DETERMINING THE SCALAR POTENTIAL IN A PERIODIC QUANTUM WAVEGUIDE FROM THE DN MAP

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Abstract. We prove logarithmic stability in the determination of the time-dependent scalar potential in a periodic quantum cylindrical waveguide, from the boundary measurements of the solution to the dynamic Schrödinger equation.

Dedicated to the memory of Alfredo Lorenzi (1944-2013).

Keywords : Schrödinger equation, periodic scalar potential, infinite cylindrical quantum waveguide, stability inequality.

Mathematics subject classification 2010 : 25R30.

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1. Introduction

In this short paper we review the main ideas and results developped in [CKS].

1.1. **Statement of the problem.** Let ω be a bounded connected open subset of $\mathbb{R}^2$ that contains the origin, with C^2-boundary $\partial\omega$. We put $\Omega = \mathbb{R} \times \omega$ and write $x = (x_1, x')$ with $x' = (x_2, x_3)$ for every $x = (x_1, x_2, x_3) \in \Omega$ throughout this text. Given $T > 0$, we consider the following initial boundary value problem (IBVP in short)

$$\begin{cases} (-i\partial_t - \Delta u + V(t,x))u = 0 & \text{in } Q = (0,T)\times\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \\ u = g & \text{on } \Sigma = (0,T)\times\partial\Omega, \end{cases} \tag{1.1}$$

where the time dependent electric potential V is 1-periodic with respect to the infinite variable x_1:

$$V(t, x_1 + 1, x') = V(t, x_1, x'),\ (t, x_1, x') \in Q. \tag{1.2}$$

In the present paper we examine the stability issue in the determination of V from the knowledge of the "boundary" operator

$$\Lambda_V : (u_0, g) \longrightarrow (\partial_\nu u_{|\Sigma}, u(T,\cdot)), \tag{1.3}$$

where the measure of $\partial_\nu u_{|\Sigma}$ (resp., $u(T,\cdot)$) is performed on Σ (resp., in Ω). Here $\nu(x)$, $x \in \partial\Omega$, denotes the outward unit normal to Ω and $\partial_\nu u(t,x) = \nabla u(t,x)\cdot\nu(x)$, $t \in (0,T)$.

1.2. **What is known so far.** There are only a few results available in the mathematical literature on the identification of time-dependent coefficients appearing in an IBVP, such as [Es1, GK, Cho, CK]. All these results were obtained in bounded domains. Several authors considered the problem of recovering time independent coefficients in an unbounded domain from boundary measurements. In most of the cases the unbounded domain under consideration is either an half space [Ra, Na] or an infinite slab [Ik, SW, LU].

The case of an infinite cylindrical waveguide was adressed in [CS, K]. For inverse problems with time-independent coefficients in unbounded domains we also refer to [DKLS]. In [Es2], uniqueness modulo gauge invariance was proved in the inverse problem of determining the time-dependent electric and magnetic potentials from the Dirichlet-to-Neumann map for the Schrödinger equation in a simply-connected bounded or unbounded domain. More specifically the inverse problem of determining periodic coefficients in the Helmholz equation was recently examined in [Fl].

1.3. **Boundary operator.** We define the trace operator τ_0 by

$$\tau_0 w = \left(w_{|\Sigma}, w(0,\cdot)\right) \text{ for all } w \in C_0^\infty([0,T]\times\mathbb{R}, C^\infty(\overline{\omega})),$$

and extend it to a bounded operator from $H^2(0,T;H^2(\Omega))$ into

$$L^2((0,T)\times\mathbb{R}; H^{3/2}(\partial\omega)) \times L^2(\Omega)$$

Then the space $X_0 = \tau_0(H^2(0,T;H^2(\Omega)))$ is easily seen to be Hilbertian for the norm

$$\|w\|_{X_0} = \inf\{\|W\|_{H^2(0,T;H^2(\Omega))}, W \in H^2(0,T;H^2(\Omega)) \text{ such that } \tau_0 W = w\},$$

and we recall from [CKS][Corollary 2.1] the following useful existence and uniqueness result:

Proposition 1.1. *Fix $M > 0$ and let $V \in C([0,T], W^{2,\infty}(\Omega))$ be such that*

$$\|V\|_{C([0,T];W^{2,\infty}(\Omega))} \leq M.$$

Then for every $(g, u_0) \in X_0$, the IBVP (1.1) admits a unique solution

$$\mathfrak{s}(g,u_0) \in Z = L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

and there is a constant $C > 0$, depending only on ω, T and M, such that we have

$$\|\mathfrak{s}(g,u_0)\|_Z \leq C\|(g,u_0)\|_{X_0}. \tag{1.4}$$

Armed with Proposition 1.1 we turn now to defining the operator Λ_V appearing in (1.3). To do that we introduce the linear bounded operator τ_1 from $L^2((0,T)\times\mathbb{R}; H^2(\omega)) \cap H^1(0,T;L^2(\Omega))$ into $X_1 = L^2(\Sigma) \times L^2(\Omega)$, obeying

$$\tau_1 w = \left(\partial_\nu w_{|\Sigma}, w(T,\cdot)\right) \text{ for } w \in C_0^\infty([0,T]\times\mathbb{R}; C^\infty(\overline{\omega})).$$

In view of (1.4) we have $\|\tau_1\mathfrak{s}(g,u_0)\|_{X_1} \leq C\|\mathfrak{s}(g,u_0)\|_Z \leq C\|(g,u_0)\|_{X_0}$, where, as in the remaining part of this text, C denotes some generic positive constant. As a consequence the operator $\Lambda_V = \tau_1 \circ \mathfrak{s}$ is bounded from X_0 into X_1 and $\|\Lambda_V\| = \|\Lambda_V\|_{\mathcal{B}(X_0,X_1)} \leq C$.

1.4. **Main result.** The main result of this paper is borrowed from [CKS][Theorem 1.1] and claims logarithmic stability in the determination of V from Λ_V. Putting $\Omega' = (0,1) \times \omega$, $Q' = (0,T) \times \Omega'$ and $\Sigma'_* = (0,T) \times (0,1) \times \partial\omega$, it may be stated as follows.

Theorem 1.1. *For $M > 0$ fixed, let $V_1,\ V_2 \in W^{2,\infty}(0,T; W^{2,\infty}(\Omega))$ fulfill (1.2) together with the three following conditions:*

$$(V_2 - V_1)(T,\cdot) = (V_2 - V_1)(0,\cdot) = 0 \ \text{in}\ \Omega', \tag{1.5}$$

$$V_2 - V_1 = 0 \ \text{in}\ \Sigma'_*, \tag{1.6}$$

$$\|V_j\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega'))} \leq M,\ \ j = 1, 2. \tag{1.7}$$

Then there are two constants $C > 0$ and $\gamma^* > 0$, depending only on T, ω and M, such that the estimate

$$\|V_2 - V_1\|_{L^2(Q')} \leq C \left(\ln \left(\frac{1}{\|\Lambda_{V_2} - \Lambda_{V_1}\|_{\mathcal{B}(X_0,X_1)}} \right) \right)^{-2/5},$$

holds whenever $0 < \|\Lambda_{V_2} - \Lambda_{V_1}\|_{\mathcal{B}(X_0,X_1)} < \gamma^*$.

1.5. **Outline.** The paper is organized as follows. In section 2 we introduce the Floquet-Bloch-Gel'fand transform, that is used to decompose the IBVP (1.1)-(1.2) into a collection of IBVPs in Q', with quasi-periodic boundary conditions on $(0,T) \times \{0,1\} \times \omega$. Section 3 is devoted to building suitable optics geometric solutions (abbreviated as OGS in the sequel) for each of these problems. Finally a sketch of the proof of Theorem 1.1, which is by means of the OGS defined in section 3, is given in section 4.

2. Floquet-Bloch-Gel'fand analysis

The main tool in the analysis of the periodic system (1.1)-(1.2) is the partial Floquet-Bloch-Gel'fand transform (abbreviated to FBG in the sequel) with respect to the x_1-direction, that is described below.

2.1. **Partial FBG transform.** For any arbitrary open subset Y of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, we define the partial FBG transform with respect to x_1 of $f \in C_0^\infty(\mathbb{R} \times Y)$ by

$$\check{f}_{Y,\theta}(x_1, y) = (\mathcal{U}_Y f)_\theta(x_1, y) = \sum_{k=-\infty}^{+\infty} e^{-ik\theta} f(x_1 + k, y),\ \ x_1 \in \mathbb{R},\ \ y \in Y,\ \ \theta \in [0, 2\pi). \tag{2.1}$$

With reference to [RS2][§XIII.16], $\mathcal{U}_Y$ extends to a unitary operator, still denoted by $\mathcal{U}_Y$, from $L^2(\mathbb{R} \times Y)$ onto the Hilbert space

$$\int_{(0,2\pi)}^{\oplus} L^2((0,1) \times Y) d\theta/(2\pi) = L^2((0,2\pi) d\theta/(2\pi); L^2((0,1) \times Y)).$$

Let $H^s_{\sharp,loc}(\mathbb{R} \times Y)$, $s = 1, 2$, denote the subspace of distributions f in $\mathbb{R} \times Y$ such that $f_{|I \times Y} \in H^s(I \times Y)$ for any bounded open subset $I \subset \mathbb{R}$. Then a function $f \in H^s_{\sharp,loc}(\mathbb{R} \times Y)$ is said to be 1-periodic with respect to x_1 if it satisfies $f(x_1 + 1, y) = f(x_1, y)$ for a.e. $(x_1, y) \in (0,1) \times Y$. The subspace of functions of $H^s_{\sharp,loc}(\mathbb{R} \times Y)$, that are 1-periodic with respect to x_1, is denoted by $H^s_{\sharp,per}(\mathbb{R} \times Y)$. Such a function being obviously determined by its values on $(0,1) \times Y$, we put $H^s_{\sharp,per}((0,1) \times Y) = \{u_{|(0,1)\times Y},\ u \in H^s_{\sharp,per}((0,1) \times Y)\}$. Since $\check{f}_{Y,\theta}(x_1 + 1, y) = e^{i\theta} \check{f}_{Y,\theta}(x_1, y)$ for a.e. $(x_1, y) \in \mathbb{R} \times Y$ and all $\theta \in [0, 2\pi)$, by (2.1), we next set $H^s_{\sharp,\theta}((0,1) \times Y) = \{e^{i\theta x_1} u, u \in H^s_{\sharp,per}((0,1) \times Y)\}$ and then derive from [Di][Chap. II, §1, Définition 1] that

$$\mathcal{U}_Y H^s(\mathbb{R} \times Y) = \int_{(0,2\pi)}^{\oplus} H^s_{\sharp,\theta}((0,1) \times Y) \frac{d\theta}{2\pi},\ \ s = 1, 2.$$

For the sake of simplicity we will systematically omit the subscript Y in $\mathcal{U}_Y$ and $\check{f}_{Y,\theta}$ in the remaining part of this text.

2.2. **FBG decomposition.** Let τ_0' denote the linear bounded operator from $H^2(0,T;H^2(\Omega'))$ into $L^2((0,T)\times(0,1);H^{3/2}(\partial\omega))\times L^2(\Omega')$ such that $\tau_0' w = \left(w_{|\Sigma_*'}, w(0,\cdot)\right)$ for $w \in C_0^\infty((0,T)\times(0,1);C^\infty(\overline{\omega}))$. Thus, putting $\mathscr{X}_{0,\theta}' = \tau_0'(H^2(0,T;H^2_{\sharp,\theta}(\Omega')))$ for all $\theta\in[0,2\pi)$, it is easy to check that $\mathscr{X}_0 = \mathcal{U}X_0 = \int_{(0,2\pi)}^{\oplus} \mathscr{X}_{0,\theta}' d\theta/(2\pi)$ and

$$\mathcal{U}\tau_0\mathcal{U}^{-1} = \int_{(0,2\pi)}^{\oplus} \tau_0' d\theta/(2\pi)$$

, where the notation τ_0' stands for the operator τ_0' restricted to $H^2(0,T;H^2_{\sharp,\theta}(\Omega'))$. The last identity means that $(\mathcal{U}\tau_0 f)_\theta = \tau_0'(\mathcal{U}f)_\theta$ for all $f\in H^2(0,T;H^2(\Omega))$ and a. e. $\theta\in(0,2\pi)$.

Further, we have $\mathscr{Z} = \mathcal{U}Z = \int_{(0,2\pi)}^{\oplus} \mathscr{Z}_\theta' d\theta/(2\pi)$, where

$$\mathscr{Z}_\theta' = L^2(0,T;H^2_{\sharp,\theta}(\Omega'))\cap H^1(0,T;L^2(\Omega')).$$

Thus, applying the transform $\mathcal{U}$ to (1.1), we immediately get the:

Proposition 2.1. *Let $V\in W^{2,\infty}(0,T;W^{2,\infty}(\Omega))$ fulfill (1.2) and let $(g,u_0)\in X_0$. Then u is the solution $\mathfrak{s}(g,u_0)\in Z$ to (1.1) defined in Proposition 1.1 if and only if $\mathcal{U}u\in\mathscr{Z}$ and each $\check{u}_\theta = (\mathcal{U}u)_\theta \in \mathscr{Z}_\theta$, for $\theta\in[0,2\pi)$, is solution to the following IBVP*

$$\left\{ \begin{array}{ll} (-i\partial_t - \Delta + V)v = 0 & \text{in } Q' = (0,T)\times\Omega', \\ v(0,\cdot) = \check{u}_{0,\theta} & \text{in } \Omega', \\ v = \check{g}_\theta & \text{on } \Sigma_*', \end{array} \right. \tag{2.2}$$

where $\check{g}_\theta$ (resp. $\check{u}_{0,\theta}$) stands for $(\mathcal{U}g)_\theta$ (resp. $(\mathcal{U}u_0)_\theta$), that is

$$(\check{g}_\theta, \check{u}_{0,\theta}) = (\mathcal{U}(g,u_0))_\theta.$$

The existence and uniqueness of solutions to (2.2) for $\theta\in[0,2\pi)$ is guaranteed by [CKS][Lemma 2.1]:

Lemma 2.1. *Assume that V obeys the conditions of Proposition 2.1 and satisfies*

$$\|V\|_{W^2(0,T;W^{2,\infty}(\Omega'))} \leq M,$$

for some $M>0$. Then for every $(\check{g}_\theta,\check{u}_{0,\theta})\in\mathscr{X}_{0,\theta}'$, $\theta\in[0,2\pi)$, there exists a unique solution $\mathfrak{s}_\theta(\check{g}_\theta,\check{u}_{0,\theta})\in\mathscr{Z}_\theta'$ to (2.2), such that the estimate

$$\|\mathfrak{s}_\theta(\check{g}_\theta,\check{u}_{0,\theta})\|_{\mathscr{Z}_\theta'} \leq C\|(\check{g}_\theta,\check{u}_{0,\theta})\|_{\mathscr{X}_{0,\theta}'}, \tag{2.3}$$

holds for some constant $C>0$ depending only on T, ω and M.

2.3. **Boundary operators.** In view of Lemma 2.1 the linear operator $\mathfrak{s}_\theta$, $\theta\in[0,2\pi)$, is bounded from $\mathscr{X}_{0,\theta}'$ into $\mathscr{Z}_\theta'$, with

$$\|\mathfrak{s}_\theta\| = \|\mathfrak{s}_\theta\|_{\mathcal{B}(\mathscr{X}_{0,\theta}',\mathscr{Z}_\theta')} \leq C,\ \theta\in[0,2\pi). \tag{2.4}$$

Let τ_1' be the linear bounded operator from

$$L^2((0,T)\times(0,1);H^2(\Omega'))\cap H^1(0,T;L^2(\Omega')) \longrightarrow \mathscr{X}_1' = L^2((0,T)\times(0,1)\times\partial\omega)\times L^2(\Omega'),$$

satisfying $\tau_1' w = \left(\partial_\nu w_{|\Sigma_*'}, w(T,\cdot)\right)$ for all $w\in C_0^\infty((0,T)\times(0,1);C^\infty(\overline{\omega}))$, in such a way that $\mathscr{X}_1 = \mathcal{U}X_1 = \int_{(0,2\pi)}^{\oplus}\mathscr{X}_1' d\theta/(2\pi)$ and $\mathcal{U}\tau_1\mathcal{U}^{-1} = \int_{(0,2\pi)}^{\oplus}\tau_1' d\theta/(2\pi)$. Then we have $\|\tau_1'\mathfrak{s}_\theta(\check{g}_\theta,\check{u}_{0,\theta})\|_{\mathscr{X}_1'} \leq C\|\mathfrak{s}_\theta(\check{g}_\theta,\check{u}_{0,\theta})\|_{\mathscr{Z}_\theta'} \leq C\|(\check{g}_\theta,\check{u}_{0,\theta})\|_{\mathscr{X}_{0,\theta}'}$ for every $\theta\in[0,2\pi)$, from (2.3), so the reduced boundary operator $\Lambda_{V,\theta} = \tau_1'\circ\mathfrak{s}_\theta \in \mathcal{B}(\mathscr{X}_{0,\theta}',\mathscr{X}_1')$. Further, it folllows from Proposition 2.1 and Lemma 2.1 that

$$\mathcal{U}\Lambda_V\mathcal{U}^{-1} = \int_{(0,2\pi)}^{\oplus}\Lambda_{V,\theta} d\theta/(2\pi),$$

hence [Di][Chap. II, §2, Proposition 2] yields:

$$\|\Lambda_V\|_{\mathcal{B}(X_0,X_1)} = \sup_{\theta\in(0,2\pi)}\|\Lambda_{V,\theta}\|_{\mathcal{B}(\mathscr{X}_{0,\theta}',\mathscr{X}_1')}. \tag{2.5}$$

3. Optics geometric solutions

For each $\theta \in [0, 2\pi)$ we aim to build solutions to the system

$$\begin{cases} (-i\partial_t - \Delta + V)v = 0 & \text{in } Q', \\ u(\cdot, 1, \cdot) = e^{i\theta} u(\cdot, 0, \cdot) & \text{on } (0,T) \times \omega, \\ \partial_{x_1} u(\cdot, 1, \cdot) = e^{i\theta} \partial_{x_1} u(\cdot, 0, \cdot) & \text{on } (0,T) \times \omega. \end{cases} \tag{3.1}$$

Specifically, for $r > 0$ fixed, we seek solutions $u_{k,\theta}$, $k \in \mathbb{Z}$, to (3.1) of the form

$$u_{k,\theta}(t,x) = \left(e^{i\theta x_1} + w_{k,\theta}(t,x) \right) e^{-i\left((\xi\cdot\xi + 4\pi^2 k^2)t + 2\pi k x_1 + x'\cdot\xi \right)},\ (t,x) = (t, x_1, x') \in Q', \tag{3.2}$$

where $w_{k,\theta} \in H^2(0,T; H^2_{\sharp,\theta}(\Omega'))$ obeys

$$\|w_{k,\theta}\|_{H^2(0,T;H^2(\Omega'))} \leq \frac{c}{r} \left(1 + |k|\right), \tag{3.3}$$

for some constant $c > 0$ independent of r, k and θ, and $\xi \in \mathbb{C}^2 \setminus \mathbb{R}^2$ is such that

$$\Im\xi \cdot \Re\xi = 0. \tag{3.4}$$

The main issue here is the quasi-periodic condition imposed on $w_{k,\theta}$ (through the requirement that $w_{k,\theta}(t,\cdot)$ is in $H^2_{\sharp,\theta}(\Omega')$ for a.e. $t \in (0,T)$). This problem may be overcomed upon adapting the framework introduced in [Ha] for the defininiton of OGS in periodic media, giving (see [CKS][Lemma 3.2]):

Lemma 3.1. *Let $\xi \in \mathbb{C}^2 \setminus \mathbb{R}^2$ obey (3.4) and let $f \in H^2(0,T; H^2(\Omega'))$. Then for all $\theta \in [0, 2\pi)$ and all $k \in \mathbb{Z}$, there exists $E_{k,\theta} \in \mathcal{B}(H^2(0,T;H^2(\Omega')); H^2(0,T; H^2_{\sharp,\theta}(\Omega')))$ such that $\varphi = E_{k,\theta} f$ is solution to the equation*

$$(-i\partial_t - \Delta + 4i\pi k \partial_{x_1} + 2i\xi \cdot \nabla_{x'})\varphi = f \ \text{ in } Q'. \tag{3.5}$$

Moreover we have

$$\|E_{k,\theta}\|_{\mathcal{B}(H^2(0,T;H^2(\Omega')))} \leq \frac{c_0}{|\Im\xi|}, \tag{3.6}$$

for some constant $c_0 > 0$, which is independent of ξ, k and θ.

The occurence of (3.5) in Lemma 3.1 follows from a direct calculation showing that $u_{k,\theta}$ fulfills (3.1) if and only if $w_{k,\theta}$ is solution to

$$\begin{cases} (-i\partial_t - \Delta + 4i\pi k \partial_{x_1} + 2i\xi \cdot \nabla_{x'} + V)w + e^{i\theta x_1} W_{k,\theta} = 0 & \text{in } Q', \\ w(\cdot, 1, \cdot) = e^{i\theta} w(\cdot, 0, \cdot) & \text{on } (0,T) \times \omega, \\ \partial_{x_1} w(\cdot, 1, \cdot) = e^{i\theta} \partial_{x_1} w(\cdot, 0, \cdot) & \text{on } (0,T) \times \omega, \end{cases} \tag{3.7}$$

with

$$W_{k,\theta} = V + \theta^2 - 4\pi k\theta.$$

Taking $r = |\Im\xi|$ so large (relative to c_0 and $\|V\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega))}$) that

$$\begin{array}{rccc} G_{k,\theta}: & H^2(0,T; H^2_{\sharp,\theta}(\Omega')) & \longrightarrow & H^2(0,T; H^2_{\sharp,\theta}(\Omega')) \\ & q & \longmapsto & -E_{k,\theta}\left(Vq + e^{i\theta x_1} W_{k,\theta} \right). \end{array}$$

is a contraction mapping, we may apply Lemma 3.1 with $f = -\left(V w_{k,\theta} + e^{i\theta x_1} W_{k,\theta} \right) \in H^2(0,T; H^2(\Omega'))$. In light of (3.5), $w_{k,\theta} = E_{k,\theta} f$ is thus a solution to (3.7) and fulfills (3.3). As a consequence we have (see [CKS][Proposition 3.1]) obtained:

Proposition 3.1. *We assume that $V \in W^{2,\infty}(0,T; W^{2,\infty}(\Omega))$ satisfies (1.2) and*

$$\|V\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega))} \leq M$$

for some $M \geq 0$. Pick $r \geq r_0 = c_0(1+M)$, where c_0 is the same as in (3.6), and let $\xi \in \mathbb{C}^2 \setminus \mathbb{R}^2$ fulfill (3.4) and $|\Im\xi| = r$. Then for all $\theta \in [0, 2\pi)$ and $k \in \mathbb{Z}$, there exists $w_{k,\theta} \in H^2(0,T; H^2_{\sharp,\theta}(\Omega'))$ obeying (3.3) such that the function $u_{k,\theta}$ defined by (3.2) is a $H^2(0,T; H^2_{\sharp,\theta}(\Omega'))$-solution to the equation (3.1).

4. Stability estimate

This section contains the proof of Theorem 1.1.

4.1. **Auxiliary result.** Fix $r > 0$ and let $\zeta = (\eta, \ell) \in \mathbb{R}^2 \times \mathbb{R}$ be such that $\eta \neq 0_{\mathbb{R}^2}$. Then there exists $\zeta_j = \zeta_j(r, \eta, \ell) = (\xi_j, \tau_j) \in \mathbb{C}^2 \times \mathbb{R}$, $j = 1, 2$, such that we have

$$|\Im \xi_j| = r, \quad \tau_j = \xi_j \cdot \xi_j, \quad \zeta_1 - \overline{\zeta_2} = \zeta, \ \Re \xi_j \cdot \Im \xi_j = 0, \tag{4.1}$$

and

$$|\xi_j| \leq \frac{1}{2}\left(|\eta| + \frac{|\ell|}{|\eta|}\right) + r, \ |\tau_j| \leq |\eta|^2 + \frac{\ell^2}{|\eta|^2} + 2r^2. \tag{4.2}$$

This can be checked by direct calculation upon setting

$$\xi_j = \frac{1}{2}\left((-1)^{j+1} + \frac{\ell}{|\eta|^2}\right)\eta + (-1)^j i \eta_r^\perp,$$
$$\tau_j = \frac{1}{4}\left((-1)^{j+1} + \frac{\ell}{|\eta|^2}\right)^2 |\eta|^2 - r^2, \ j = 1, 2,$$

where $\eta^\perp$ is any non zero $\mathbb{R}^2$-vector, orthogonal to η and $\eta_r^\perp = r\eta^\perp / |\eta^\perp|$.

This, combined with Proposition 3.1, immediately yields the:

Lemma 4.1. *Assume that $V_j \in W^{2,\infty}(0, T; W^{2,\infty}(\Omega))$, $j = 1, 2$, fulfill (1.2) and fix $r \geq r_0 = c_0(1+M) > 0$, where $M \geq \max_{j=1,2} \|V_j\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega))}$ and c_0 is the same as in (3.6). Pick $\zeta = (\eta, \ell) \in \mathbb{R}^2 \times \mathbb{R}$ with $\eta \neq 0_{\mathbb{R}^2}$, and let $\zeta_j = (\xi_j, \tau_j) \in \mathbb{C}^2 \times \mathbb{R}$, $j = 1, 2$, obey (4.1)-(4.2). Then, there is a constant $C > 0$ depending only on T, $|\omega|$ and M, such that for every $k \in \mathbb{Z}$ and $\theta \in [0, 2\pi)$, the function $u_{j,k,\theta}$, $j = 1, 2$, defined in Proposition 3.1 by substituting ξ_j for ξ, satisfies the estimate*

$$\|u_{j,k,\theta}\|_{H^2(0,T;H^2(\Omega'))} \leq C(1 + \mathfrak{q}(\zeta, k))^{\frac{13}{2}} \frac{(1 + r^2)^3}{r} e^{|\omega| r}, \ k \in \mathbb{Z}, \ \theta \in [0, 2\pi), \ r \geq r_0,$$

with

$$\mathfrak{q}(\zeta, k) = \mathfrak{q}(\eta, \ell, k) = |\eta|^2 + \frac{|\ell|}{|\eta|} + k^2.$$

4.2. **Sketch of the proof.** Let $\zeta = (\eta, \ell)$, r and $\zeta_j = (\xi_j, \tau_j)$, $j = 1, 2$, be as in Lemma 4.1, fix $k \in \mathbb{Z}$, and put

$$(k_1, k_2) = \begin{cases} (k/2, -k/2) & \text{if } k \text{ is even} \\ ((k+1)/2, -(k-1)/2) & \text{if } k \text{ is odd.} \end{cases}$$

Further we pick $\theta \in [0, 2\pi)$ and note u_j, $j = 1, 2$, the OGS $u_{j,k_j,\theta}$, defined by Lemma 4.1. In light of Lemma 2.1 there is a unique solution $v \in L^2(0, T; H^2_{\sharp,\theta}(\Omega')) \cap H^1(0, T; L^2(\Omega'))$ to the IBVP

$$\begin{cases} (-i\partial_t + \Delta + V_2)v = 0 & \text{in } Q' \\ v(0, \cdot) = u_1(0, \cdot) & \text{in } \Omega', \\ v = u_1 & \text{on } \Sigma'_*. \end{cases} \tag{4.3}$$

Hence $u = v - u_1$ is solution to the following system

$$\begin{cases} (-i\partial_t + \Delta + V_2)u = (V_1 - V_2)u_1 & \text{in } Q' \\ u(0, \cdot) = 0 & \text{in } \Omega', \\ u = 0 & \text{on } \Sigma'_*, \\ u(\cdot, 1, \cdot) = e^{i\theta} u(\cdot, 0, \cdot) & \text{on } (0, T) \times \omega \\ \partial_{x_1} u(\cdot, 1, \cdot) = e^{i\theta} \partial_{x_1} u(\cdot, 0, \cdot) & \text{on } (0, T) \times \omega, \end{cases} \tag{4.4}$$

so we get

$$\int_{Q'} (V_1 - V_2) u_1 \overline{u_2} \, dt dx = \int_{\Sigma'_*} \partial_\nu u \overline{u_2} \, dt d\sigma(x) - i \int_{\Omega'} u(T, \cdot) \overline{u_2(T, \cdot)} \, dx, \tag{4.5}$$

by integrating by parts and taking into account the quasi-periodic boundary conditions satisfied by u and u_2. Notice from (4.3)-(4.4) that $\partial_\nu u = \left(\Lambda^1_{V_2,\theta} - \Lambda^1_{V_1,\theta}\right)(\mathfrak{g}_1)$ and $u(T,.) = \left(\Lambda^2_{V_2,\theta} - \Lambda^2_{V_1,\theta}\right)(\mathfrak{g}_1)$, where

$$\mathfrak{g}_1 = \left(u_{1|\Sigma'_*}, u_1(0,.)\right) \in \mathscr{X}'_{0,\theta}.$$

Thus, putting

$$\beta_k = \begin{cases} 0 & \text{if } k \text{ is even or } k \in \mathbb{R} \setminus \mathbb{Z} \\ 4\pi^2 & \text{if } k \text{ is odd,} \end{cases}$$

for all $k \in \mathbb{Z}$, and

$$\varrho = \varrho_{k,\theta} = e^{-i\theta x_1} w_1 + e^{i\theta x_1}\overline{w_2} + w_1 \overline{w_2}, \tag{4.6}$$

we deduce from (4.1), (3.2) and (4.5) that

$$\int_{Q'} (V_1 - V_2) e^{-i\left((\ell+\beta_k k)t + 2\pi k x_1 + x'\cdot\eta\right)} dt dx = A + B + C, \tag{4.7}$$

with

$$A = -\int_{Q'} (V_2 - V_1)\varrho(t,x) e^{-i\left(((\ell+\beta_k k)t + 2\pi k x_1 + x'\cdot\eta\right)} dt dx, \tag{4.8}$$

$$B = \int_{\Sigma'_*} \left(\Lambda^1_{V_2,\theta} - \Lambda^1_{V_1,\theta}\right)(\mathfrak{g}_1)\overline{u_2}\, dt d\sigma(x), \tag{4.9}$$

$$C = -i \int_{\Omega'} \left(\Lambda^2_{V_2,\theta} - \Lambda^2_{V_1,\theta}\right)(\mathfrak{g}_1)\overline{u_2(T,\cdot)}\, dx. \tag{4.10}$$

Upon setting

$$V(t,x) = \begin{cases} (V_2 - V_1)(t,x) & \text{if } (t,x) \in Q, \\ 0 & \text{if } (t,x) \in \mathbb{R}^4 \setminus Q, \end{cases} \quad \text{and } \phi_k(x_1) = e^{i2\pi k x_1},\ x_1 \in \mathbb{R},\ k \in \mathbb{Z},$$

we may rewrite (4.7) as

$$\int_{Q'} (V_1 - V_2) e^{-i\left(((\ell+\beta_k k)t + 2\pi k x_1 + x'\cdot\eta\right)} dt dx = \left\langle \widehat{V}(\ell + \beta_k k, \eta), \phi_k \right\rangle_{L^2(0,1)}, \tag{4.11}$$

where $\widehat{V}$ stands for the partial Fourier transform of V with respect to $t \in \mathbb{R}$ and $x' \in \mathbb{R}^2$. Further, due to (3.3) and (4.6), we have $\|\varrho\|_{L^1(Q')} \leq c'(1+|k|)^2/r^2$, where the constant $c' > 0$ depends only on T, $|\omega|$ and M. Since $\|V_1 - V_2\|_\infty \leq 2M$, it follows from this and (4.8) (upon substituting c' for $4Mc'$) in the above estimate that

$$|A| \leq \|V_1 - V_2\|_\infty \|\varrho\|_{L^1(Q')} \leq c' \frac{(1 + \mathfrak{q}(\zeta,k))}{r^2}, \tag{4.12}$$

where $\mathfrak{q}$ is defined in Lemma 4.1. Moreover, we have

$$|B| + |C| \leq C^2 \|\Lambda^1_{V_2,\theta} - \Lambda^1_{V_1,\theta}\|_{\mathcal{B}(\mathscr{X}'_{0,\theta},\mathscr{X}'_1)} (1 + \mathfrak{q}(\zeta,k))^{13} \frac{(1+r^2)^6}{r^2} e^{2|\omega| r},\ r \geq r_0. \tag{4.13}$$

from (4.9)-(4.10) and Lemma 4.1. Now, putting (4.7) and (4.11)–(4.13) together, we end up getting that

$$\left| \left\langle \widehat{V}(\ell + \beta_k k, \eta), \phi_k \right\rangle_{L^2(0,1)} \right| \leq c'' \frac{(1 + \mathfrak{q}(\zeta,k))}{r^2} \left(1 + \gamma(1 + \mathfrak{q}(\zeta,k))^{12}(1+r^2)^6 e^{2|\omega| r}\right)$$

for $r \geq r_0$, where $\gamma = \|\Lambda_{V_2,\theta} - \Lambda_{V_1,\theta}\|_{\mathcal{B}(\mathscr{X}'_{0,\theta},\mathscr{X}'_1)}$ and the constant $c'' > 0$ is independent of k, r and $\zeta = (\eta, \ell)$. From this and the Parseval-Plancherel theorem, entailing

$$\|V_2 - V_1\|^2_{L^2(Q')} = \|V\|^2_{L^2(\mathbb{R}\times(0,1)\times\mathbb{R}^2)} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^3} |\langle \widehat{V}(\ell,\eta), \phi_k \rangle_{L^2(0,1)}|^2 d\zeta,$$

then follows the:

Theorem 4.1. *Let M and V_j, $j=1,2$, be the same as in Theorem 1.1. Then we may find two constants $C>0$ and $\gamma^*>0$, depending on T, ω and M, such that we have*

$$\|V_2-V_1\|_{L^2(Q')} \leq C\left(\ln\left(\frac{1}{\|\Lambda_{V_2,\theta}-\Lambda_{V_1,\theta}\|_{\mathcal{B}(\mathscr{X}_0',\mathscr{X}_1')}}\right)\right)^{-2/5},$$

for any $\theta\in[0,2\pi)$, provided $0<\|\Lambda_{V_2,\theta}-\Lambda_{V_1,\theta}\|_{\mathcal{B}(\mathscr{X}_{0,\theta}',\mathscr{X}_1'))}<\gamma^*$.

Finally, putting (2.5) together with Theorem 4.1, we obtain Theorem 1.1.

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