2, obtained upon replacing (2.13) by a more natural condition that can be checked through direct observation of the quantum state of the system governed by $\tilde{H} := -\partial_x^2 + (-i\partial_y - \tilde{a}(x))^2$.

Corollary 2.3. *Let* b *and* $\tilde{b}$ *be as in Theorem 2.2. Assume* (2.12) *and suppose that*

$$\forall \chi \in C_0^\infty(\mathbb{R}),\ \forall x \in \mathbb{R},\ e^{-it_0 H} u_{0,\chi}(x,y_0) = e^{-it_0\tilde{H}} \tilde{u}_{0,\chi}(x,y_0), \tag{2.14}$$

for some $(y_0,t_0) \in \mathbb{R} \times (0,T)$. *Then, we have* $\tilde{a} = a$.

Notice that since $\tilde{\vartheta}(\chi)$ is the net quantum current (carried by $\tilde{u}_{0,\chi}$) flowing across the horizontal line $y = y_0$, it can be measured at the exact same place (and at the same time t_0) where the quantum state $e^{-it_0\tilde{H}}\tilde{u}_{0,\chi}(\cdot,y_0)$ appearing in Corollary 2.3 is observed.

3. Preliminaries: definition of the current operator ϑ

In this section we study the transport properties of quantum devices described by the system

$$\begin{cases} (-i\partial_t + H)u(x,y,t) = 0, & (x,y,t) \in \mathbb{R}^2 \times (0,\infty) \\ u(x,y,0) = u_0(x,y), & (x,y) \in \mathbb{R}^2, \end{cases} \tag{3.1}$$

where H is the self-adjoint realization introduced in Section 2.1 of the Iwatsuka Hamiltonian defined by (2.5) on $C_0^\infty(\mathbb{R}^2)$, and u_0 is taken in $D(H)$, the domain of H. More precisely, we aim to relate the current operator ϑ defined in (2.9) to the second component of the quantum velocity operator associated with H.

3.1. The forward problem: Energy concentration and fast decaying property.

For all $u \in L^2(\mathbb{R}^2)$, we have

$$\hat{u}(x,\xi) = \sum_{j=1}^{\infty} u_j(\xi)\varphi_j(x,\xi),\ (x,\xi) \in \mathbb{R}^2,$$

where $u_j(\xi) := \langle \hat{u}(\cdot,\xi), \varphi_j(\cdot,\xi)\rangle_2$, since $\{\varphi_j(\cdot,\xi),\ j \in \mathbb{N}\}$, $\xi \in \mathbb{R}$, is an orthonormal basis of $L^2(\mathbb{R})$. Thus, for all $\lambda \in (0,\infty)$ and all $u \in D(H)$, it holds true that

$$\|(\lambda + iH)u\|^2_{L^2(\mathbb{R}^2)} = \sum_{j=1}^{\infty} \int_{\mathbb{R}} |\lambda + i\lambda_j(\xi)|^2 |u_j(\xi)|^2 \,\mathrm{d}\xi$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}} (\lambda^2 + \lambda_j(\xi)^2) \, |u_j(\xi)|^2 \, \mathrm{d}\xi \\
&\geq \lambda^2 \|u\|_{L^2(\mathbb{R}^2)}^2,
\end{aligned}
$$

and hence the operator $-iH$ is dissipative in $L^2(\mathbb{R}^2)$. Further, $\lambda + iH = i(H - i\lambda)$ being surjective since the spectrum of H is embedded in $[b_-, \infty)$, the operator $-iH$ is maximally dissipative in $L^2(\mathbb{R}^2)$. Therefore, (3.1) admits a unique solution $u \in C^0([0,\infty), D(H)) \cap C^1([0,\infty), L^2(\mathbb{R}^2))$, which is expressed as

$$
u(x,y,t) = e^{-itH} u_0(x,y), \ (x,y) \in \mathbb{R}^2, \ t \in [0,\infty), \tag{3.2}
$$

from [6, Lemma 2.1].

For $\chi \in C_0^\infty(\mathbb{R})$, let $u_{0,\chi}$ be the same as in (2.9). Then we have $\hat{u}_{0,\chi} \in D(h(\xi))$ for all $\xi \in \mathbb{R}$, and

$$
\int_{\mathbb{R}} \left(\|\hat{u}_{0,\chi}(\cdot,\xi)\|_2^2 + \|h(\xi)\hat{u}_{0,\chi}(\cdot,\xi)\|_2^2 \right) \mathrm{d}\xi = \int_{\mathbb{R}} \left(1 + \lambda_1(\xi)^2\right) \chi(\xi)^2 \mathrm{d}\xi < \infty,
$$

whence $u_{0,\chi} \in D(H)$ by [23, Section XIII.16]. Therefore, it follows from (3.2) that

$$
u_\chi(x,y,t) := e^{-itH} u_{0,\chi}(x,y), \ (x,y) \in \mathbb{R}^2, \ t \in [0,\infty), \tag{3.3}
$$

is well-defined.

For further reference we shall establish that, 1) the quantum state $u_\chi(\cdot,\cdot,t)$, $t \in [0,\infty)$, has energy concentration in the first spectral band (b_-, b_+), of H, and 2) $\partial_t^k u(\cdot,\cdot,t)$, $k = 0,1$, together with its partial derivatives with respect to y, decay faster than any polynomials in the y-direction. For this purpose we introduce the Schwartz space $\mathcal{S}_y(\mathbb{R}, L_x^2(\mathbb{R}))$ of smooth functions $y \mapsto f(\cdot,y)$ from $\mathbb{R}$ into $L^2(\mathbb{R})$, whose derivatives are rapidly decreasing, as:

$$
\mathcal{S}_y(\mathbb{R}, L_x^2(\mathbb{R})) := \{ f \in C_y^\infty(\mathbb{R}, L_x^2(\mathbb{R})), \ \forall (m,n) \in \mathbb{N}_0^2, \ \sup_{y \in \mathbb{R}} |y|^m \, \|\partial_y^n f(\cdot,y)\|_2 < \infty \}.
$$

Here and below, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where, as usual, $\mathbb{N} := \{1,2,3,\ldots\}$ is the set of positive integers. Next, we recall that the spectral projection of H associated with a Borel set $I \subset \mathbb{R}$, reads

$$
\mathbb{P}_I w(x,y) = \mathcal{F}^* \left(\sum_{j=1}^{\infty} \mathbb{1}_{\lambda_j^{-1}(I)} w_j \varphi_j(x,\cdot) \right)(y), \ w \in L^2(\mathbb{R}^2), \tag{3.4}
$$

where $\mathbb{1}_I$ is the characteristic function of I and $w_j(\xi) := \langle \hat{w}(\cdot,\xi), \varphi_j(\cdot,\xi) \rangle_2$. Then, the expected result is as follows.

Lemma 3.1. *Let* u_χ, $\chi \in C_0^\infty(\mathbb{R})$, *be defined by* (3.3), *where* $u_{0,\chi}$ *is as in* (2.9). *Then, we have*

$$u_\chi \in C^1([0,\infty), \mathcal{S}_y(\mathbb{R}, L_x^2(\mathbb{R}))) \tag{3.5}$$

and

$$\mathbb{P}_{(b_-,b_+)} u_\chi(\cdot,\cdot,t) = u_\chi(\cdot,\cdot,t),\ t \in [0,\infty). \tag{3.6}$$

Proof. We start by proving (3.5). To this end, we infer from (2.9) and (3.3) that

$$\hat{u}_\chi(x,\xi,t) = e^{-it\lambda_1(\xi)}\chi(\xi)\varphi_1(x,\xi),\ (x,\xi) \in \mathbb{R}^2,\ t \in [0,\infty).$$

Thus, using that $\|\varphi_1(\cdot,\xi)\|_2 = 1$ for all $\xi \in \mathbb{R}$, we have $\|\hat{u}_\chi(\cdot,\xi,t)\|_2 = |\chi(\xi)|$ and $\|\partial_t \hat{u}_\chi(\cdot,\xi,t)\|_2 = \lambda_1(\xi)\,|\chi(\xi)|$ for all $t \in [0,\infty)$. As a consequence we have

$$\hat{u}_\chi \in C^1([0,\infty), \mathcal{S}_\xi(\mathbb{R}, L_x^2(\mathbb{R}))),$$

and (3.5) follows from this since the partial Fourier transform $\mathcal{F}$ is an automorphism of the Schwartz space $\mathcal{S}(\mathbb{R})$. As for (3.6), this a straightforward consequence of (2.9) and (3.3)- (3.4), because we have $\lambda_1^{-1}(b_-,b_+) = \mathbb{R}$ by virtue of (2.6)-(2.7). □

3.2. **Quantum velocity.** Let u be given by (3.2). Assume moreover that

$$u \in C^1([0,\infty), \mathcal{S}_y(\mathbb{R}, L_x^2(\mathbb{R}))).$$

Then, the expectation of the y-component of the velocity operator of the system in the quantum state u is (well-) defined by

$$v(u_0,t) := \frac{\mathrm{d}}{\mathrm{d}t}\langle yu(\cdot,t), u(\cdot,t)\rangle_{L^2(\mathbb{R}^2)},\ t \in [0,\infty),$$

where $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^2)}$ is the usual scalar product in $L^2(\mathbb{R}^2)$ and the notation y stands for the multiplication operator by y. Otherwise stated, $v(u_0,t)$ is the velocity, i.e. the first time derivative, of the quantum realization in the state $e^{-itH}u_0$, of the second component y of the position observable. Hence $v(u_0,t)$ can be interpreted as the quantum current flowing in the y-direction, that is carried by the state $e^{-itH}u_0$. We refer the reader to [9, 10, 16] and the references therein, for an extensive mathematical study of the transport properties of Iwatsuka Hamiltonians.

Further, since $u = e^{-itH}u_0 \in C^1([0,\infty), \mathcal{S}_y(\mathbb{R}, L_x^2(\mathbb{R})))$ yields that

$$Hu \in C^0([0,\infty), \mathcal{S}_y(\mathbb{R}, L_x^2(\mathbb{R}))), \tag{3.7}$$

we see that

$$\begin{aligned} v(u_0,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle y e^{-itH}u_0, e^{-itH}u_0\rangle_{L^2(\mathbb{R}^2)} \\ &= -i\left(\langle yHe^{-itH}u_0, e^{-itH}u_0\rangle_{L^2(\mathbb{R}^2)} - \langle ye^{-itH}u_0, He^{-itH}u_0\rangle_{L^2(\mathbb{R}^2)}\right) \\ &= -i\langle [y,H]e^{-itH}u_0, e^{-itH}u_0\rangle_{L^2(\mathbb{R}^2)}, \end{aligned}$$

where $[y,H]$ denotes the commutator of y with H. Next, using that $[y,H]=[y,(-i\partial_y-a)^2]$ and that $[y,(-i\partial_y-a)]=i$, we find that $[y,H]=2i(-i\partial_y-a)$, and hence that

$$v(u_0,t)=2\langle(-i\partial_y-a)e^{-itH}u_0, e^{-itH}u_0\rangle_{L^2(\mathbb{R}^2)},\ t\in[0,\infty). \tag{3.8}$$

Notice from (3.7) that $(-i\partial_y-a)e^{-itH}u_0\in L^2(\mathbb{R}^2)$ and hence that the right-hand side of (3.8) is well-defined, as we have

$$\|\partial_x e^{-itH}u_0\|^2_{L^2(\mathbb{R}^2)}+\|(-i\partial_y-a)e^{-itH}u_0\|^2_{L^2(\mathbb{R}^2)}=\langle He^{-itH}u_0,e^{-itH}u_0\rangle_{L^2(\mathbb{R}^2)}.$$

Now, the transform $\mathcal{F}$ being unitary in $L^2(\mathbb{R}^2)$, we deduce from the identity

$$2\mathcal{F}(-i\partial_y-a(x))\mathcal{F}^*=2(\xi-a(x))=v(x,\xi),\ (x,\xi)\in\mathbb{R}^2,$$

and from (3.8) that

$$v(u_0,t)=\langle v\mathcal{F}(e^{-itH}u_0),\mathcal{F}(e^{-itH}u_0)\rangle_{L^2(\mathbb{R}^2)},\ t\in[0,\infty). \tag{3.9}$$

Let us now express $v(u_0,t)$ when $u_0\in\mathbb{P}_I(L^2(\mathbb{R}^2))$ for some $I\subset\mathbb{R}$, that is to say when $u_0=\mathbb{P}_I u_0$. In this case, we have

$$\mathcal{F}(e^{-itH}u_0)(x,\xi)=\sum_{j=1}^{\infty}\mathbb{1}_{\lambda_j^{-1}(I)}(\xi)e^{-it\lambda_j(\xi)}u_{0,j}(\xi)\varphi_j(x,\xi),\ (x,\xi)\in\mathbb{R}^2,$$

from (3.4), where $u_{0,j}(\xi):=\langle\hat{u}_0(\cdot,\xi),\varphi_j(\cdot,\xi)\rangle_2$. Putting this into (3.9), we obtain that

$$v(u_0,t)=\sum_{j,k=1}^{\infty}\int_{\lambda_j^{-1}(I)\cap\lambda_k^{-1}(I)}e^{-it(\lambda_j(\xi)-\lambda_k(\xi))}u_{0,j}(\xi)\overline{u_{0,k}(\xi)}\langle v(\cdot,\xi)\varphi_j(\cdot,\xi),\varphi_k(\cdot,\xi)\rangle_2\mathrm{d}\xi. \tag{3.10}$$

Now, suppose that $I\subset(b_-,b_+)$, in such a way that we have

$$\lambda_j^{-1}(I)=\emptyset,\ j\geqslant 2,$$

by virtue of (2.2) and (2.6). Then, it follows from (3.10) that

$$v(u_0,t)=\int_{\lambda_1^{-1}(I)}|u_{0,1}(\xi)|^2\langle v(\cdot,\xi)\varphi_1(\cdot,\xi),\varphi_1(\cdot,\xi)\rangle_2\mathrm{d}\xi. \tag{3.11}$$

Therefore, the quantum current $\upsilon(u_0, t)$ carried by a state u_0 with energy concentration in $I \subset (b_-, b_+)$, is independent of t. For the sake of notational simplicity, we write $\upsilon(u_0)$ instead of $\upsilon(u_0, t)$ in the following.

Now, with reference to Lemma 3.1, we deduce from (2.9) and (3.11) that $\vartheta(\chi)$ is the quantum current carried by $e^{-itH} u_{0,\chi}$, $t \in [0, \infty)$, or equivalently by $u_{0,\chi}$:

$$\vartheta(\chi) = \upsilon(u_{0,\chi}),\ \chi \in C_0^\infty(\mathbb{R}).$$

3.3. **Comments on the data ϑ and the formulation of the inverse problem.** It might seem surprising at first sight that we probe the system (3.1) with the frequency profile χ of the initial state $u_{0,\chi}$, rather than with the initial state itself. But there are at least two reasons why the current operator ϑ should not be defined as a function of the initial states u_0, the first one being that this would be physically irrelevant. Indeed, since any initial state of the quantum system (3.1) governed by the Iwatsuka Hamiltonian H, with energy concentration between b_- and b_+, is expressed as $\mathcal{F}^*(\chi\varphi_1(x, \cdot))(y)$, $(x, y) \in \mathbb{R}^2$, for some suitable $L^2(\mathbb{R})$-function χ, and since all the ground states $\varphi_1(\cdot, \xi)$, $\xi \in \mathbb{R}$, are determined by H, it is clear that only the frequency profile χ can be prescribed.

Secondly, it turns out that from a mathematical viewpoint, the inverse problem of recovering the magnetic potential a by triggering the system (3.1) with a suitable set of initial states u_0, is pointless. This can be understood (upon using Proposition 4.1 below) from the following lines.

Given two magnetic fields b and $\tilde{b}$ in $\mathbb{B}$, we aim to compare the quantum currents $\upsilon(u_0)$ and $\tilde{\upsilon}(u_0)$ induced by (3.1) associated with, respectively, b and $\tilde{b}$, and endowed with a non-zero initial state $u_0 \in U$, where $U := \mathbb{P}_{(b_-,b_+)}(L^2(\mathbb{R}^2)) \cap \tilde{\mathbb{P}}_{(b_-,b_+)}(L^2(\mathbb{R}^2))$. Here, $\tilde{\mathbb{P}}_{(b_-,b_+)}$ denotes the spectral projection on (b_-, b_+) of the Iwatsuka Hamiltonian $\tilde{H}$ obtained upon substituting $\tilde{b}$ for b in (2.4)-(2.5). If such a state exists, that is to say if there exists $u_0 \in L^2(\mathbb{R}^2) \setminus \{0\}$ such that

$$u_0 = \mathbb{P}_{(b_-,b_+)} u_0 = \tilde{\mathbb{P}}_{(b_-,b_+)} u_0,$$

then we have

$$\hat{u}_0(x, \xi) = u_{0,1}(\xi)\varphi_1(x, \xi) = \tilde{u}_{0,1}(\xi)\tilde{\varphi}_1(x, \xi),\ (x, \xi) \in \mathbb{R}^2, \tag{3.12}$$

where $u_{0,1}(\xi) := \langle \hat{u}_0(\cdot, \xi), \varphi_1(\cdot, \xi)\rangle_2$, $\tilde{u}_{0,1}(\xi) := \langle \hat{u}_0(\cdot, \xi), \tilde{\varphi}_1(\cdot, \xi)\rangle_2$, and $\tilde{\varphi}_1(\cdot, \xi)$ is defined as in Therorem 2.2. Thus, upon squaring both sides of (3.12) and then integrating with respect to

x over $\mathbb{R}$, we get that $u_{0,1}(\xi)^2 = \tilde{u}_{0,1}(\xi)^2$ for a.e. $\xi \in \mathbb{R}$, and hence that $|u_{0,1}(\xi)| = |\tilde{u}_{0,1}(\xi)|$. Therefore, the set

$$\mathcal{N}(u_0) := u_{0,1}^{-1}(\{0\}) = \{\xi \in \mathbb{R},\ u_{0,1}(\xi) = 0\}$$

can be equivalently defined as $\mathcal{N}(u_0) := \tilde{u}_{0,1}^{-1}(\{0\}) = \{\xi \in \mathbb{R},\ \tilde{u}_{0,1}(\xi) = 0\}$, and it holds true that $|\varphi_1(\cdot,\xi)| = |\tilde{\varphi}_1(\cdot,\xi)|$ a.e. in $\mathbb{R}$, whenever $\xi \in \mathbb{R} \setminus \mathcal{N}(u_0)$. As a consequence we have

$$\varphi_1(x,\xi) = \tilde{\varphi}_1(x,\xi),\ (x,\xi) \in \mathbb{R} \times (\mathbb{R} \setminus \mathcal{N}(u_0)), \tag{3.13}$$

since $\varphi_1(\cdot,\xi)$ and $\tilde{\varphi}_1(\cdot,\xi)$ are positive functions for all $\xi \in \mathbb{R}$.

Let us now assume for a while that the function

$$F(\xi) := \|\varphi_1(\cdot,\xi) - \tilde{\varphi}_1(\cdot,\xi)\|^2_{L^2(\mathbb{R})},\ \xi \in \mathbb{R},$$

is not identically zero. Since F is real analytic and the Lebesgue measure of the zero set of a non-trivial real analytic function is zero, the Lebesgue measure of $\mathbb{R} \setminus \mathcal{N}(u_0)$ should be zero, according to (3.13). This would mean that $u_{0,1}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}$, and hence that $\hat{u}_0 = 0$ in $L^2(\mathbb{R}^2)$, according to (3.12), which is contradiction the fact that u_0 is non-zero. Therefore, we have $F(\xi) = 0$ for all $\xi \in \mathbb{R}$, and consequently $\varphi_1(\cdot,\xi) = \tilde{\varphi}_1(\cdot,\xi)$ in $L^2(\mathbb{R})$ for all $\xi \in \mathbb{R}$. Summing up, we have proved that the following equivalence holds:

$$U \neq \{0\} \Longleftrightarrow \forall \xi \in \mathbb{R},\ \varphi_1(\cdot,\xi) = \tilde{\varphi}_1(\cdot,\xi) \text{ in } L^2(\mathbb{R}). \tag{3.14}$$

Having seen this, let us suppose that $U \neq \{0\}$, and assume in addition that

$$\upsilon(u_0) = \tilde{\upsilon}(u_0),\ u_0 \in U.$$

Then, we have $\varphi_1(\cdot,\xi) = \tilde{\varphi}_1(\cdot,\xi)$ for all $\xi \in \mathbb{R}$, from (3.14), and it is clear for all $\chi \in C_0^\infty(\mathbb{R})$ that $u_0(x,y) = (\mathcal{F}^*(\chi\varphi_1(x,\cdot))(y) = (\mathcal{F}^*(\chi\tilde{\varphi}_1(x,\cdot))(y) \in U$. Moreover, since $\upsilon(u_0) = \vartheta(\chi)$ and $\tilde{\upsilon}(u_0) = \tilde{\vartheta}(\chi)$, where $\tilde{\vartheta}$ is the same as in Theorem 2.2, we get that

$$\vartheta(\chi) = \tilde{\vartheta}(\chi),\ \chi \in C_0^\infty(\mathbb{R}).$$

Therefore, we have $\lambda_1 = \tilde{\lambda}_1$ by Proposition 4.1, whence

$$\left((\xi - \tilde{a}(x))^2 - (\xi - a(x))^2\right)\varphi_1(x,\xi) = 0,\ (x,\xi) \in \mathbb{R}^2.$$

Since $\varphi_1(\cdot,\xi)$ is positive for all $\xi \in \mathbb{R}$, this entails that

$$(\xi - \tilde{a}(x))^2 = (\xi - a(x))^2,\ (x,\xi) \in \mathbb{R}^2.$$

Upon differentiating the above identity with respect to ξ, we get that $\xi - \tilde{a}(x) = \xi - a(x)$ for all $(x,\xi) \in \mathbb{R}^2$, and hence that $a = \tilde{a}$ in $\mathbb{R}$.

4. Analysis of the inverse problem

We start with a technical result needed by the proof of Theorems 2.1 and 2.2.

4.1. **Preliminaries.** We aim to establish that knowledge of the current operator uniquely determines the first band function. With reference to the notations of Section 2.4, the corresponding result can be stated as follows.

Proposition 4.1. *Let b and $\tilde{b}$ be in $\mathbb{B}$. Assume (2.12), i.e. assume that $\vartheta(\chi) = \tilde{\vartheta}(\chi)$ for all $\chi \in C_0^\infty(\mathbb{R})$. Then, we have*

$$\lambda_1(\xi) = \tilde{\lambda}_1(\xi),\ \xi \in \mathbb{R}.$$

Proof. Applying (3.11) with $I = (b_-, b_+)$ and $u_{0,1} = \chi$, we obtain that

$$\vartheta(\chi) = \int_{\mathbb{R}} \chi(\xi)^2 \langle v(\cdot, \xi)\varphi_1(\cdot, \xi), \varphi_1(\cdot, \xi)\rangle_2 \mathrm{d}\xi,$$

from (2.6)-(2.7). Further, since $v(\cdot, \xi) = 2(\xi - a)$ is the (formal) derivative of $h(\xi)$ with respect to ξ, we have

$$\langle v(\cdot, \xi)\varphi_1(\cdot, \xi), \varphi_1(\cdot, \xi)\rangle_2 = \lambda_1'(\xi),\ \xi \in \mathbb{R},$$

by the Feynman-Hellmann theorem (see, e.g. [19, Chapter VII, Problem 4.19]), and hence

$$\vartheta(\chi) = \int_{\mathbb{R}} \chi(\xi)^2 \lambda_1'(\xi)\mathrm{d}\xi.$$

Similarly, we have $\tilde{\vartheta}(\chi) = \int_{\mathbb{R}} \chi(\xi)^2 \tilde{\lambda}_1'(\xi)\mathrm{d}\xi$, and consequently

$$\int_{\mathbb{R}} (\lambda_1 - \tilde{\lambda}_1)'(\xi)\chi(\xi)^2 \mathrm{d}\xi = 0,\ \chi \in C_0^\infty(\mathbb{R}), \tag{4.1}$$

from (2.12).

Having seen this, we will prove by contradiction that $(\lambda_1 - \tilde{\lambda}_1)'$ is identically zero in $\mathbb{R}$. For this purpose we assume existence of $\xi_0 \in \mathbb{R}$ such that $(\lambda_1 - \tilde{\lambda}_1)'(\xi_0) \neq 0$. Since λ_1 and $\tilde{\lambda}_1$ play symmetric roles here, we may assume without loss of generality that $\delta := (\lambda_1 - \tilde{\lambda}_1)'(\xi_0) > 0$. Thus, by continuity of $\xi \mapsto (\lambda_1 - \tilde{\lambda}_1)'(\xi)$ at ξ_0, there exists $\epsilon > 0$ such that $(\lambda_1 - \tilde{\lambda}_1)'(\xi) \geq \delta/2$ whenever $|\xi - \xi_0| \leq \varepsilon$. As a consequence we have

$$\int_{\mathbb{R}} (\lambda_1 - \tilde{\lambda}_1)'(\xi)\chi^2(\xi)\mathrm{d}\xi \geq \frac{\delta}{2} \int_{\mathbb{R}} \chi(\xi)^2 \mathrm{d}\xi > 0,$$

for all $\chi \in C_0^\infty(\mathbb{R}) \setminus \{0\}$ that is supported in $(\xi_0 - \varepsilon, \xi_0 + \varepsilon)$. This contradicts (4.1) and shows that $(\lambda_1 - \tilde{\lambda}_1)'(\xi) = 0$ for all $\xi \in \mathbb{R}$. Therefore, there exists $C \in \mathbb{R}$ such that

$$\lambda_1(\xi) = \tilde{\lambda}_1(\xi) + C,\ \xi \in \mathbb{R}.$$

Now, since λ_1 and $\tilde{\lambda}_1$ fulfill (2.7), we get that $C = 0$ upon sending ξ to infinity in the above identity, and the result follows. $\square$

4.2. **Proof of Theorem 2.1.** The proof being quite lengthy, we split it into six steps.

Step 1: Spectral projections. Let us denote by r_ξ, $\xi \in \mathbb{R}$, the resolvent operator of $h(\xi)$, i.e.,

$$r_\xi(z) := (h(\xi) - z)^{-1},\ z \in \mathbb{C} \setminus \{\lambda_j(\xi),\ j \in \mathbb{N}\}.$$

Then, the spectral projection of $h(\xi)$ associated with $\lambda_1(\xi)$ can be expressed as

$$p_1(\xi) = -\frac{1}{2i\pi} \int_{C(\lambda_1(\xi),\rho)} r_\xi(z) \mathrm{d}z, \tag{4.2}$$

where ρ is arbitrarily fixed in $(0, 3b_- - b_+)$ and $C(\lambda_1(\xi), \rho) := \{\lambda_1(\xi) + \rho e^{i\theta},\ \theta \in [0, 2\pi)\}$ is the circle centered at $\lambda_1(\xi)$ with radius ρ, oriented counterclockwise, see e.g., [19, Section VII.3, Eq. 1.3].

With reference to (2.11) and the notations introduced in the following line, we have $q_\varepsilon(x, \xi) = q(x, \xi) + \ell_\varepsilon(x, \xi)$ for all $(x, \xi) \in \mathbb{R}^2$, where

$$\ell_\varepsilon(x, \xi) := \varepsilon\omega(x, \xi) + \varepsilon^2 w(x)^2,\ \omega(x, \xi) := -v(x, \xi)w(x) \text{ and } w(x) := \tilde{a}(x) - a(x). \tag{4.3}$$

Thus, putting $\delta := \|\tilde{a} - a\|_{L^\infty(\mathbb{R})}$ and $M(\xi) := \|v(\cdot, \xi)\|_{L^\infty(K)} = 2\|\xi - a\|_{L^\infty(K)} < \infty$, where $K := [-r, r]$, we infer from (4.3) that

$$\|\ell_\varepsilon(\cdot, \xi)\|_{L^\infty(\mathbb{R})} \leq \varepsilon C(\xi),\ C(\xi) := \delta(M(\xi) + \delta),\ \varepsilon \in (0, 1),\ \xi \in \mathbb{R}. \tag{4.4}$$

From this and the MinMax principle, it then follows that

$$|\lambda_1(\xi) - \lambda_{\varepsilon,1}(\xi)| \leqslant \varepsilon C(\xi),\ \varepsilon \in (0, 1),\ \xi \in \mathbb{R}, \tag{4.5}$$

where $\lambda_{\varepsilon,j}(\xi)$, $j \in \mathbb{N}$, denotes the j-th eigenvalue of the operator $h_\varepsilon(\xi)$. Therefore, taking $\varepsilon \in (0, 1)$ so small that $\varepsilon C(\xi) < 3b_- - b_+ - \rho$, we deduce from (4.5) that

$$\overline{D}(\lambda_1(\xi), \rho) \cap \{\lambda_{\varepsilon,j}(\xi), j \in \mathbb{N}\} = \{\lambda_{\varepsilon,1}(\xi)\}, \tag{4.6}$$

where $\overline{D}(\lambda_1(\xi), \rho) := \{z \in \mathbb{C}, |z - \lambda_1(\xi)| \leqslant \rho\}$.

Set $r_{\varepsilon,\xi}(z) := (h_\varepsilon(\xi) - z)^{-1}$ for all $z \in \mathbb{C} \setminus \{\lambda_{\varepsilon,j}(\xi), j \in \mathbb{N}\}$, and put $\varepsilon_* := \varepsilon_*(\xi, \rho) = \min(1, C(\xi)^{-1}(3b_- - b_+ - \rho))$. Then, with reference to (4.6) and the path independence of contour integration of the meromorphic function $z \mapsto r_{\varepsilon,\xi}(z)$ around $\lambda_{\varepsilon,1}(\xi)$, the spectral projection of $h_\varepsilon(\xi)$ associated with $\lambda_{\varepsilon,1}(\xi)$,

$$p_{\varepsilon,1}(\xi) = -\frac{1}{2i\pi} \int_{C(\lambda_{\varepsilon,1}(\xi),\rho)} r_{\varepsilon,\xi}(z) \mathrm{d}z,$$

can be equivalently rewritten as

$$p_{\varepsilon,1}(\xi) = -\frac{1}{2i\pi}\int_{C(\lambda_1(\xi),\rho)} r_{\varepsilon,\xi}(z)\mathrm{d}z,\ \xi\in\mathbb{R},\ \varepsilon\in(0,\varepsilon_*). \tag{4.7}$$

Step 2: Relating $\lambda_{\varepsilon,1}$ to λ_1. Having established (4.7), we turn now to relating $\lambda_{\varepsilon,1}(\xi)$ to $\lambda_1(\xi)$ with the aid the identity $h_\varepsilon(\xi) = h(\xi) + \ell_\varepsilon(\cdot,\xi)$. To do that, we start from the eigenvalue equality $h_\varepsilon(\xi)p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi) = \lambda_{1,\varepsilon}(\xi)p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi)$, recall that the operators $h(\xi)$ and $\ell_\varepsilon(\cdot,\xi)$ are self-adjoint in $L^2(\mathbb{R})$, and obtain for all $\xi\in\mathbb{R}$ and all $\varepsilon\in(0,1)$, that

$$\begin{aligned}
& \lambda_{1,\varepsilon}(\xi)\langle p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\varphi_1(\cdot,\xi)\rangle_2\\
= \ & \langle h_\varepsilon(\xi)p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\varphi_1(\cdot,\xi)\rangle_2\\
= \ & \langle h(\xi)p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\varphi_1(\cdot,\xi)\rangle_2 + \langle \ell_\varepsilon(\xi)p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\varphi_1(\cdot,\xi)\rangle_2\\
= \ & \langle p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),h(\xi)\varphi_1(\cdot,\xi)\rangle_2 + \langle p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\ell_\varepsilon(\xi)\varphi_1(\cdot,\xi)\rangle_2\\
= \ & \lambda_1(\xi)\langle p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\varphi_1(\cdot,\xi)\rangle_2 + \langle p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\ell_\varepsilon(\xi)\varphi_1(\cdot,\xi)\rangle_2.
\end{aligned}$$

Therefore, for all $\xi\in\mathbb{R}$ and all $\varepsilon\in(0,1)$, we have

$$\begin{aligned}
F_\xi(\varepsilon) &:= \langle p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\ell_\varepsilon(\xi)\varphi_1(\cdot,\xi)\rangle_2\\
&= (\lambda_{1,\varepsilon}(\xi)-\lambda_1(\xi))\langle p_{\varepsilon,1}(\xi)\varphi_1(\cdot,\xi),\varphi_1(\cdot,\xi)\rangle_2.
\end{aligned}\tag{4.8}$$

Step 3: Resolvent formula. With reference to the second resolvent formula, we have

$$r_{\varepsilon,\xi}(z) = r_\xi(z) - r_\xi(z)\ell_\varepsilon(\cdot,\xi)r_{\varepsilon,\xi}(z),\ \varepsilon\in(0,\varepsilon_*),\ \xi\in\mathbb{R},\ z\in C(\lambda_1(\xi),\rho). \tag{4.9}$$

Notice that for all $\xi\in\mathbb{R}$ and all $z\in C(\lambda_1(\xi),\rho)$, we have

$$\mathrm{dist}\,(z,\{\lambda_j(\xi),\ j\in\mathbb{N}\}) = \min(|\lambda_1(\xi)-z|,|\lambda_2(\xi)-z|) \geq \min(\rho, 3b_- - b_+ - \rho),$$

whence

$$\begin{aligned}
\|\ell_\varepsilon(\cdot,\xi)r_\xi(z)\|_{\mathcal{B}(L^2(\mathbb{R}))} &\leq \|\ell_\varepsilon(\cdot,\xi)\|_{L^\infty(\mathbb{R})}\|r_\xi(z)\|_{\mathcal{B}(L^2(\mathbb{R}))}\\
&\leq \frac{\varepsilon C(\xi)}{\min(\rho,3b_- - b_+ - \rho)},
\end{aligned}$$

from (4.4), where $\mathcal{B}(L^2(\mathbb{R})$ denotes the space of linear bounded operators in $L^2(\mathbb{R})$. Therefore, we get that $\|\ell_\varepsilon(\cdot,\xi)r_\xi(z)\|_{\mathcal{B}(L^2(\mathbb{R}))} < 1$ for all $\xi\in\mathbb{R}$ and all $z\in C(\lambda_1(\xi),\rho)$, provided that

$$\varepsilon\in(0,\varepsilon_\star),\ \varepsilon_\star = \varepsilon_\star(\xi,\rho) := \min(\varepsilon_*, C(\xi)^{-1}\rho).$$

Thus, by iterating (4.9) we get for all $\xi \in \mathbb{R}$ and all $z \in C(\lambda_1(\xi), \rho)$, that

$$r_{\varepsilon,\xi}(z) = \sum_{n=0}^{\infty} (-1)^n r_\xi(z) \left(\ell_\varepsilon(\cdot,\xi) r_\xi(z)\right)^n, \ \varepsilon \in (0, \varepsilon_\star),$$

where the series converges in $\mathcal{B}(L^2(\mathbb{R}))$. In view of (4.3), this leads to

$$r_{\varepsilon,\xi}(z) = \sum_{n=0}^{\infty} (-1)^n \theta_{n,\xi}(z) \varepsilon^n, \ \xi \in \mathbb{R}, \ \varepsilon \in (0, \varepsilon_\star), \tag{4.10}$$

the series being convergent in $\mathcal{B}(L^2(\mathbb{R}))$, uniformly in $z \in C(\lambda_1(\xi), \rho)$. Here, each $\theta_{n,\xi}(z) \in \mathcal{B}(L^2(\mathbb{R}))$, $n \in \mathbb{N} \cup \{0\}$, can be expressed in terms of $r_\xi(z)$, $\omega(\cdot, \xi)$ and w only. For instance, we get through elementary computations that

$$\theta_{0,\xi}(z) = r_\xi(z), \theta_{1,\xi}(z) = r_\xi(z)\omega(\cdot,\xi) r_\xi(z) \text{ and } \theta_{2,\xi}(z) = r_\xi(z)(w^2 r_\xi(z) - (\omega(\cdot,\xi) r_\xi(z))^2). \tag{4.11}$$

Step 4: Analytic expansion of F_ξ. By inserting (4.10) into (4.7), we obtain with the aid of (4.2) that

$$\begin{aligned} p_{\varepsilon,1}(\xi) &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2i\pi} \left(\int_{C(\lambda_1(\xi),\rho)} \theta_{n,\xi}(z) \mathrm{d}z \right) \varepsilon^n \\ &= p_1(\xi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2i\pi} \left(\int_{C(\lambda_1(\xi),\rho)} \theta_{n,\xi}(z) \mathrm{d}z \right) \varepsilon^n, \ \xi \in \mathbb{R}, \ \varepsilon \in (0, \varepsilon_\star). \end{aligned}$$

It follows readily from this and (4.3) that the function F_ξ defined in (4.8) can be brought into the form

$$F_\xi(\varepsilon) = \sum_{n=1}^{\infty} A_n(\xi) \varepsilon^n, \ \xi \in \mathbb{R}, \ \varepsilon \in (0, \varepsilon_\star), \tag{4.12}$$

where each $A_n : \mathbb{R} \to \mathbb{C}$, $n \in \mathbb{N}$, is independent of ε. As a matter of fact, in the special case when $n = 2$, we find with the aid of (4.11) by elementary calculation, that for all $\xi \in \mathbb{R}$,

$$\begin{aligned} A_2(\xi) &= \langle p_1(\xi)\varphi_1(\cdot,\xi), w^2 \varphi_1(\cdot,\xi) \rangle_2 \\ &\quad + \frac{1}{2i\pi} \int_{C(\lambda_1(\xi),\rho)} \langle r_\xi(z)\omega(\cdot,\xi) r_\xi(z) \varphi_1(\cdot,\xi), \omega(\cdot,\xi)\varphi_1(\cdot,\xi) \rangle_2 \mathrm{d}z \\ &= \| w \varphi_1(\cdot,\xi) \|_2^2 + \frac{1}{2i\pi} \int_{C(\lambda_1(\xi),\rho)} \langle (\omega(\cdot,\xi) r_\xi(z))^2 \varphi_1(\cdot,\xi), \varphi_1(\cdot,\xi) \rangle_2 \mathrm{d}z. \end{aligned} \tag{4.13}$$

In light of (4.12), each F_ξ, where $\xi \in \mathbb{R}$ is fixed, can be extended to a real-analytic function at $\varepsilon = 0$. Moreover, by assumption, $\varepsilon = 0$ is an accumulation point of the zeros of F_ξ. Therefore, the function F_ξ is necessarily identically zero by the principle of isolated zeros, and we have

$$A_n(\xi) = 0, \ \xi \in \mathbb{R}, \ n \in \mathbb{N}, \tag{4.14}$$

according to (4.12).

Step 5: Computation of A_2. With reference to (4.13) we start by decomposing $\omega(\cdot,\xi) r_\xi(z)\varphi_1(\cdot,\xi)$, $\xi \in \mathbb{R}$, on the $L^2(\mathbb{R})$-orthonormal basis $\{\varphi_j(\cdot,\xi),\ j \geqslant 1\}$. We get that

$$\omega(\cdot,\xi) r_\xi(z)\varphi_1(\cdot,\xi) = \frac{\omega(\cdot,\xi)}{\lambda_1(\xi)-z}\varphi_1(\cdot,\xi) = \frac{1}{\lambda_1(\xi)-z}\sum_{j=1}^{\infty}\omega_j(\xi)\varphi_j(\cdot,\xi),$$

where $\omega_j(\xi) := \langle \omega(\cdot,\xi)\varphi_1(\cdot,\xi), \varphi_j(\cdot,\xi)\rangle_2$. Therefore, we have

$$\langle (\omega(\cdot,\xi) r_\xi(z))^2 \varphi_1(\cdot,\xi), \varphi_1(\cdot,\xi)\rangle_2 = \frac{1}{\lambda_1(\xi)-z}\sum_{j=1}^{\infty}\frac{|\omega_j(\xi)|^2}{\lambda_j(\xi)-z},$$

and (4.13) then yields that

$$\begin{aligned} A_2(\xi) &= \|w\varphi_1(\cdot,\xi)\|_2^2 + \sum_{j=1}^{\infty}\frac{|\omega_j(\xi)|^2}{2i\pi}\int_{C(\lambda_1(\xi),\rho)}\frac{\mathrm{d}z}{(\lambda_1(\xi)-z)(\lambda_j(\xi)-z)} \\ &= \|w\varphi_1(\cdot,\xi)\|_2^2 - \sum_{j=2}^{\infty}\frac{|\omega_j(\xi)|^2}{\lambda_j(\xi)-\lambda_1(\xi)},\ \xi \in \mathbb{R}, \end{aligned} \tag{4.15}$$

by using the residue theorem. The next step is to make the second term on the right-hand side of (4.15) sufficiently small relative to $\|w\varphi_1(\cdot,\xi)\|_2^2$ by choosing ξ suitably in $\mathbb{R}$.

Step 6: End of the proof. We refer to (2.11) and notice that there exists $\kappa \in (0,1)$ such that

$$r < r(\kappa) := (1-\kappa)^{1/2}\frac{(3b_- - b_+)^{1/2}}{2b_+}.$$

Further, since $|a(x)| \leq b_+ |x|$ for all $x \in \mathbb{R}$, by (2.3)-(2.4), we get that

$$|v(x,\xi)| \leq 2(|\xi| + b_+ r),\ x \in K,\ \xi \in \mathbb{R},$$

and hence that

$$\|v(\cdot,\xi)\|_{L^\infty(K)} \leq (1-\kappa)^{1/2}(3b_- - b_+)^{1/2},\ \xi \in [-\xi(\kappa), \xi(\kappa)], \tag{4.16}$$

where $\xi(\kappa) := b_+(r(\kappa) - r)$. Moreover, we see from (2.6) that

$$\lambda_j(\xi) - \lambda_1(\xi) \geq (2j-1)b_- - b_+ \geq 3b_- - b_+,\ \xi \in \mathbb{R},\ j \geq 2,$$

and consequently

$$\sum_{j=2}^{\infty}\frac{|\omega_j(\xi)|^2}{\lambda_j(\xi)-\lambda_1(\xi)} \leq \sum_{j=2}^{\infty}\frac{|\omega_j(\xi)|^2}{3b_- - b_+} \leq \frac{\|\omega(\cdot,\xi)\varphi_1(\cdot,\xi)\|_2^2}{3b_- - b_+},$$

by the Plancherel formula. It follows from this and the identity $\omega(\cdot,\xi) = -v(\cdot,\xi)w$ for all $\xi \in \mathbb{R}$, that

$$\sum_{j=2}^{\infty} \frac{|\omega_j(\xi)|^2}{\lambda_j(\xi) - \lambda_1(\xi)} \leq \frac{\|v(\cdot,\xi)\|_{L^\infty(K)}^2}{3b_- - b_+} \|w\varphi_1(\cdot,\xi)\|_2^2,\ \xi \in \mathbb{R}.$$

Thus, taking $\xi \in [-\xi(\kappa), \xi(\kappa)]$ in the above line, we deduce from (4.16) that

$$\sum_{j=2}^{\infty} \frac{|\omega_j(\xi)|^2}{\lambda_j(\xi) - \lambda_1(\xi)} \leq (1-\kappa)\|w\varphi_1(\cdot,\xi)\|_2^2.$$

Therefore, we get from (4.15) that

$$A_2(\xi) \geq \kappa \|w\varphi_1(\cdot,\xi)\|_2^2,\ \xi \in [-\xi(\kappa), \xi(\kappa)],$$

and then from (4.14) that $w\varphi_1(\cdot,\xi) = 0$ in $L^2(\mathbb{R})$. Finally, bearing in mind that $\varphi_1(x,\xi) > 0$ for a.e. $x \in \mathbb{R}$ and all $\xi \in [-\xi(\kappa), \xi(\kappa)]$, we obtain that $w = 0$ a.e. in $\mathbb{R}$. This proves the desired result.

4.3. **Proof of Theorem 2.2 and Corollary 2.3.** The proof of Theorem 2.2 boils down to the following lemma.

Lemma 4.2. *Assume that* $\mathrm{supp}\,(\tilde{a} - a)$ *is compact and that* $\lambda_1(\pm\xi_0) = \tilde{\lambda}_1(\pm\xi_0)$ *for some* $\xi_0 \in (0,+\infty)$*. Then, we have* $q(\cdot,\pm\xi_0) = \tilde{q}(\cdot,\pm\xi_0)$ *if and only if* $\varphi_1(\cdot,\pm\xi_0) = \tilde{\varphi}_1(\cdot,\pm\xi_0)$.

Proof. For notational convenience we set $q(x) := (\pm\xi_0 - a(x))^2$, $\tilde{q}(x) := (\pm\xi_0 - \tilde{a}(x))^2$, $x \in \mathbb{R}$, $\lambda_j := \lambda_j(\pm\xi_0)$, $\varphi_j(x) := \varphi_j(x,\pm\xi_0)$, $j \in \mathbb{N}$, and $\tilde{\varphi}_1(x) = \tilde{\varphi}_1(x,\pm\xi_0)$.

We start from the following identity $-\varphi_1'' + (q-\lambda_1)\varphi_1 = -\tilde{\varphi}_1'' + (\tilde{q}-\lambda_1)\tilde{\varphi}_1$, put $\varphi := \varphi_1 - \tilde{\varphi}_1$ and get that

$$-\varphi'' + (q-\lambda_1)\varphi = (\tilde{q}-q)\tilde{\varphi}_1.$$

Multiplying both sides of this equality by φ_j, $j \geq 1$, and integrating over $\mathbb{R}$ yields that

$$-\int_{\mathbb{R}} \varphi''(x)\varphi_j(x)\mathrm{d}x + \langle (q-\lambda_1)\varphi, \varphi_j\rangle_2 = \langle (\tilde{q}-q)\tilde{\varphi}_1, \varphi_j\rangle_2.$$

Next, upon integrating by parts twice in the first term, we obtain that

$$\langle \varphi, -\varphi_j'' + (q-\lambda_1)\varphi_j\rangle_2 = \langle (\tilde{q}-q)\tilde{\varphi}_1, \varphi_j\rangle_2,\ j \in \mathbb{N}.$$

Taking into account that $-\varphi_j'' + q\varphi_j = \lambda_j\varphi_j$, we deduce from the above line that

$$(\lambda_j - \lambda_1)\langle \varphi, \varphi_j\rangle_2 = \langle (\tilde{q}-q)\tilde{\varphi}_1, \varphi_j\rangle_2,\ j \in \mathbb{N}. \tag{4.17}$$

Further, taking into account that $\langle \varphi, \varphi_j \rangle_2 = -\langle \tilde{\varphi}_1, \varphi_j \rangle_2$ for all $j \geq 2$, (4.17) can be equivalently rewritten as

$$-(\lambda_j - \lambda_1)\langle \tilde{\varphi}_1, \varphi_j \rangle_2 = \langle (\tilde{q} - q)\tilde{\varphi}_1, \varphi_j \rangle_2,\ j \in \mathbb{N}.$$

Now, supp $(\tilde{q} - q)$ being compact, we have $(\tilde{q} - q)\tilde{\varphi}_1 \in L^2(\mathbb{R})$ and consequently

$$\sum_{j=2}^{\infty} (\lambda_j - \lambda_1)^2 \langle \tilde{\varphi}_1, \varphi_j \rangle_2^2 = \|(\tilde{q} - q)\tilde{\varphi}_1\|_2^2. \tag{4.18}$$

by the Plancherel theorem. Since $\lambda_j - \lambda_1 > 0$ for all $j \geq 2$, by (2.6)-(2.7), and since $\tilde{\varphi}_1(x) > 0$ for all $x \in \mathbb{R}$, it follows readily from (4.18) that

$$\begin{aligned} \tilde{q} - q = 0 \quad &\Longleftrightarrow \quad \langle \tilde{\varphi}_1, \varphi_j \rangle_2 = 0,\ j \geq 2 \\ &\Longleftrightarrow \quad \tilde{\varphi}_1 = \langle \tilde{\varphi}_1, \varphi_1 \rangle_2 \varphi_1. \end{aligned}$$

The two functions φ_1 and $\tilde{\varphi}_1$ being positive and normalized in $L^2(\mathbb{R})$, this last equality is equivalent to $\tilde{\varphi}_1 = \varphi_1$. $\square$

Under the conditions of Theorem 2.2, it follows readily from Proposition 4.1 and Lemma 4.2 that

$$(\pm\xi_0 - a(x))^2 = (\pm\xi_0 - \tilde{a}(x))^2,\ x \in \mathbb{R}. \tag{4.19}$$

Now, since $\pm a(x) \geqslant 0$ and $\pm\tilde{a}(x) \geq 0$ whenever $\pm x \geq 0$, it is apparent that (4.19) yields that $\tilde{a} = a$.

Finally, we prove Corollary 2.3 by noticing that (2.14) can be equivalently rewritten as

$$\int_{\mathbb{R}} e^{i(y_0\xi - t_0\lambda_1(\xi))} \chi(\xi)(\varphi_1(x,\xi) - \tilde{\varphi}_1(x,\xi))\mathrm{d}x = 0,\ \chi \in C_0^\infty(\mathbb{R}),\ x \in \mathbb{R}.$$

This entails that $e^{i(y_0\xi - t_0\lambda_1(\xi))}(\varphi_1(\cdot,\xi) - \tilde{\varphi}_1(\cdot,\xi)) = 0$ for all $\xi \in \mathbb{R}$, and hence that $\varphi_1(\cdot,\xi) = \tilde{\varphi}_1(\cdot,\xi)$, which yields (2.13).

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