STABLE DETERMINATION OF TIME-DEPENDENT SCALAR POTENTIAL FROM BOUNDARY MEASUREMENTS IN A PERIODIC QUANTUM WAVEGUIDE

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ABSTRACT. We prove logarithmic stability in the determination of the time-dependent scalar potential in a 1-periodic quantum cylindrical waveguide, from the boundary measurements of the solution to the dynamic Schrödinger equation.

Keywords : Schrödinger equation, periodic scalar potential, infinite cylindrical quantum waveguide, stability inequality.

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CONTENTS

1. INTRODUCTION

1.1. **Statement of the problem and existing papers.** In the present paper we consider an infinite waveguide $\Omega = \mathbb{R} \times \omega$, where ω is a bounded domain of $\mathbb{R}^2$ with C^2-boundary $\partial\omega$. We assume without limiting the generality of the foregoing that ω contains the origin. For shortness sake we write $x = (x_1, x')$ with $x' = (x_2, x_3)$ for every $x = (x_1, x_2, x_3) \in \Omega$. Given $T > 0$, we examine the following initial-boundary value problem (abbreviated to IBVP in what follows)

$$(1.1) \qquad \begin{cases} (-\mathrm{i}\partial_t - \Delta + V)u = 0 & \text{in } Q = (0,T) \times \Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \\ u = g & \text{on } \Sigma = (0,T) \times \partial\Omega, \end{cases}$$

where the electric potential $V = V(t,x)$ is 1-periodic with respect to the infinite variable x_1:

$$V(\cdot, x_1 + 1, \cdot) = V(\cdot, x_1, \cdot),\ x_1 \in \mathbb{R}. \tag{1.2}$$

The main purpose of this paper is to prove stability in the recovery of V from the "boundary" operator

$$\Lambda_V : (u_0, g) \longrightarrow \left(\partial_\nu u_{|\Sigma}, u(T,\cdot)\right), \tag{1.3}$$

where the measurement of $\partial_\nu u_{|\Sigma}$ (resp. $u(T,\cdot)$) is performed on Σ (resp. Ω). Here $\nu(x)$, $x \in \partial\Omega$, denotes the outward unit normal to Ω and $\partial_\nu u(t,x) = \nabla u(t,x) \cdot \nu(x)$.

The inverse problem of determining a time-independent coefficient appearing in a partial differential equation by boundary measurements has been studied by numerous authors. In the framework of the dynamical Schrödinger equation acting on a bounded domain, Lipschitz stable determination of the time-independent electric potential from a single partial Neumann observation of the solution was proved by Baudouin and Puel in [BP]. Their proof relies on the celebrated Bukhgeim-Klibanov method presented in [BK], and the measurement is performed on any sub-boundary fulfilling the observability condition introduced by Bardos, Lebeau and Rauch in [BLR]. The same problem was solved by Bellassoued and Choulli in [BC1] from a single observation of the solution on an arbitrary sub-part of the boundary, at the expense of lesser generality (the unknown coefficient has to be known in a neighborhood of the whole boundary) and weaker (i.e. logarithmic) stability. In [CriS], the time-independent part of the magnetic field associated with the potential appearing in the dynamic Schrödinger equation was Lipschitz stably retrieved from a finite number of partial Neumann boundary observations of the solution. We also refer to [A] for logarithmic stable determination of a conductivity coefficient of an elliptic PDE by boundary measurements, and to [S] for continuous dependence of inverse initial boundary problems with time-independent coefficients of the wave equation, in bounded domains. Here and henceforth we do not intend any comprehensive list of references. In a more similar fashion to the one we aim for developing in this paper, Bellassoued and Choulli considered in [BC2], the problem of determining the time-independent magnetic field of the dynamic Schrödinger equation from the knowledge of the corresponding Dirichlet-to-Neumann (abbreviated to DN in the following) map. Actually, their result is for the magnetic Schrödinger operator in a bounded domain, but their analysis works for the operator examined in this article as well. In the context of (1.1) their claim is that the map $V \mapsto \Lambda_V$ is Hölder continuous. We aim for extending this result (upon weakening Hölder continuity to logarithmic Hölder continuity) to the inverse problem of determining a time-dependent potential living in a periodic waveguide.

There are only a few results available in the mathematical literature on the identification of time-dependent coefficients appearing in an IBVP. Stefanov proved in [St] that scattering data (or equivalently boundary data) uniquely determine the time-dependent potential of the wave equation. In [RS], Ramm and Sjöstrand established uniqueness in the identification of a time-dependent coefficient appearing in a hyperbolic equation from boundary observations, measured on an infinite time-span, of the solution. This result was adapted to the case of a finite time-span $[0,T]$, $T > 0$, by Rakesh and Ramm in [RR]. Assuming that T is greater than the diameter of the bounded spatial domain under consideration, they claim uniqueness for potentials supported outside the cone $\{(t,x),\ t \in [0,T], |x| \leq T\}$. Recently, the result of [RS] was generalized in [Sa] to a wider class of unknown coefficients. Isakov showed in [Is] that the time-dependent potential of a wave equation can be uniquely recovered from boundary observations of the solution on ∂Q, where Q is defined in (1.1). His proof relies on a suitable design of complex geometric optics solutions inspired from the ones introduced in [SU] for elliptic operators. In [Es1] Eskin established by means of a unique continuation result borrowed from [T] that time analytic coefficients of hyperbolic equations are uniquely determined from the knowledge of partial Neumann data. The case of a bounded cylindrical domain was addressed in [GK] where the time-dependent coefficient of order zero appearing in a parabolic equation is stably determined from a single Neumann boundary data. In [Cho, Section 3.6.3], Choulli proved logarithmic stability in the recovery of zero order time-dependent coefficients of a parabolic equation from partial boundary measurements of the solution. Lipschitz stability was derived in [CK] for the same problem with coefficients depending only on time, from a single measurement of the solution.

All the above mentioned results were obtained in a bounded spacial domain. Several authors addressed the problem of recovering time-independent coefficients in an unbounded domain from boundary measurements. In most of the cases the unbounded domain under consideration was either a half-space or an infinite slab. Namely, in [Ra], Rakesh examined the uniqueness issue in the problem of determining a scalar potential of the wave equation in a half-space, from the Neumann-to-Dirichlet map. By means of a unique continuation theorem for the Cauchy problem associated with the wave equation at constant speed and the X-ray transform method developed by Hamaker, Smith, Solmon and Wagner in [HSSW], he obtained uniqueness for electric potentials that are constant outside some *a priori* fixed compact set. This result was generalized by Nakamura in [Na], to a wider class of coefficients. The inverse problem of identifying an embedded object in an infinite slab was treated in [Ik, SW]. In [LU], the compactly supported coefficients appearing in a stationary Schrödinger equation were identified from the knowledge of partial DN data. For an infinite cylindrical waveguide, the stability issue with respect to the DN map was treated in [CS, K] in absence of any assumption on the behavior of the unknown coefficient outside an *a priori* fixed compact set. In [KPS1], the compactly supported electric potential of the dynamic Schrödinger equation acting in an unbounded cylindrical domain was Lispchitz stably retrieved from a finite number of boundary Neumann data. This result was extended to non-compactly supported potentials in [KPS2], at the expense of weaker (i.e. Hölder continuous) stability.

For inverse problems with time-independent coefficients in unbounded domains we also refer to [DKLS]. Moreover, in [Es2], Eskin proved uniqueness modulo gauge invariance in the determination of time-dependent electric and magnetic potentials of the Schrödinger equation, from the DN map. His result is valid for simply-connected bounded or unbounded domains. More specifically, we point out that the inverse problem of determining periodic coefficients in the Helmholtz equation was recently examined in [Fl].

1.2. **Main result.** Let us introduce the trace operator τ_0,

$$\tau_0 w = \left(w_{|\Sigma}, w(0,\cdot)\right),\ w \in C_0^\infty([0,T]\times\mathbb{R}, C^\infty(\overline{\omega})),$$

and extend it to a bounded map from $H^2(0,T;H^2(\Omega))$ into $L^2((0,T)\times\mathbb{R}, H^{3/2}(\partial\omega))\times L^2(\Omega)$. Then the space $X_0 = \tau_0(H^2(0,T;H^2(\Omega)))$, endowed with the norm

$$\|w\|_{X_0} = \inf\{\|W\|_{H^2(0,T;H^2(\Omega))};\ W \in H^2(0,T;H^2(\Omega)) \text{ such that } \tau_0 W = w\},$$

is Hilbertian. Moreover, as will appear in the Section 2, the linear operator Λ_V, defined in (1.3), is bounded from X_0 into $X_1 = L^2(\Sigma)\times L^2(\Omega)$.

Next, bearing in mind that the potential V appearing in the first line of (1.1) is 1-periodic with respect to x_1, by assumption (1.2), we put

$$\check{\Omega} = (0,1)\times\omega,\ \check{Q} = (0,T)\times\check{\Omega},\ \check{\Sigma} = (0,T)\times(0,1)\times\partial\omega. \tag{1.4}$$

The main result of the paper may be stated as follows.

Theorem 1.1. *For $M > 0$ fixed, let $V_1,\ V_2 \in W^{2,\infty}(0,T;W^{2,\infty}(\Omega))$ fulfill (1.2) together with the three following conditions:*

$$(V_2 - V_1)(T,\cdot) = (V_2 - V_1)(0,\cdot) = 0 \text{ in } \check{\Omega}, \tag{1.5}$$

$$V_2 - V_1 = 0 \text{ in } \check{\Sigma}, \tag{1.6}$$

$$\|V_j\|_{W^{2,\infty}(0,T;W^{2,\infty}(\check{\Omega}))} \leq M,\ j = 1,2. \tag{1.7}$$

Then there is a constant $C > 0$, depending only on T, ω and M, such that we have

$$\|V_2 - V_1\|_{L^2(\check{Q})} \leq C\left(\|\Lambda_{V_2} - \Lambda_{V_1}\|_{\mathcal{B}(X_0,X_1)} + \left|\ln\|\Lambda_{V_2} - \Lambda_{V_1}\|_{\mathcal{B}(X_0,X_1)}\right|^{-\frac{1}{5}}\right). \tag{1.8}$$

Remark 1.1. The stability estimate (1.8) remains valid for $\|\Lambda_{V_2} - \Lambda_{V_1}\|_{\mathcal{B}(X_0,X_1)} = 0$, upon extending (by 0) the mapping $\mathbb{R}_+^* \ni t \mapsto t + |\ln t|^{-1/5}$ to a continuous function at $t = 0$. As a consequence (1.8) yields unique determination of V from Λ_V, in the sense that the identity $V_1 = V_2$ holds whenever $\Lambda_{V_1} = \Lambda_{V_2}$.

1.3. **Outline.** The remaining part of this text is organized as follows. In Section 2 we analyze the direct problem associated with the IBVP (1.1) and we prove that the boundary operator Λ_V is bounded. In Section 3 we introduce the family of Cauchy problems with quasi-periodic boundary conditions, indexed by the real parameter $\theta \in [0, 2\pi)$,

$$\left\{ \begin{array}{ll} (-\mathrm{i}\partial_t - \Delta + V)v = 0 & \text{in } \check{Q}, \\ v(0,\cdot) = v_0 & \text{in } \check{\Omega}, \\ v = h & \text{on } \check{\Sigma}, \\ v(\cdot,1,\cdot) = \mathrm{e}^{\mathrm{i}\theta} v(\cdot,0,\cdot) & \text{on } (0,T)\times\omega, \\ \partial_{x_1} v(\cdot,1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}\partial_{x_1} v(\cdot,0,\cdot) & \text{on } (0,T)\times\omega, \end{array} \right. \tag{1.9}$$

associated with a suitable initial condition v_0 and Dirichlet data h. To each θ in $[0, 2\pi)$, we associate the boundary operator

$$\Lambda_{V,\theta} : (v_0, h) \mapsto \left(\partial_\nu v_{|\check{\Sigma}}, v(T,.) \right), \tag{1.10}$$

where v is the solution to (1.9). We state in Theorem 3.1 for all $\theta \in [0, 2\pi)$, that the unknown function V can be stably determined from the knowledge of $\Lambda_{V,\theta}$. The proof of Theorem 3.1 is given in Section 5 and relies on a sufficiently rich set of suitable geometric optics solutions to (1.9), which are built in Section 4. Finally, in Section 6, we relate, by means of the partial Floquet-Bloch-Gel'fand (abbreviated to FBG in the sequel) transform, the IBVP (1.1) to the family (1.9) of Cauchy problems introduced in Section 3. Then we prove, upon showing that the boundary operator Λ_V is unitarily equivalent to the direct integral with fibers $\{\Lambda_{V,\theta},\ \theta \in [0, 2\pi)\}$, that Theorem 1.1 is a byproduct of Theorem 3.1.

Remark 1.2. The method of the proofs of Theorems 1.1 and 3.1, given in the remaining part of this text, can be easily adapted to the case of the inverse elliptic problem of recovering the (time-independent) periodic scalar potential V in the stationary Schrödinger equation

$$\left\{ \begin{array}{ll} (-\Delta + V)u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{array} \right.$$

from the knowledge of the DN map $g \mapsto \partial_\nu u$. Nevertheless, in order to prevent the inadequate expense of the size of this paper, we shall not extend the technique developed in the following sections to this peculiar framework.

2. Boundedness of the boundary operator

In this section we prove that the boundary operator Λ_V is bounded from X_0 into X_1. This preliminarily requires that the existence, uniqueness and smoothness properties of the solution to the IBVP (1.2) be appropriately established in Corollary 2.1. To this end we start by proving the following technical lemma.

Lemma 2.1. *Let X be a Banach space, U be a m-dissipative operator in X with dense domain $D(U)$ and $B \in C([0,T], \mathcal{B}(D(U)))$. Then for all $v_0 \in D(U)$ and $f \in C([0,T], X) \cap L^1(0,T; D(U))$ (resp. $f \in W^{1,1}(0,T; X)$) there is a unique solution $v \in Z_0 = C([0,T], D(U)) \cap C^1([0,T], X)$ to the following Cauchy problem*

$$\left\{ \begin{array}{l} v'(t) = Uv(t) + B(t)v(t) + f(t), \\ v(0) = v_0, \end{array} \right. \tag{2.1}$$

such that

$$\|v\|_{Z_0} = \|v\|_{C^0([0,T],D(U))} + \|v\|_{C^1([0,T],X)} \leq C(\|v_0\|_{D(U)} + \|f\|_*). \tag{2.2}$$

Here C is some positive constant depending only on T and $\|B\|_{C([0,T],\mathcal{B}(D(U)))}$, and $\|f\|_*$ stands for the norm $\|f\|_{C([0,T],X)\cap L^1(0,T;D(U))}$ (resp. $\|f\|_{W^{1,1}(0,T;X)}$).

Proof. Put $Y = C([0,T], D(U))$ and define

$$\begin{array}{llll} K : & Y & \to & Y \\ & v & \mapsto & \left[t \mapsto (Kv)(t) = \int_0^t S(t-s)B(s)v(s)\mathrm{d}s \right], \end{array}$$

where $S(t)$ denotes the contraction semi-group generated by U. The operator K is well defined from [CH, Proposition 4.1.6] and we have

$$\|Kv(t)\|_X \leq tM\|v\|_Y,\ t \in [0,T],\ M = \|B\|_{C([0,T],\mathcal{B}(D(U)))}. \tag{2.3}$$

Therefore $K \in \mathcal{B}(Y)$ and we get

$$\|K^n v(t)\|_X \leq \frac{t^n M^n}{n!}\|v\|_Y,\ t \in [0,T], \tag{2.4}$$

by iterating (2.3). Fix $F \in Y$ and put $\widetilde{K}v = Kv + F$ for all $v \in Y$. Thus, since

$$\widetilde{K}^n v - \widetilde{K}^n w = K^n(v-w),\ v,w \in Y,\ n \in \mathbb{N},$$

(2.4) entails that $\widetilde{K}^n$ is strictly contractive for some $n \in \mathbb{N}^*$. Hence $\widetilde{K}$ admits a unique fixed point in Y, which is the unique solution $v \in Y$ to the following Volterra integral equation

$$v(t) = \int_0^t S(t-s)B(s)v(s)\mathrm{d}t + F(t),\ t \in [0,T]. \tag{2.5}$$

As a consequence we have

$$\|v\|_Y \leq \mathrm{e}^{MT}\|F\|_Y, \tag{2.6}$$

by Gronwall lemma.

The last step of the proof is to choose $F(t) = S(t)v_0 + \int_0^t S(t-s)f(s)\mathrm{d}s$ for $t \in [0,T]$ and to apply [CH, Proposition 4.1.6] twice, so we find out that $F \in Y$. Therefore the function v given by (2.5) belongs to $C^1([0,T],X)$ and it is the unique solution to (2.1). Finally we complete the proof by noticing that (2.2) follows readily from (2.6). □

Prior to solving the IBVP (1.1) with the aid of Lemma 2.1, we define the Dirichlet Laplacian A in $L^2(\Omega)$ as the self-adjoint operator generated in $L^2(\Omega)$ by the closed quadratic form

$$a(u) = \int_\Omega |\nabla u(x)|^2 \mathrm{d}x,\ u \in D(a) = H_0^1(\Omega),$$

and establish the coming:

Lemma 2.2. *The domain of the operator A is $H_0^1(\Omega) \cap H^2(\Omega)$ and the norm associated to $D(A)$ is equivalent to the usual one in $H^2(\Omega)$.*

Proof. We have

$$\mathscr{F} A \mathscr{F}^{-1} = \int_{\mathbb{R}}^{\oplus} \widehat{A}_k \mathrm{d}k, \tag{2.7}$$

where $\mathscr{F}$ denotes the partial Fourier with respect to x_1, i.e.

$$(\mathscr{F}u)(k,x') = \widehat{u}(k,x') = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}kx_1} u(x_1,x')\mathrm{d}x_1,\ (k,x') \in \Omega,$$

and $\widehat{A}_k = -\Delta_{x'} + k^2$, $k \in \mathbb{R}$, is the self-adjoint operator in $L^2(\omega)$ generated by the closed quadratic form $\widehat{a}_k(v) = \int_\omega (|\nabla_{x'} v(x')|^2 + k^2|v(x')|^2)\mathrm{d}x'$, $v \in D(\widehat{a}_k) = H_0^1(\omega)$. Here $\Delta_{x'}$ (resp., $\nabla_{x'}$) denotes the Laplace (resp., gradient) operator with respect to the variable x'. Since ω is a bounded domain with C^2-boundary, we have $D(\widehat{A}_k) = H_0^1(\omega) \cap H^2(\omega)$ for each $k \in \mathbb{R}$, by [Ag].

Further, bearing in mind that (2.7) reads

$$\begin{cases} D(A) = \{u \in L^2(\Omega),\ \widehat{u}(k) \in D(\widehat{A}_k) \text{ a.e. } k \in \mathbb{R} \text{ and } \int_{\mathbb{R}} \|\widehat{A}_k \widehat{u}(k)\|_{L^2(\omega)}^2 \mathrm{d}k < \infty\}, \\ (\mathscr{F}Au)(k) = \widehat{A}_k \widehat{u}(k) \text{ a.e. } k \in \mathbb{R}, \end{cases}$$

and noticing that

$$\|\widehat{A}_k v\|_{L^2(\omega)}^2 = \sum_{j=0}^2 C_2^j k^{2j} \|\nabla_{x'}^{2-j} v\|_{L^2(\omega)}^2,\ v \in H_0^1(\omega) \cap H^2(\omega),\ k \in \mathbb{R},$$

with $C_2^j = 2!/(j!(2-j)!)$, $j=0,1,2$, we see that $D(A)$ is made of functions $u \in L^2(\Omega)$ satisfying simultaneously $\widehat{u}(k) \in H_0^1(\omega) \cap H^2(\omega)$ for a.e. $k \in \mathbb{R}$, and $k \mapsto (1+k^2)^{j/2} \|\widehat{u}(k)\|_{H^{2-j}(\omega)} \in L^2(\mathbb{R})$ for $j=0,1,2$. Finally, $\|\widehat{u}(k)\|_{H^2(\omega)}$ being equivalent to $\|\Delta_{x'} \widehat{u}(k)\|_{L^2(\omega)}$ by [Ev, Section 6.3, Theorem 4], we obtain the result. □

Let B denote the multiplier by $V \in C([0,T], W^{2,\infty}(\Omega))$. Due to Lemma 2.2 we have $B \in C([0,T], \mathcal{B}(D(A)))$ with $\|B\|_{C([0,T],\mathcal{B}(D(A)))} \leq \|V\|_{C([0,T],W^{2,\infty}(\Omega))}$. Therefore, applying Lemma 2.1 to $U=-iA$ we obtain the following existence and uniqueness result:

Proposition 2.1. *Let $M>0$ and $V \in C([0,T], W^{2,\infty}(\Omega))$ be such that $\|V\|_{C([0,T],W^{2,\infty}(\Omega))} \leq M$. Then for all $v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $f \in W^{1,1}(0,T;L^2(\Omega))$ there is a unique solution $v \in C([0,T], H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0,T], L^2(\Omega))$ to*

$$\left\{ \begin{array}{ll} (-\mathrm{i}\partial_t - \Delta + V)v = f & \textit{in } Q, \\ v(0,\cdot) = v_0 & \textit{in } \Omega, \\ v = 0 & \textit{on } \Sigma. \end{array} \right. \tag{2.8}$$

Moreover, v fulfills

$$\|v\|_{C([0,T],H^2(\Omega))} + \|v\|_{C^1([0,T],L^2(\Omega))} \leq C\left(\|v_0\|_{H^2(\Omega)} + \|f\|_{W^{1,1}(0,T;L^2(\Omega))}\right),$$

for some constant $C>0$ depending only on ω, T and M.

Finally, using a classical extension argument we now derive the coming useful consequence to Proposition 2.1.

Corollary 2.1. *Let M and V be the same as in Proposition 2.1. Then for every $(g,u_0) \in X_0$, the IBVP* (1.1) *admits a unique solution*

$$\mathfrak{s}(g,u_0) \in Z = L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)).^1$$

Moreover, we have

$$\|\mathfrak{s}(g,u_0)\|_Z \leq C\|(g,u_0)\|_{X_0}, \tag{2.9}$$

for some constant $C>0$ depending only on ω, T and M.

Proof. Consider the space $Y_0 = H^2(0,T;H^2(\Omega))/\mathrm{Ker}(\tau_0)$, equipped with its natural quotient norm and recall that Y_0 is Hilbertian according to [Sc, Section XXIII.4.2, Theorem 2]. Moreover, the mapping $\widetilde{\tau}_0 : Y_0 \ni \dot{W} \mapsto \tau_0 W \in \tau_0(H^2(0,T;H^2(\Omega)))$, where W is arbitrary in $\dot{W}$, being bijective, it turns out that $X_0 = \tau_0(H^2(0,T;H^2(\Omega)))$ is an Hilbert space for the norm

$$\|w\|_{X_0} = \|\widetilde{\tau}_0^{-1} w\|_{Y_0},\ w \in X_0.$$

Therefore, we can choose $G \in H^2(0,T;H^2(\Omega))$ obeying $\tau_0 G = (g,u_0)$ and $\|G\|_{H^2(0,T;H^2(\Omega))} = \|(g,u_0)\|_{X_0}$. Then u is solution to (1.1) if and only if $u-G$ is solution to (2.8) with $f = \mathrm{i}\partial_t G + \Delta G - VG$ and $v_0 = u_0 - G(0,.)$. The result follows from this and Proposition 2.1. □

Armed with Corollary 2.1 we turn now to defining Λ_V. We preliminarily need to introduce the trace operator τ_1, defined as the linear bounded operator from $L^2((0,T)\times\mathbb{R}, H^2(\omega)) \cap H^1(0,T;L^2(\Omega))$ into $L^2(\Sigma) \times L^2(\Omega)$, which coincides with the mapping

$$w \mapsto \left(\partial_\nu w_{|\Sigma}, w(T,\cdot)\right) \text{ for } w \in C_0^\infty([0,T]\times\mathbb{R}, C^\infty(\overline{\omega})).$$

Evidently, we have $\|\tau_1 \mathfrak{s}(g,u_0)\|_{X_1} \leq C\|\mathfrak{s}(g,u_0)\|_Z \leq C\|(g,u_0)\|_{X_0}$, by (2.9), hence the linear operator

$$\Lambda_V = \tau_1 \circ \mathfrak{s} \in \mathcal{B}(X_0, X_1), \tag{2.10}$$

satisfies $\|\Lambda_V\| = \|\Lambda_V\|_{\mathcal{B}(X_0,X_1)} \leq C$, for some suitable positive constant C depending on $\|V\|_{C([0,T],W^{2,\infty}(\Omega))}$.

[1]Actually we shall see that $\mathfrak{s}(g,u_0)$ belongs to $C([0,T],H^2(\Omega)) \cap C^1([0,T],L^2(\Omega))$.

Remark 2.1. In light of [LM2, Section 4.2] and since $\overline{\Omega}$ is a smooth manifold with boundary $\partial\Omega$, we may as well define $\Lambda_V(g,u_0)$ in a similar way as before for all $u_0 \in H^2(\Omega)$ and all

$$g \in H^{3/2,3/2}(\Sigma) = L^2(0,T;H^{3/2}(\partial\Omega)) \cap H^{3/2}(0,T;L^2(\partial\Omega)),$$

fulfilling the compatibility conditions [LM2, Section 4, Eq. (2.47)-(2.48)]. Nevertheless there is no need to impose these conditions in our approach since they are automatically verified by any $(g,u_0) \in X_0$.

3. Direct and inverse fibered problems

In this section we reformulate the inverse problem under consideration as to whether the unknown function V can be determined from the knowledge of any fibered boundary operators $\Lambda_{V,\theta}$, $\theta \in [0,2\pi)$, defined in (1.10). As a preamble we first examine the direct problem associated with the fibered IBVP (1.9) and we establish that $\Lambda_{V,\theta}$ is linear bounded in appropriate functional spaces we shall make precise below.

3.1. **Analysis of the direct fibered problem.** Let θ be in $[0,2\pi)$. In accordance with the definition given in Section 6 of the functional spaces $\mathcal{H}^s_\theta(\check{\Omega})$ for $s \in [0,2]$, we put

$$\mathcal{H}^2_\theta(\check{\Omega}) = \{u \in H^2(\check{\Omega});\ u(1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}u(0,\cdot) \text{ and } \partial_{x_1}u(1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}\partial_{x_1}u(0,\cdot) \text{ on } \omega\}.$$

We denote by $\check{\tau}_0$ the linear bounded operator from $H^2(0,T;H^2(\check{\Omega}))$ into $L^2((0,T)\times(0,1),H^{3/2}(\partial\omega))\times L^2(\check{\Omega})$, such that

$$\check{\tau}_0 w = \left(w_{|\check{\Sigma}}, w(0,\cdot)\right) \text{ for } w \in C_0^\infty((0,T)\times(0,1),C^\infty(\overline{\omega})).$$

Then the space $\check{X}_{0,\theta} = \check{\tau}_0(H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})))$, endowed with the norm

$$\|w\|_{\check{X}_{0,\theta}} = \inf\{\|W\|_{H^2(0,T;H^2(\check{\Omega}))};\ W \in H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})) \text{ satisfies } \check{\tau}_0 W = w\},$$

is Hilbertian.

Having seen this, we may now prove the following existence and uniqueness result.

Lemma 3.1. *Let $V \in W^{2,\infty}(0,T;W^{2,\infty}(\check{\Omega}))$ fulfill (1.2) and satisfy $\|V\|_{W^2(0,T;W^{2,\infty}(\check{\Omega}))} \leq M$ for some $M>0$. Then for every $(h,v_0) \in \check{X}_{0,\theta}$, $\theta \in [0,2\pi)$, there exists a unique solution*

$$\mathfrak{s}_\theta(h,v_0) \in \check{Z}_\theta = L^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})) \cap H^1(0,T;L^2(\check{\Omega}))$$

to (1.9). Moreover, we may find a constant $C = C(T,\omega,M) > 0$ such that the estimate

$$\|\mathfrak{s}_\theta(h,v_0)\|_{\check{Z}_\theta} \leq C\|(h,v_0)\|_{\check{X}_{0,\theta}}, \tag{3.1}$$

holds for every $\theta \in [0,2\pi)$.

Proof. Let A_θ be the self-adjoint operator in $L^2(\check{\Omega})$ generated by the closed quadratic form

$$a_\theta(u) = \int_{\check{\Omega}} |\nabla u(x)|^2 \mathrm{d}x,\ u \in D(a_\theta) = L^2(0,1;H^1_0(\omega)) \cap \mathcal{H}^1_\theta(0,1;L^2(\omega)),$$

where $\mathcal{H}^1_\theta(0,1;L^2(\omega)) = \{u \in H^1(0,1;L^2(\omega));\ u(1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}u(0,\cdot) \text{ on } \omega\}$. Then A_θ acts as $(-\Delta)$ on its domain $D(A_\theta) = \mathcal{H}^2_\theta(\check{\Omega}) \cap L^2(0,1;H^1_0(\omega))$. This can be easily seen by arguing in the exact same way as in the derivation of Lemma 2.2. Let B denote the multiplier by $V \in C([0,T],W^{2,\infty}(\check{\Omega}))$. Due to (1.2) we have $B \in C([0,T],\mathcal{B}(D(A_\theta)))$ and $\|B\|_{C([0,T],\mathcal{B}(D(A_\theta)))} \leq \|V\|_{C([0,T],W^{2,\infty}(\check{\Omega}))}$. Therefore, by Lemma 2.1, for every $f \in W^{1,1}(0,T;L^2(\check{\Omega}))$ and $w_0 \in \mathcal{H}^2_\theta(\check{\Omega}) \cap L^2(0,1;H^1_0(\omega))$, there is a unique solution $w \in L^2(0,T;L^2(0,1;H^1_0(\omega)) \cap \mathcal{H}^2_\theta(\check{\Omega})) \cap H^1(0,T;L^2(\check{\Omega}))$ to the IBVP

$$\begin{cases} (-\mathrm{i}\partial_t - \Delta + V)w = f & \text{in } \check{Q}, \\ w(0,\cdot) = w_0 & \text{in } \check{\Omega}, \\ w = 0 & \text{on } \check{\Sigma}, \end{cases} \tag{3.2}$$

satisfying

$$\|w\|_{L^2(0,T;\mathcal{H}^2_\theta(\check{\Omega}))} + \|w\|_{H^1(0,T;L^2(\check{\Omega}))} \leq C\left(\|w_0\|_{H^2(\check{\Omega})} + \|f\|_{W^{1,1}(0,T;L^2(\check{\Omega}))}\right).$$

Further, from the very definition of $\check{X}_{0,\theta}$ we may find $W \in H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega}))$ such that $\check{\tau}_{0,\theta} W = (h, v_0)$ and $\|W\|_{H^2(0,T;H^2(\check{\Omega}))} = \|(h,v_0)\|_{\check{X}_{0,\theta}}$. Thus, taking $f = (\mathrm{i}\partial_t + \Delta - V)W$ and $w_0 = 0$ in (3.2), it is obvious that $w + W$ is solution to (1.9) if and only if w is solution to (3.2). This yields the desired result. $\square$

In virtue of Lemma 3.1 the linear operator $\mathfrak{s}_\theta$ is bounded from $\check{X}_{0,\theta}$ into $\check{Z}_\theta$, with

$$\|\mathfrak{s}_\theta\| = \|\mathfrak{s}_\theta\|_{\mathcal{B}(\check{X}_{0,\theta},\check{Z}_\theta)} \leq C,\ \theta \in [0, 2\pi).$$

Let $\check{\tau}_1$ be the linear bounded operator from $L^2((0,T)\times(0,1), H^2(\check{\Omega})) \cap H^1(0,T;L^2(\check{\Omega}))$ into $\check{X}_1 = L^2(\check{\Sigma}) \times L^2(\check{\Omega})$, obeying

$$\check{\tau}_1 w = \left(\partial_\nu w_{|\check{\Sigma}}, w(T,\cdot)\right) \text{ for } w \in C_0^\infty((0,T)\times(0,1), C^\infty(\overline{\omega})).$$

Then, for every $(h, v_0) \in \check{X}_{0,\theta}$, $\theta \in [0, 2\pi)$, we have

$$\|\check{\tau}_1 \mathfrak{s}_\theta(h,v_0)\|_{\check{X}_1} \leq C\|\mathfrak{s}_\theta(h,v_0)\|_{\check{Z}_\theta} \leq C\|(h,v_0)\|_{\check{X}_{0,\theta}},$$

by (3.1). Here and in the remaining part of this text, C denotes some suitable generic positive constant. Thus we may deduce from (1.10) that the reduced boundary operator

$$\Lambda_{V,\theta} = \check{\tau}_1 \circ \mathfrak{s}_\theta \in \mathcal{B}(\check{X}_{0,\theta}, \check{X}_1),\ \theta \in [0,2\pi). \tag{3.3}$$

Having seen this, we turn now to examining the inverse problem of determining V from the knowledge of $\Lambda_{V,\theta}$, where the real number θ is arbitrary in $[0, 2\pi)$.

3.2. **Stability estimate for the inverse fibered problem.** Much of the technical work displayed in this paper is devoted to the analysis of the identification of V by any arbitrary boundary operator of the family $\{\Lambda_{V,\theta},\ \theta \in [0,2\pi)\}$. With reference to (3.3), we shall actually derive the following inverse result.

Theorem 3.1. *Let M and V_j, $j = 1, 2$, be the same as in Theorem 1.1. Then we may find a constant $C > 0$, depending on T, ω and M, such that we have*

$$\|V_2 - V_1\|_{L^2(\check{Q})} \leq C\left(\|\Lambda_{V_2,\theta} - \Lambda_{V_1,\theta}\|_{\mathcal{B}(\check{X}_{0,\theta},\check{X}_1)} + \left|\ln\|\Lambda_{V_2,\theta} - \Lambda_{V_1,\theta}\|_{\mathcal{B}(\check{X}_{0,\theta},\check{X}_1)}\right|^{-\frac{1}{5}}\right), \tag{3.4}$$

for any $\theta \in [0, 2\pi)$.

In a similar fashion to Remark 1.1, we point out that (3.4) holds for $\|\Lambda_{V_2,\theta} - \Lambda_{V_1,\theta}\|_{\mathcal{B}(\check{X}_{0,\theta},\check{X}_1)} = 0$ as well, provided the function $\mathbb{R}_+^* \ni t \mapsto t + |\ln t|^{-1/5}$ is extended by continuity at $t = 0$. This guarantees that V is uniquely determined by the knowledge of $\Lambda_{V,\theta}$.

The proof of Theorem 3.1 is presented in Sections 4 and 5. Moreover, we establish in Proposition 6.2 (upon preliminarily decomposing the IBVP (1.1) into the collection of Cauchy problems (1.9), indexed by $\theta \in [0, 2\pi)$) that the boundary operator Λ_V, defined in (1.3), is unitarily equivalent to the direct integral with fibers $\{\Lambda_{V,\theta},\ \theta \in (0, 2\pi)\}$. The proof of this result being rather technical, we postpone it to Section 6.

With reference to [Di, Section II.2, Proposition 2], we deduce from the statement of Proposition 6.2, that

$$\|\Lambda_V\|_{\mathcal{B}(X_0,X_1)} = \sup_{\theta\in(0,2\pi)} \|\Lambda_{V,\theta}\|_{\mathcal{B}(\check{X}_{0,\theta},\check{X}_1)}. \tag{3.5}$$

Thus, putting (3.5) together with Theorem 3.1, we obtain that Theorem 1.1 is a byproduct of Theorem 3.1.

We may now focus on the derivation of Theorem 3.1 in Sections 4 and 5, and on the proof of Proposition 6.2 in Section 6.

4. Geometric optics solutions

Fix $\theta \in [0, 2\pi)$. This section is devoted to building geometric optics solutions to the system

$$\begin{cases} (-\mathrm{i}\partial_t - \Delta + V)v = 0 & \text{in } \check{Q}, \\ u(\cdot,1,\cdot) = \mathrm{e}^{\mathrm{i}\theta} u(\cdot,0,\cdot) & \text{on } (0,T)\times\omega, \\ \partial_{x_1} u(\cdot,1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}\partial_{x_1} u(\cdot,0,\cdot) & \text{on } (0,T)\times\omega. \end{cases} \tag{4.1}$$

Specifically, given $\xi \in \mathbb{C}^2 \setminus \mathbb{R}^2$ and $r > 0$, we seek solutions $u_{k,\theta}$, $k \in \mathbb{Z}$, to (4.1) of the form

$$u_{k,\theta}(t,x) = \left(\mathrm{e}^{\mathrm{i}\theta x_1} + w_{k,\theta}(t,x)\right) \mathrm{e}^{-\mathrm{i}\left((\xi\cdot\xi+4\pi^2k^2)t + x'\cdot\xi + 2\pi k x_1\right)},\ x = (x_1, x') \in \check{\Omega},$$

where $w_{k,\theta} \in H^2(0,T;\mathcal{H}_\theta^2(\check{\Omega}))$ scales at best like the inverse of the above (large) parameter r, i.e.

$$\|w_{k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))} \leq \frac{c}{r}(1+|k|),$$

for some constant $c > 0$ which is independent of r, k and θ. The main issue here is the quasi-periodic condition imposed on $w_{k,\theta}$. To overcome this problem we shall adapt the framework introduced in [Ha] for defining geometric optics solutions in periodic media.

4.1. **Geometric optics solutions in periodic media.** Fix $R > 0$ and put $\mathcal{O} = (-R,R)\times(0,1)\times(-R,R)^2$. We recall that $u \in H^1_{loc}(\mathbb{R}^4)$ is $\mathcal{O}$-periodic if it satisfies

$$u(y + 2R\mathcal{E}_j) = u(y),\ j = 0,2,3,\ \text{and}\ u(y+\mathcal{E}_1) = u(y),\ \text{a.e. } y = t\mathcal{E}_0 + \sum_{j=1}^{3} x_j \mathcal{E}_j \in \mathcal{O},$$

where $\{\mathcal{E}_j\}_{j=0}^3$ denotes the canonical basis of $\mathbb{R}^4$. We note $H^1_{\mathrm{per}}(\mathcal{O})$ the subset of $\mathcal{O}$-periodic functions in $H^1_{\mathrm{loc}}(\mathbb{R}^4)$, endowed with the scalar product of $H^1(\mathcal{O})$. Similarly we define $H^2_{per}(\mathcal{O}) = \{u \in H^1_{\mathrm{per}}(\mathcal{O}),\ \partial_k u \in H^1_{\mathrm{per}}(\mathcal{O}),\ k = 0,1,2,3\}$.

Further we introduce the space

$$\mathscr{H}_\theta = \left\{ \mathrm{e}^{\mathrm{i}\theta x_1} \mathrm{e}^{\mathrm{i}\frac{\pi x_2}{2R}} u;\ u \in H^2_{\mathrm{loc}}(\mathbb{R}_t; H^2_{\mathrm{loc}}(\mathbb{R}^3)) \cap H^2_{per}(\mathcal{O}) \right\},\ \theta \in [0,2\pi),$$

which is Hilbertian for the natural norm of $\mathscr{H}^2 = H^2(-R,R;H^2((0,1)\times(-R,R)^2))$, and mimic the proof of [Ha, Theorem 1] or [Cho, Proposition 2.19] to claim the coming technical result.

Lemma 4.1. *Let $s > 0$, let $\kappa \in \mathbb{R}^4$ be such that $\kappa \cdot \mathcal{E}_2 = 0$, and set $\vartheta = s\mathcal{E}_2 + i\kappa$. Then for every $h \in \mathscr{H}^2$ the equation*

$$-\mathrm{i}\partial_t \psi - \Delta\psi + 2\vartheta\cdot\nabla\psi = h \text{ in } \mathcal{O}, \tag{4.2}$$

admits a unique solution $\psi \in \mathscr{H}_\theta$. Moreover, it holds true that

$$\|\psi\|_{\mathscr{H}^2} \leq \frac{R}{s\pi}\|h\|_{\mathscr{H}^2}. \tag{4.3}$$

Proof. For all $\alpha \in \mathbb{Z}_\theta = \theta\mathcal{E}_1 + \frac{\pi}{2R}\mathcal{E}_2 + \left(\frac{\pi}{R}\mathbb{Z}\right)\times\mathbb{Z}\times\left(\frac{\pi}{R}\mathbb{Z}\right)^2$, put

$$\phi_\alpha(y) = \frac{1}{(2R)^{\frac{3}{2}}}\mathrm{e}^{\mathrm{i}\alpha\cdot y},\ y = (t,x) \in \mathcal{O},$$

in such a way that $\{\phi_\alpha\}_{\alpha\in\mathbb{Z}_\theta}$ is a Hilbert basis of $L^2(\mathcal{O})$. Assume that $\psi \in \mathscr{H}_\theta$ is solution to (4.2). Then for each $\alpha \in \mathbb{Z}_\theta$ it holds true that $\langle h, \phi_\alpha\rangle_{L^2(\mathcal{O})} = \langle -\mathrm{i}\partial_t\psi - \Delta\psi + 2\vartheta\cdot\nabla\psi, \phi_\alpha\rangle_{L^2(\mathcal{O})}$ whence

$$\begin{aligned}\langle h, \phi_\alpha\rangle_{L^2(\mathcal{O})} &= \langle \psi, -i\partial_t\phi_\alpha - \Delta\phi_\alpha - 2\overline{\vartheta}\cdot\nabla\phi_\alpha\rangle_{L^2(\mathcal{O})} \\ &= \left(\alpha_0 + \sum_{j=1}^{3}\alpha_j^2 - 2\kappa\cdot\alpha + 2is\alpha_2\right)\langle\psi,\phi_\alpha\rangle_{L^2(\mathcal{O})},\end{aligned}$$

by integrating by parts, with

$$\left|\Im\left(\alpha_0 + \sum_{j=1}^{3}\alpha_j^2 - 2\kappa\cdot\alpha + 2is\alpha_2\right)\right| = 2s|\alpha_2| \geq \frac{s\pi}{R}. \tag{4.4}$$

Therefore we necessarily have

$$\psi = \sum_{\alpha\in Z_\theta} \frac{\langle h,\phi_\alpha\rangle_{L^2(\mathcal{O})}}{\alpha_0 + \sum_{j=1}^{3}\alpha_j^2 - 2\kappa\cdot\alpha + 2is\alpha_2}\phi_\alpha. \tag{4.5}$$

On the other hand the function ψ defined by the right hand side of (4.5) is in $\mathscr{H}^2$ since

$$\begin{aligned}\|\psi\|^2_{\mathscr{H}^2} &= \sum_{\alpha\in\mathbb{Z}_\theta} \frac{(1+\alpha_0^2+\alpha_0^4)\left(\sum_{1\leq j,l\leq 3}\alpha_j^2\alpha_l^2\right)|\langle h,\phi_\alpha\rangle_{L^2(\mathcal{O})}|^2}{\left|\alpha_0+\sum_{j=1}^3\alpha_j^2-2\kappa\cdot\alpha+2is\alpha_2\right|^2}\\ &\leq \frac{R^2}{s^2\pi^2}\sum_{\alpha\in\mathbb{Z}_\theta}(1+\alpha_0^2+\alpha_0^4)\left(\sum_{1\leq j,l\leq 3}\alpha_j^2\alpha_l^2\right)|\langle h,\phi_\alpha\rangle_{L^2(\mathcal{O})}|^2<+\infty.\end{aligned}$$

Here we used the fact that the last sum over $\alpha\in\mathbb{Z}_\theta$ is equal to $\|h\|^2_{\mathscr{H}^2}$, which incidentally entails (4.3). Finally the trace operators $w\mapsto\left(\partial_{x_1}^m w\right)_{|[-R,R]\times\{0,1\}\times[-R,R]^2}$ being continuous on $\mathscr{H}^2$ for $m=0,1$, we end up getting that $\psi\in\mathscr{H}_\theta$. $\square$

Remark 4.1. It should be noticed that in contrast to [Ha, Theorem 1] where the fundamental H^2-solutions ψ to the Faddeev-type equation are obtained from any L^2-right hand side h, it is actually required in Lemma 4.1 that h be taken in $\mathscr{H}^2$. This boils down to the fact that the elliptic regularity of the Faddeev equation does not hold for the Schrödinger equation (4.2).

4.2. **Building quasi-periodic geometric optics solutions.** We first deduce from Lemma 4.1 the:

Lemma 4.2. *Let $\xi\in\mathbb{C}^2\setminus\mathbb{R}^2$ verify*

$$\Im\xi\cdot\Re\xi=0. \tag{4.6}$$

Then, for all $\theta\in[0,2\pi)$ and $k\in\mathbb{Z}$, there exists $E_{k,\theta}\in\mathcal{B}(H^2(0,T;H^2(\check{\Omega})),H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})))$ such that $\varphi=E_{k,\theta}f$, where $f\in H^2(0,T;H^2(\check{\Omega}))$, is solution to the equation

$$(-\mathrm{i}\partial_t-\Delta+4i\pi k\partial_{x_1}+2\mathrm{i}\xi\cdot\nabla_{x'})\varphi=f \text{ in } \check{Q}. \tag{4.7}$$

Moreover, we have

$$\|E_{k,\theta}\|_{\mathcal{B}(H^2(0,T;H^2(\check{\Omega})))}\leq\frac{c_0}{|\Im\xi|}, \tag{4.8}$$

for some constant $c_0>0$ independent of ξ, k and θ.

Proof. Pick $R>0$ so large that any planar rotation around the origin of $\mathbb{R}^2$ maps ω into $(-R,R)^2$. Next, bearing in mind that $r=|\Im\xi|>0$, we call S the unique planar rotation around $0_{\mathbb{R}^2}\in\omega$, mapping the second vector $\mathfrak{e}_2$ in the canonical basis of $\mathbb{R}^2$ onto $(-\Im\xi)/r$:

$$S\mathfrak{e}_2=-\frac{\Im\xi}{r}. \tag{4.9}$$

Further, pick $f\in H^2(0,T;H^2(\check{\Omega}))$, and put

$$\tilde{f}(t,x_1,x')=f(t,x_1,S^*x'),\ (t,x_1,x')\in(0,T)\times(0,1)\times S\omega, \tag{4.10}$$

where S^* denotes the inverse transformation to S. Evidently, $\tilde{f}\in H^2(0,T;H^2((0,1)\times S\omega))$. Moreover, as $\partial\omega$ is C^2, there exists

$$P\in\mathcal{B}(H^2(0,T;H^2((0,1)\times S\omega)),H^2(\mathbb{R};H^2((0,1)\times\mathbb{R}^2))),$$

such that $(P\tilde{f})_{|(0,T)\times(0,1)\times S\omega}=\tilde{f}$, by [LM1, Section 1, Theorems 2.2 & 8.1]. Let $\chi=\chi(t,x')\in C_0^\infty((-R,R)^3)$ fulfill $\chi=1$ in a neighborhood of $[0,T]\times\overline{S\omega}$. Then the function

$$h(t,x_1,x')=\chi(t,x')(P\tilde{f})(t,x_1,x'),\ (t,x_1,x')\in\mathcal{O}, \tag{4.11}$$

belongs to $\mathscr{H}^2$. Moreover, it holds true that $h_{|(0,T)\times(0,1)\times S\omega}=\tilde{f}$.

The next step of the proof is to choose $\kappa=(0,2\pi k,S^*\Re\xi)\in\mathbb{R}^4$ so we get

$$\kappa\cdot\mathcal{E}_2=S^*\Re\xi\cdot\mathfrak{e}_2=\Re\xi\cdot S\mathfrak{e}_2=-\frac{\Re\xi\cdot\Im\xi}{r}=0,$$

by combining (4.6) with (4.9). We call ψ the $\mathscr{H}_\theta$-solution to (4.2) obtained by applying Lemma 4.1 with $\vartheta = r\mathcal{E}_2 + i\kappa$ and h given by (4.9)–(4.11), and put

$$(E_{k,\theta} f)(t,x_1,x') = \psi(t,x_1,Sx'),\ (t,x_1,x') \in \check{Q}. \tag{4.12}$$

Obviously, $E_{k,\theta} f \in H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega}))$ and (4.3) yields

$$\|E_{k,\theta} f\|_{H^2(0,T;H^2(\check{\Omega}))} \leq C\|\psi\|_{\mathscr{H}^2} \leq \frac{CR}{r\pi}\|h\|_{\mathscr{H}^2}, \tag{4.13}$$

Furthermore, in light of (4.10)-(4.11) we have

$$\|h\|_{\mathscr{H}^2} \leq \|P\tilde{f}\|_{H^2(\mathbb{R};H^2(\mathbb{R}^2\times(0,1)))} \leq \|P\|\|\tilde{f}\|_{H^2(0,T;H^2((0,1)\times S\omega))} \leq C\|f\|_{H^2(0,T;H^2(\check{\Omega}))},$$

where $\|P\|$ stands for the norm of P in the space of linear bounded operators acting from $H^2(0,T;H^2((0,1)\times S\omega))$ into $H^2(\mathbb{R};H^2((0,1)\times\mathbb{R}^2))$. Putting this together with (4.13), we end up getting (4.8).

This being said, it remains to show that $\varphi = E_{k,\theta} f$ is solution to (4.7). To see this we notice from (4.12) that $\varphi = \psi \circ F$, where F is the unitary transform $(t,x_1,x') \mapsto (t,x_1,Sx')$ in $\mathbb{R}^4$. As a consequence we have $\nabla\varphi = F\nabla\psi \circ F$, whence

$$\vartheta \cdot \nabla\psi \circ F = F\vartheta \cdot \nabla\varphi = \mathrm{i}2\pi k\partial_{x_1}\varphi + \mathrm{i}\xi \cdot \nabla_{x'}\varphi, \tag{4.14}$$

and

$$-\Delta\varphi = -\nabla\cdot\nabla\varphi = -F\nabla\cdot F\nabla\psi \circ F = -\nabla\cdot\nabla\psi \circ F = -\Delta\psi \circ F. \tag{4.15}$$

Moreover, we have $h \circ F = \tilde{f} \circ F = f$ in $\check{Q}$, directly from (4.10)-(4.11), and $\partial_t\varphi = \partial_t\psi \circ F$, so (4.7) follows readily from this, (4.2) and (4.14)-(4.15). □

Armed with Lemma 4.2 we are now in position to establish the main result of this section.

Proposition 4.1. *Assume that* $V \in W^{2,\infty}(0,T;W^{2,\infty}(\Omega))$ *satisfies* (1.2). *Pick*

$$r \geq r_0 = 2c_0\|V\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega))},$$

where c_0 *is the same as in* (4.8), *and let* $\xi \in \mathbb{C}^2 \setminus \mathbb{R}^2$ *fulfill* (4.6) *and* $|\Im\xi| = r$. *Then for all* $\theta \in [0,2\pi)$ *and* $k \in \mathbb{Z}$, *there exists* $w_{k,\theta} \in H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega}))$ *obeying*

$$\|w_{k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))} \leq \frac{c}{r}\left(1 + \|V\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega))}\right)(1+|k|), \tag{4.16}$$

for some [2]*constant* $c > 0$, *independent of* r, k *and* θ, *such that the function*

$$u_{k,\theta}(t,x) = \left(\mathrm{e}^{\mathrm{i}\theta x_1} + w_{k,\theta}(t,x)\right)\mathrm{e}^{-\mathrm{i}\left((\xi\cdot\xi + 4\pi^2k^2)t + 2\pi k x_1 + x'\cdot\xi\right)},\ (t,x) = (t,x_1,x') \in \check{Q}, \tag{4.17}$$

is a $H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega}))$*-solution to the equation* (4.1).

Proof. A direct calculation shows that $u_{k,\theta}$ fulfills (4.1) if and only if $w_{k,\theta}$ is solution to

$$\left\{\begin{array}{ll} (-\mathrm{i}\partial_t - \Delta + 4\mathrm{i}\pi k\partial_{x_1} + 2i\xi\cdot\nabla_{x'} + V)w + \mathrm{e}^{\mathrm{i}\theta x_1}W_{k,\theta} = 0 & \text{in } \check{Q}, \\ w(\cdot,1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}w(\cdot,0,\cdot) & \text{on } (0,T)\times\omega, \\ \partial_{x_1}w(\cdot,1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}\partial_{x_1}w(\cdot,0,\cdot) & \text{on } (0,T)\times\omega, \end{array}\right. \tag{4.18}$$

with

$$W_{k,\theta} = V + \theta^2 - 4\pi k\theta. \tag{4.19}$$

In light of (4.18)-(4.19) we introduce the map

$$\begin{array}{rccc} G_{k,\theta}: & H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})) & \longrightarrow & H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})) \\ & q & \longmapsto & -E_{k,\theta}\left(Vq + \mathrm{e}^{\mathrm{i}\theta x_1}W_{k,\theta}\right), \end{array}$$

set

$$M_0 = 12\pi^2(3T|\omega|)^{1/2}(4\pi^2 + \|V\| + 8\pi^2|k|), \tag{4.20}$$

[2]Actually c depends only on T and $|\omega|$.

where $\|V\|$ is a shorthand for $\|V\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega))}$, and notice that

$$\left\|\mathrm{e}^{\mathrm{i}\theta x_1}W_{k,\theta}\right\|_{H^2(0,T;H^2(\check{\Omega}))}\leq M_0. \tag{4.21}$$

Then we have

$$\|G_{k,\theta}q\|_{H^2(0,T;H^2(\check{\Omega}))}\leq \frac{c_0}{r}\left(\|V\|\|q\|_{H^2(0,T;H^2(\check{\Omega}))}+M_0\right),\ q\in H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})),$$

in virtue of (4.8) and (4.21) . From this and the condition $r\geq r_0$ then follows that $\|G_{k,\theta}q\|_{H^2(0,T;H^2(\check{\Omega}))}\leq M_0$ for all q taken in the ball B_{M_0} centered at the origin with radius M_0 in $H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega}))$. Moreover, it holds true that

$$\|G_{k,\theta}q-G_{k,\theta}\tilde{q}\|_{H^2(0,T;H^2(\check{\Omega}))}\leq \frac{\|q-\tilde{q}\|_{H^2(0,T;H^2(\check{\Omega}))}}{2},\ q,\tilde{q}\in B_{M_0},$$

hence $G_{k,\theta}$ has a unique fixed point $w_{k,\theta}\in H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega}))$. Further, by applying Lemma 4.2 with

$$f=-\left(Vw_{k,\theta}+\mathrm{e}^{\mathrm{i}\theta x_1}W_{k,\theta}\right)\in H^2(0,T;H^2(\check{\Omega})),$$

we deduce from (4.7) that $w_{k,\theta}=E_{k,\theta}f$ is a solution to (4.18). Last, taking into account the identity $\|w_{k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))}=\|G_{k,\theta}w_{k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))}$, we get that

$$\begin{aligned}\|w_{k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))} &\leq \|G_{k,\theta}w_{k,\theta}-G_{k,\theta}\underline{0}\|_{H^2(0,T;H^2(\check{\Omega}))}+\|G_{k,\theta}\underline{0}\|_{H^2(0,T;H^2(\check{\Omega}))}\\ &\leq \|E_{k,\theta}(Vw_{k,\theta})\|_{H^2(0,T;H^2(\check{\Omega}))}+\left\|E_{k,\theta}\left(\mathrm{e}^{\mathrm{i}\theta x_1}W_{k,\theta}\right)\right\|_{H^2(0,T;H^2(\check{\Omega}))}\\ &\leq \frac{c_0}{r}\left(\|V\|\|w_{k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))}+M_0\right),\end{aligned}$$

directly from (4.8) and (4.21), where $\underline{0}$ denotes the function which is identically zero in $\check{\Omega}$. Hence we have $\|w_{k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))}\leq (2c_0/r)M_0$, by the hypothesis $r\geq r_0$, and the estimate $M_0\leq c(1+\|V\|)(1+|k|)$, arising from (4.20), entails (4.16). $\square$

5. Stability estimate

This section contains the proof of Theorem 3.1. We start by establishing two auxiliary results.

5.1. Auxiliary results.

In view of deriving Lemma 5.2 from Proposition 4.1, we first prove the following technical result.

Lemma 5.1. *For all $r>0$ and $\zeta=(\eta,\ell)\in\mathbb{R}^2\times\mathbb{R}$ with $\eta\neq 0_{\mathbb{R}^2}$, there exists $\zeta_j=\zeta_j(r,\eta,\ell)=(\xi_j,\tau_j)\in\mathbb{C}^2\times\mathbb{R}$, $j=1,2$, such that we have*

$$|\Im\xi_j|=r,\quad \tau_j=\xi_j\cdot\xi_j,\quad \zeta_1-\overline{\zeta_2}=\zeta,\ \Re\xi_j\cdot\Im\xi_j=0, \tag{5.1}$$

and

$$|\xi_j|\leq\frac{1}{2}\left(|\eta|+\frac{|\ell|}{|\eta|}\right)+r,\ |\tau_j|\leq|\eta|^2+\frac{\ell^2}{|\eta|^2}+2r^2. \tag{5.2}$$

Proof. Let $\eta^\perp$ be any non zero $\mathbb{R}^2$-vector, orthogonal to η and put $\eta_r^\perp=r\eta^\perp/|\eta^\perp|$. Then, a direct calculation shows that

$$\xi_j=\frac{1}{2}\left((-1)^{j+1}+\frac{\ell}{|\eta|^2}\right)\eta+(-1)^j i\eta_r^\perp,\ \tau_j=\frac{1}{4}\left((-1)^{j+1}+\frac{\ell}{|\eta|^2}\right)^2|\eta|^2-r^2,\ j=1,2,$$

fulfill (5.1)-(5.2). $\square$

In light of Proposition 4.1 and Lemma 5.1 we may now derive the following:

Lemma 5.2. *Let $V_j \in W^{2,\infty}(0,T;W^{2,\infty}(\Omega))$, $j=1,2$, fulfill the condition (1.2). Assume that $\max_{j=1,2}\|V_j\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega))} \leq M$ for some $M>0$ and fix $r \geq r_0 = 2c_0M > 0$, where c_0 is the same as in (4.8). Pick $\zeta = (\eta,\ell) \in \mathbb{R}^2 \times \mathbb{R}$ with $\eta \neq 0_{\mathbb{R}^2}$, and let $\zeta_j = (\xi_j,\tau_j) \in \mathbb{C}^2 \times \mathbb{R}$, $j=1,2$, be given by Lemma 5.1. Then, there is a constant $C>0$ depending only on T, $|\omega|$ and M, such that for every $j=1,2$, the function $u_{j,k,\theta}$, defined in Proposition 4.1 by substituting ξ_j for ξ, satisfies the estimate*

$$\|u_{j,k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))} \leq C(1+\mathfrak{q}(\zeta,k))^{\frac{13}{2}}\frac{(1+r^2)^3}{r}\mathrm{e}^{|\omega|r},\ k \in \mathbb{Z},\ \theta \in [0,2\pi),\ r \geq r_0,$$

with

$$\mathfrak{q}(\zeta,k) = \mathfrak{q}(\eta,\ell,k) = |\eta|^2 + \frac{|\ell|}{|\eta|} + k^2.$$

Proof. In light of (4.17) we have

$$\begin{aligned}&\|u_{j,k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))}\\ \leq\ & \left(\|\mathrm{e}^{\mathrm{i}\theta x_1}\|_{H^2(0,T;H^2(\check{\Omega}))} + \|w_{j,k,\theta}\|_{H^2(0,T;H^2(\check{\Omega}))}\right)\|\mathrm{e}^{-\mathrm{i}\left((\tau_j+4\pi^2k^2)t+2\pi kx_1+x'\cdot\xi_j\right)}\|_{W^{2,\infty}(0,T;W^{2,\infty}(\check{\Omega}))},\end{aligned}$$

with

$$\|\mathrm{e}^{-\mathrm{i}\left((\tau_j+4\pi^2k^2)t+2\pi kx_1+x'\cdot\xi_j\right)}\|_{W^{2,\infty}(0,T;W^{2,\infty}(\check{\Omega}))} \leq (1+|\tau_j|+4\pi^2k^2)^2(1+|\xi_j|^2+4\pi^2k^2)\mathrm{e}^{|\omega|r},$$

and

$$\|\mathrm{e}^{\mathrm{i}\theta x_1}\|_{H^2(0,T;H^2(\check{\Omega}))} \leq c(T|\omega|)^{1/2},$$

for some positive constant c which is independent of r, θ, ζ, k, T and ω. Thus we get the desired result by combining the three above inequalities with (4.16) and (5.2). $\square$

5.2. **Proof of Theorem 3.1.** Let $\zeta = (\eta,\ell)$, r and $\zeta_j = (\xi_j,\tau_j)$, $j=1,2$, be as in Lemma 5.2, fix $k \in \mathbb{Z}$, and put

$$(k_1,k_2) = \begin{cases} (k/2,-k/2) & \text{if } k \text{ is even}\\ ((k+1)/2,-(k-1)/2) & \text{if } k \text{ is odd.}\end{cases}$$

Further we pick $\theta \in [0,2\pi)$ and note u_j, $j=1,2$, the geometric optics solution $u_{j,k_j,\theta}$, defined by Lemma 5.2. In light of Lemma 3.1 there is a unique solution $v \in L^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})) \cap H^1(0,T;L^2(\check{\Omega}))$ to the boundary value problem

$$\begin{cases}(-\mathrm{i}\partial_t+\Delta+V_2)v = 0 & \text{in } \check{Q}\\ v(0,\cdot) = u_1(0,\cdot) & \text{in } \check{\Omega},\\ v = u_1 & \text{on } \check{\Sigma},\end{cases} \tag{5.3}$$

in such a way that $u = v - u_1$ is solution to the following system:

$$\begin{cases}(-\mathrm{i}\partial_t+\Delta+V_2)u = (V_1-V_2)u_1 & \text{in } \check{Q}\\ u(0,\cdot) = 0 & \text{in } \check{\Omega},\\ u = 0 & \text{on } \check{\Sigma},\\ u(\cdot,1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}u(\cdot,0,\cdot) & \text{on } (0,T)\times\omega\\ \partial_{x_1}u(\cdot,1,\cdot) = \mathrm{e}^{\mathrm{i}\theta}\partial_{x_1}u(\cdot,0,\cdot) & \text{on } (0,T)\times\omega.\end{cases} \tag{5.4}$$

Therefore we get

$$\int_{\check{Q}}(V_1-V_2)u_1\overline{u_2}\mathrm{d}t\mathrm{d}x = \int_{\check{\Sigma}}\partial_\nu u\overline{u_2}\mathrm{d}t\mathrm{d}\sigma(x) - i\int_{\check{\Omega}}u(T,\cdot)\overline{u_2(T,\cdot)}\mathrm{d}x, \tag{5.5}$$

by integrating by parts and taking into account the quasi-periodic boundary conditions satisfied by u and u_2. Notice from (5.3)-(5.4) that $\partial_\nu u = \left(\Lambda^1_{V_2,\theta}-\Lambda^1_{V_1,\theta}\right)(\mathfrak{g}_1)$ and $u(T,.) = \left(\Lambda^2_{V_2,\theta}-\Lambda^2_{V_1,\theta}\right)(\mathfrak{g}_1)$, where $\mathfrak{g}_1 = \left(u_{1|\check{\Sigma}},u_1(0,.)\right) \in \check{X}_{0,\theta}$ and

$$\Lambda^1_{V_j,\theta}(h,v_0) = \partial_\nu v_{j|\check{\Sigma}},\ \Lambda^2_{V_j,\theta}(h,v_0) = v_j(T,.),\ (h,v_0) \in X_{0,\theta},\ j=1,2,$$

where v_j solves (1.9) with $V = V_j$.

Thus, putting

$$\beta_k = \begin{cases} 0 & \text{if } k \text{ is even or } k \in \mathbb{R} \setminus \mathbb{Z} \\ 4\pi^2 & \text{if } k \text{ is odd,} \end{cases}$$

for all $k \in \mathbb{Z}$, and

$$\varrho = \varrho_{k,\theta} = \mathrm{e}^{-\mathrm{i}\theta x_1} w_1 + \mathrm{e}^{\mathrm{i}\theta x_1} \overline{w_2} + w_1 \overline{w_2}, \tag{5.6}$$

we deduce from (4.17), (5.1) and (5.5) that

$$\int_{\check{Q}} (V_1 - V_2) \mathrm{e}^{-\mathrm{i}\left((\ell+\beta_k k)t + 2\pi k x_1 + x'\cdot\eta\right)} \mathrm{d}t\mathrm{d}x = A + B + C, \tag{5.7}$$

with

$$A = -\int_{\check{Q}} (V_2 - V_1)\varrho(t,x) \mathrm{e}^{-\mathrm{i}\left((\ell+\beta_k k)t + 2\pi k x_1 + x'\cdot\eta\right)} \mathrm{d}t\mathrm{d}x, \tag{5.8}$$

$$B = \int_{\check{\Sigma}} \left(\Lambda^1_{V_2,\theta} - \Lambda^1_{V_1,\theta}\right)(\mathfrak{g}_1)\overline{u_2}\, \mathrm{d}t\mathrm{d}\sigma(x), \tag{5.9}$$

$$C = -i \int_{\check{\Omega}} \left(\Lambda^2_{V_2,\theta} - \Lambda^2_{V_1,\theta}\right)(\mathfrak{g}_1)\overline{u_2(T,\cdot)}\, \mathrm{d}x. \tag{5.10}$$

Next, we introduce

$$V(t,x) = \begin{cases} (V_2 - V_1)(t,x) & \text{if } (t,x) \in Q, \\ 0 & \text{if } (t,x) \in \mathbb{R}^4 \setminus Q, \end{cases}$$

and

$$\phi_k(x_1) = \mathrm{e}^{\mathrm{i}2\pi k x_1},\ x_1 \in \mathbb{R},\ k \in \mathbb{Z},$$

so the left hand side of (5.7) can be rewritten as

$$\int_{\check{Q}} (V_1 - V_2) \mathrm{e}^{-\mathrm{i}\left((\ell+\beta_k k)t + 2\pi k x_1 + x'\cdot\eta\right)} \mathrm{d}t\mathrm{d}x = \left\langle \widehat{V}(\ell+\beta_k k, \eta), \phi_k \right\rangle_{L^2(0,1)}, \tag{5.11}$$

where $\widehat{V}$ stands for the partial Fourier transform of V with respect to $t \in \mathbb{R}$ and $x' \in \mathbb{R}^2$. Further, in light of (4.16) and (5.6) it holds true that

$$\begin{aligned} \|\varrho\|_{L^1(\check{Q})} &\leq (T|\omega|)^{1/2} \left(\|w_1\|_{L^2(\check{Q})} + \|w_2\|_{L^2(\check{Q})} \right) + \|w_1\|_{L^2(\check{Q})} \|w_2\|_{L^2(\check{Q})} \\ &\leq \frac{C}{r} \left((T|\omega|)^{1/2}(2 + |k_1| + |k_2|) + \frac{C}{r}(1+|k_1|)(1+|k_2|) \right) \\ &\leq c' \frac{(1+|k|)^2}{r}, \end{aligned}$$

where the constant $c' > 0$ depends only on T, $|\omega|$ and M. Since $\|V_1 - V_2\|_\infty \leq 2M$, it follows from this and (5.8) upon substituting c' for $4Mc'$ in the above estimate that

$$|A| \leq \|V_1 - V_2\|_\infty \|\varrho\|_{L^1(\check{Q})} \leq c' \frac{(1+\mathfrak{q}(\zeta,k))}{r}, \tag{5.12}$$

where $\mathfrak{q}$ is defined in Lemma 5.2. Moreover, we have

$$|B| \leq \|(\Lambda^1_{V_2,\theta} - \Lambda^1_{V_1,\theta})(\mathfrak{g}_1)\|_{L^2(\check{\Sigma})} \|u_2\|_{L^2(\check{\Sigma})},$$

by (5.9) and

$$|C| \leq \|(\Lambda^2_{V_2,\theta} - \Lambda^2_{V_1,\theta})(\mathfrak{g}_1)\|_{L^2(\check{\Omega})} \|u_2\|_{L^2(\check{\Omega})},$$

from (5.10), whence

$$
\begin{aligned}
|B|+|C| &\leq \left(\|(\Lambda^1_{V_2,\theta}-\Lambda^1_{V_1,\theta})(\mathfrak{g}_1)\|^2_{L^2(\check{\Sigma})}+\|(\Lambda^2_{V_2,\theta}-\Lambda^2_{V_1,\theta})(\mathfrak{g}_1)\|^2_{L^2(\check{\Omega})}\right)^{1/2} \\
&\quad \times\left(\|u_2\|^2_{L^2(\check{\Sigma})}+\|u_2(T,.)\|^2_{L^2(\check{\Omega})}\right)^{1/2} \\
&\leq \|(\Lambda_{V_2,\theta}-\Lambda_{V_1,\theta})(\mathfrak{g}_1)\|_{L^2(\check{\Sigma})\times L^2(\check{\Omega})}\,\|\mathfrak{g}_2\|_{L^2(\check{\Sigma})\times L^2(\check{\Omega})} \\
&\leq \|\Lambda_{V_2,\theta}-\Lambda_{V_1,\theta}\|_{\mathcal{B}(\check{X}_{0,\theta},\check{X}_1)}\|\mathfrak{g}_1\|_{\check{X}_{0,\theta}}\|\|\mathfrak{g}_2\|_{\check{X}_1},
\end{aligned} \tag{5.13}
$$

where we note

$$
\mathfrak{g}_2=\left(u_{2|\check{\Sigma}},u_2(T,.)\right).
$$

The next step is to use that $\|\mathfrak{g}_1\|_{\check{X}_{0,\theta}}$ and $\|\mathfrak{g}_2\|_{\check{X}_{0,\theta}}$ are both upper bounded, up to some multiplicative constant depending only on T and ω, by $\|u_j\|_{H^2(0,T;H^2(\check{\Omega}))}$. Therefore (5.13) and Lemma 5.2 yield

$$
|B|+|C|\leq C^2\|\Lambda^1_{V_2,\theta}-\Lambda^1_{V_1,\theta}\|_{\mathcal{B}(\check{X}_{0,\theta},\check{X}_1)}(1+\mathfrak{q}(\zeta,k))^{13}\frac{(1+r^2)^6}{r^2}\mathrm{e}^{2|\omega|r},\ r\geq r_0. \tag{5.14}
$$

Now, putting (5.7)–(5.12) and (5.14) together, we end up getting that

$$
\left|\left\langle \widehat{V}(\ell+\beta_k k,\eta),\phi_k\right\rangle_{L^2(0,1)}\right|\leq c''\frac{(1+\mathfrak{q}(\zeta,k))}{r^2}\left(r+\gamma(1+\mathfrak{q}(\zeta,k))^{12}(1+r^2)^6\mathrm{e}^{2|\omega|r}\right),\ r\geq r_0, \tag{5.15}
$$

where

$$
\gamma=\|\Lambda_{V_2,\theta}-\Lambda_{V_1,\theta}\|_{\mathcal{B}(\check{X}_{0,\theta},\check{X}_1)}
$$

and the constant $c''>0$ is independent of k, r and $\zeta=(\eta,\ell)$.

The next step is to apply Parseval-Plancherel theorem, getting

$$
\|V_2-V_1\|^2_{L^2(\check{Q})}=\|V\|^2_{L^2(\mathbb{R}\times(0,1)\times\mathbb{R}^2)}=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^3}|\hat{\mathfrak{v}}(\zeta,k)|^2\mathrm{d}\zeta, \tag{5.16}
$$

where $\hat{\mathfrak{v}}(\zeta,k)=\hat{\mathfrak{v}}(\ell,\eta,k)$ stands for $\langle\widehat{V}(\ell,\eta),\phi_k\rangle_{L^2(0,1)}$ for all $(\zeta,k)\in\mathbb{R}^3\times\mathbb{Z}$. By splitting $\int_{\mathbb{R}^3}|\hat{\mathfrak{v}}(\zeta,k)|^2\mathrm{d}\zeta$, $k\in\mathbb{Z}$, into the sum $\int_{\mathbb{R}^3}|\hat{\mathfrak{v}}(\ell,\eta,2k)|^2\mathrm{d}\ell\mathrm{d}\eta+\int_{\mathbb{R}^3}|\hat{\mathfrak{v}}(\ell,\eta,2k+1)|^2d\ell d\eta$ and performing the change of variable $\ell'=\ell-(2k+1)$ in the last integral, we may actually rewrite (5.16) as

$$
\|V_2-V_1\|^2_{L^2(\check{Q})}=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^3}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta=\int_{\mathbb{R}^4}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k), \tag{5.17}
$$

where $\mu=\sum_{n\in\mathbb{Z}}\delta_n$. Putting $B_\rho=\{(\zeta,k)\in\mathbb{R}^3\times\mathbb{Z},\ |(\zeta,k)|<\rho\}$ for some $\rho>0$ we shall make precise below, we treat $\int_{B_\rho}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k)$ and $\int_{\mathbb{R}^4\setminus B_\rho}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2d\ell d\eta d\mu(k)$ separately. We start by examining the last integral. To do that we first notice that $(\ell,\eta,k)\mapsto|(\ell+\beta_k k,\eta,k)|$ is a norm in $\mathbb{R}^4$ so we may find a constant $C_1>0$ such that the estimate

$$
|(\ell,\eta,k)|\leq C_1|(\ell+\beta_k k,\eta,k)|.
$$

holds for every $(\ell,\eta,k)\in\mathbb{R}^4$. As a consequence we have

$$
\begin{aligned}
\int_{\mathbb{R}^4\setminus B_\rho}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k) &\leq \frac{1}{\rho^2}\int_{\mathbb{R}^4\setminus B_\rho}|(\ell,\eta,k)|^2|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k) \\
&\leq \frac{C_1^2}{\rho^2}\int_{\mathbb{R}^4\setminus B_\rho}|(\ell+\beta_k k,\eta,k)|^2|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k) \\
&\leq \frac{C_1^2}{\rho^2}\int_{\mathbb{R}^4}(1+|(\ell+\beta_k k,\eta,k)|^2)|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k).
\end{aligned}
$$

The change of variable $\ell'=\ell+\beta_k k$ in the last integral then yields

$$
\int_{\mathbb{R}^4\setminus B_\rho}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k)\leq\frac{C_1^2}{\rho^2}\int_{\mathbb{R}^4}(1+|(\zeta,k)|^2)|\hat{\mathfrak{v}}(\zeta,k)|^2\mathrm{d}\zeta\mathrm{d}\mu(k)\leq\frac{C_1^2}{\rho^2}\|V\|^2_{H^1(\check{Q})},
$$

so we end up getting that

$$\int_{\mathbb{R}^4\setminus B_\rho} |\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2 \mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k) \leq \frac{4C_1^2M^2}{\rho^2}. \tag{5.18}$$

Further, we introduce $\mathcal{C}_\rho=\{(\zeta,k)\in\mathbb{R}^4,\ |\eta|<\rho^{-1}\}$ in such a way that the integral $\int_{B_\rho\cap\mathcal{C}_\rho}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k)$ is upper bounded by

$$\begin{aligned}\int_{\mathcal{C}_\rho} |\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2 \mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k) &\leq \frac{\pi}{\rho^2}\sup_{|\eta|\leq\rho^{-1}}\int_{\mathbb{R}^2}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\mathrm{d}\mu(k)\\ &\leq \frac{\pi}{\rho^2}\sup_{|\eta|\leq\rho^{-1}}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2\mathrm{d}\ell\\ &\leq \frac{\pi}{\rho^2}\sup_{|\eta|\leq\rho^{-1}}\sum_{k\in\mathbb{Z}}\|\hat{\mathfrak{v}}(.,\eta,k)\|_{L^2(\mathbb{R})}^2,\end{aligned}$$

giving

$$\int_{B_\rho\cap\mathcal{C}_\rho} |\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2 \mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k) \leq \frac{\pi}{\rho^2}\|V\|^2_{L^\infty_{x'}(\mathbb{R}^2;L^2_{t,x_1}(\mathbb{R}\times(0,1)))}\leq\frac{4\pi M^2}{\rho^2}, \tag{5.19}$$

and

$$\mathfrak{q}(\zeta,k)\leq 3\rho^2,\ (\zeta,k)\in B_\rho\cap(\mathbb{R}^4\setminus\mathcal{C}_\rho),\ \rho\geq 1.$$

From (5.15) and the above estimate then follows that

$$|\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|\leq c''\frac{\rho^2}{r^2}\left(r+\gamma\rho^{24}r^{12}\mathrm{e}^{2|\omega|r}\right),\ (\zeta,k)\in B_\rho\cap(\mathbb{R}^4\setminus\mathcal{C}_\rho),\ \rho\geq1,\ r\geq\max(1,r_0),$$

whence

$$\int_{B_\rho\cap(\mathbb{R}^4\setminus\mathcal{C}_\rho)} |\hat{\mathfrak{v}}(\ell+\beta_k k,\eta,k)|^2 \mathrm{d}\ell\mathrm{d}\eta\mathrm{d}\mu(k) \leq c''\frac{\rho^8}{r^4}\left(r^2+\gamma^2\rho^{48}r^{24}\mathrm{e}^{4|\omega|r}\right),\ \rho\geq1,\ r\geq\max(1,r_0), \tag{5.20}$$

upon eventually substituting c'' for some suitable algebraic expression of c''.

Last, putting (5.17)–(5.20) together we find that

$$\|V_2-V_1\|^2_{L^2(\check{Q})}\leq C_2\left(\frac{1}{\rho^2}+\frac{\rho^8}{r^2}+\gamma^2\rho^{56}r^{20}\mathrm{e}^{4|\omega|r}\right),\ \rho\geq1,\ r\geq\max(1,r_0), \tag{5.21}$$

where the constant $C_2>0$ depends only on T, ω and M. Putting $\gamma^*=\mathrm{e}^{-4|\omega|\max(1,r_0)}$ and taking for each $\gamma\in(0,\gamma^*)$, $r=r_1=\frac{1}{4|\omega|}\ln\gamma^{-1}$ and $\rho=r_1^{1/5}$ in (5.21), which is permitted since $r_1>\max(1,r_0)$, we obtain

$$\|V_2-V_1\|^2_{L^2(\check{Q})}\leq C_3\left(1+\gamma\left(\ln\gamma^{-1}\right)^{158/5}\right)\left(\ln\gamma^{-1}\right)^{-2/5},$$

where C_3 is another positive constant depending only on T, ω and M. Now, since $\sup_{0<\gamma\leq\gamma^*}\left(\gamma\left(\ln\gamma^{-1}\right)^{158/5}\right)$ is just another constant depending only on T, ω and M, then we end up getting $C_4=C_4(T,\omega,M)>0$ such that the estimate

$$\|V_2-V_1\|_{L^2(\check{Q})}\leq C_4\left(\ln\gamma^{-1}\right)^{-1/5}, \tag{5.22}$$

holds whenever $\gamma\in(0,\gamma^*)$.

Finally, since we have $\|V_2-V_1\|_{L^2(\check{Q})}\leq 2MT|\omega|\leq(2MT|\omega|/\gamma^*)\gamma$ if $\gamma\geq\gamma^*$, by (1.7), then (3.4) follows readily from this and (5.22).

6. Fiber decomposition

This section contains the proof of Proposition 6.2, establishing that the boundary operator Λ_V is, up to the partial FBG transform, the direct integral with fibers $\{\Lambda_{V,\theta},\ \theta\in(0,2\pi)\}$. We start by recalling the definition of the partial FBG transform and collecting some useful properties that are needed in the derivation of Proposition 6.2.

6.1. **Partial Floquet-Bloch-Gel'fand transform.** The main tool for the analysis of 1-periodic structures such as waveguides $\mathbb{R} \times Y$, where Y denotes a C^2 open subset or sub-manifold of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, is the partial FBG transform $\mathcal{U}_Y$, defined for every $f \in C_0^\infty(\mathbb{R} \times Y)$ as

$$\check{f}_{Y,\theta}(x_1, y) = (\mathcal{U}_Y f)_\theta(t, x) = \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k\theta} f(x_1 + k, y),\ (x_1, y) \in \mathbb{R} \times Y,\ \theta \in [0, 2\pi). \tag{6.1}$$

For notational simplicity, we systematically drop the Y in the above notations and write $\check{f}_\theta$ (resp., $\mathcal{U}$) instead of $\check{f}_{Y,\theta}$ (resp., $\mathcal{U}_Y$). We notice from (6.1) that

$$\check{f}_\theta(x_1 + 1, y) = \mathrm{e}^{\mathrm{i}\theta} \check{f}_\theta(x_1, y),\ (x_1, y) \in \mathbb{R} \times Y,\ \theta \in [0, 2\pi), \tag{6.2}$$

and

$$\left(\mathcal{U} \frac{\partial^m f}{\partial z^m} \right)_\theta = \frac{\partial^m \check{f}_\theta}{\partial z^m},\ m \in \mathbb{N}^*,\ \theta \in [0, 2\pi), \tag{6.3}$$

whenever $z = x_1$ or y_j, $j = 1, 2, \ldots, n$, where $y = (y_j)_{1 \leq j \leq n}$.

With reference to [RS2, Section XIII.16], $\mathcal{U}$ extends to a unitary operator, still denoted by $\mathcal{U}$, from $L^2(\mathbb{R} \times Y)$ onto the Hilbert space

$$\int_{(0,2\pi)}^{\oplus} L^2((0,1) \times Y) \frac{\mathrm{d}\theta}{2\pi} = L^2 \left((0, 2\pi) \frac{\mathrm{d}\theta}{2\pi}, L^2((0,1) \times Y) \right), \tag{6.4}$$

i.e. the direct integral (see e.g. [Di, Section II.1, Définition 1] or [RS2, Section XIII.16]) over $(0, 2\pi)$ with constant fibers $L^2((0,1) \times Y)$.

In the particular case where $Y = \omega$, we have $\mathbb{R} \times Y = \Omega$, hence $\mathcal{U}$ maps $L^2(\Omega)$ onto $\int_{(0,2\pi)}^{\oplus} L^2(\check{\Omega}) \mathrm{d}\theta/(2\pi)$, according to (1.4). Similarly, if $Y = (0,T) \times \omega$ (resp. $Y = (0,T) \times \partial\omega$) so that the cylinder $\mathbb{R} \times Y$ is simply denoted by Q (resp. Σ), we see from (1.4) that $\mathcal{U}$ maps $L^2(Q)$ onto $\int_{(0,2\pi)}^{\oplus} L^2(\check{Q}) \mathrm{d}\theta/(2\pi)$ (resp., $L^2(\Sigma)$ onto $\int_{(0,2\pi)}^{\oplus} L^2(\check{\Sigma}) \mathrm{d}\theta/(2\pi)$).

Further, bearing in mind that the trace operators $w \mapsto \partial_{x_1}^j w(0, \cdot)$ and $w \mapsto \partial_{x_1}^j w(1, \cdot)$ are bounded from $H^s((0,1) \times Y)$ onto $H^{s-j-1/2}(Y)$ for all $s \in (1/2, 2]$ and all natural number $j < s - 1/2$, by [LM1, Section 1, Theorem 9.4], we set for every $\theta \in [0, 2\pi)$,

$$\mathcal{H}_\theta^s((0,1) \times Y) = \left\{ u \in H^s((0,1) \times Y);\ \partial_{x_1}^j u(1, .) = \mathrm{e}^{\mathrm{i}\theta} \partial_{x_1}^j u(0, .),\ 0 \leq j < s - 1/2 \right\},\ s \in (1/2, 2]. \tag{6.5}$$

On the other hand, when s is not greater than $1/2$, we put

$$\mathcal{H}_\theta^s((0,1) \times Y) = H^s((0,1) \times Y),\ s \in [0, 1/2], \tag{6.6}$$

for all $\theta \in [0, 2\pi)$. In view of (6.2)–(6.4), we deduce from (6.5)-(6.6) that

$$\mathcal{U} H^k(0, T; H^s(\mathbb{R} \times Y)) = \int_{(0,2\pi)}^{\oplus} H^k(0, T; \mathcal{H}_\theta^s((0,1) \times Y)) \frac{\mathrm{d}\theta}{2\pi},\ k \in \mathbb{N},\ s \in [0, 2], \tag{6.7}$$

where the notation H^0 stands for L^2.

6.2. **Fiber decomposition of the boundary operator.** We start by decomposing the IBVP (1.1) into the collection of Cauchy problems (1.9) indexed by $\theta \in [0, 2\pi)$.

Proposition 6.1. *Let $V \in W^{2,\infty}(0, T; W^{2,\infty}(\Omega))$ fulfill (1.2) and let $(g, u_0) \in X_0$. Then u is the solution $\mathfrak{s}(g, u_0) \in Z$ to (1.1) defined in Corollary 2.1 if and only if each $\check{u}_\theta = (\mathcal{U}u)_\theta \in \check{Z}_\theta$, $\theta \in [0, 2\pi)$, is solution to the following IBVP*

$$\begin{cases} (-\mathrm{i}\partial_t - \Delta + V)v = 0 & \text{in } \check{Q}, \\ v(0, \cdot) = \check{u}_{0,\theta} & \text{in } \check{\Omega}, \\ v = \check{g}_\theta & \text{on } \check{\Sigma}, \end{cases} \tag{6.8}$$

where $\check{g}_\theta$ (resp. $\check{u}_{0,\theta}$) stands for $(\mathcal{U}g)_\theta$ (resp. $(\mathcal{U}u_0)_\theta$), i.e. $(\check{g}_\theta, \check{u}_{0,\theta}) = (\mathcal{U}(g, u_0))_\theta$.

Proof. Bearing in mind that $\check{X}_{0,\theta} = \check{\tau}_0(H^2(0,T;\mathcal{H}^2_\theta(\check{\Omega})))$ for every $\theta \in [0,2\pi)$, where we recall that $\check{\tau}_0$ is the linear bounded operator from $H^2(0,T;H^2(\check{\Omega}))$ into $L^2((0,T)\times(0,1),H^{3/2}(\partial\omega))\times L^2(\check{\Omega})$ such that

$$\check{\tau}_0 w = \left(w_{|\check{\Sigma}}, w(0,\cdot)\right) \text{ for } w \in C_0^\infty((0,T)\times(0,1),C^\infty(\overline{\omega})),$$

it is apparent that $\mathcal{U}X_0 = \int_{(0,2\pi)}^{\oplus} \check{X}_{0,\theta}\mathrm{d}\theta/(2\pi)$ and $\mathcal{U}\tau_0\mathcal{U}^{-1} = \int_{(0,2\pi)}^{\oplus} \check{\tau}_0\mathrm{d}\theta/(2\pi)$. We refer to [RS2, Section XIII.16] for the definition and the main properties of fiber decomposable operators. Thus, we have $\mathcal{U}Z = \int_{(0,2\pi)}^{\oplus} \check{Z}_\theta\mathrm{d}\theta/(2\pi)$, by (6.7), and we get the desired result upon applying the partial FBG transform $\mathcal{U}$ to both sides of the three lines in (1.1). □

Armed with Proposition 6.1, we may now decompose Λ_V in terms of the fibered boundary operators $\Lambda_{V,\theta}$, $\theta \in [0,2\pi)$.

Proposition 6.2. *Let V be the same as in Proposition 6.1. Then we have:*

$$\mathcal{U}\Lambda_V\mathcal{U}^{-1} = \int_{(0,2\pi)}^{\oplus} \Lambda_{V,\theta}\frac{d\theta}{2\pi}.$$

Proof. For every $\theta \in [0,2\pi)$, we know from Lemma 3.1 that the IBVP (6.8) admits a unique solution $\mathfrak{s}_\theta(\check{g}_\theta,\check{u}_{0,\theta}) \in \check{Z}_\theta$. Further, it holds true that $\Lambda_{V,\theta} = \check{\tau}_1 \circ \mathfrak{s}_\theta$, by (3.3), where we recall that $\check{\tau}_1$ is the linear bounded operator from $L^2((0,T)\times(0,1),H^2(\check{\Omega}))\cap H^1(0,T;L^2(\check{\Omega}))$ into $\check{X}_1 = L^2(\check{\Sigma})\times L^2(\check{\Omega})$, obeying

$$\check{\tau}_1 w = \left(\partial_\nu w_{|\check{\Sigma}}, w(T,\cdot)\right) \text{ for } w \in C_0^\infty((0,T)\times(0,1),C^\infty(\overline{\omega})).$$

Since it is obvious that $\mathcal{U}X_1 = \int_{(0,2\pi)}^{\oplus} \check{X}_1\mathrm{d}\theta/(2\pi)$ and $\mathcal{U}\tau_1\mathcal{U}^{-1} = \int_{(0,2\pi)}^{\oplus} \check{\tau}_1\mathrm{d}\theta/(2\pi)$, the desired result follows readily by combining the identity $\Lambda_V = \tau_1 \circ \mathfrak{s}$, arising from (2.10), with (3.3) and Proposition 6.1. □

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References

[A] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal., **27** (1988), 153–172.
[Ag] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965.
[BLR] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization from the boundary*, SIAM J. Control Optim., **30** (1992), 1024–1065.
[BP] L. Baudouin and J.-P. Puel, *Uniqueness and stability in an inverse problem for the Schrödinger equation*, Inverse Probl., **18** (2002), 1537–1554.
[BC1] M. Bellassoued and M. Choulli, *Logarithmic stability in the dynamical inverse problem for the Schrodinger equation by an arbitrary boundary observation*, J. Math. Pures Appl., **91** (3) (2009), 233–255.
[BC2] M. Bellassoued and M. Choulli, *Stability estimate for an inverse problem for the magnetic Schrödinger equation from the Dirichlet-to-Neumann map*, J. Funct. Anal., **258** (2010), 161–195.
[BK] A.L. Bukhgeim and M.V. Klibanov, *Uniqueness in the large of a class of multidimensional inverse problems*, Sov. Math. Dokl., **17** (1981), 244–247.
[CH] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, Clarendon Press, Oxford, 1998.
[Cho] M. Choulli, *Une introduction aux problèmes inverses elliptiques et paraboliques*, Mathématiques et Applications, Vol. 65, Springer-Verlag, Berlin, 2009.
[CK] M. Choulli and Y. Kian, *Stability of the determination of a time-dependent coefficient in parabolic equations*, Math. Control and Related Fields, **3** (2) (2013), 143–160.
[CS] M. Choulli and E. Soccorsi, *An inverse anisotropic conductivity problem induced by twisting a homogeneous cylindrical domain*, J. Spec. Theory., **5** (2) (2015), 295–329.
[CriS] M. Cristofol and E. Soccorsi, *Stability estimate in an inverse problem for non-autonomous Schrödinger equations*, Appl. Anal., **90** (10) (2011), 1499–1520.
[Di] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de Von Neumann)*, Cahiers scientifiques, Vol. 25, Gauthier-Villars, Paris, 1957.

[DKLS] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas and M. Salo, *The Calderon problem in transversally anisotropic geometries*, to appear in Journal of European Mathematical Society, arXiv:1305.1273.

[Es1] G. Eskin, *Inverse hyperbolic problems with time-dependent coefficients*, Comm. PDE, **32** (11) (2007), 1737–1758.

[Es2] G. Eskin, *Inverse problem for the Schrödinger equation with time-dependent electromagnetic potentials and the Aharonov-Bohm effect*, J. Math. Phys., **49** (2) (2008), 1–18.

[Ev] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.

[Fl] S. Fliss, *A Dirichlet-to-Neumann approach for the exact computation of guided modes in photonic crystal waveguides*, SIAM J. Sci. Comput., **35** (2) (2013), 438–461.

[GK] P. Gaitan and Y. Kian, *Stability result for a time-dependent potential in a cylindrical domain*, Inverse Problems, **29** (6) (2013), 065006.

[HSSW] C. Hamaker, K. T. Smith, D. C. Solomon and S. C. Wagner, *The divergent beam x-ray transform*, Rocky Mountain J. Math., **10** (1980), 253–283.

[Ha] P. Hähner, *A periodic Faddeev-type operator*, J. Diff. Equat., **128** (1996), 300–308.

[Ik] M. Ikehata, *Inverse conductivity problem in the infinite slab*, Inverse Problems, **17** (2001), 437–454.

[Is] V. Isakov, *Completness of products of solutions and some inverse problems for PDE*, J. Diff. Equat., **92** (1991), 305–316.

[K] Y. Kian, *Stability of the determination of a coefficient for wave equations in an infinite waveguide*, Inverse Probl. Imaging, **8** (3) (2014), 713–732.

[KPS1] Y. Kian, Q. S. Phan and E. Soccorsi, *Carleman estimate for infinite cylindrical quantum domains and application to inverse problems*, Inverse Problems, **30** (5) (2014), 055016.

[KPS2] Y. Kian, Q. S. Phan and E. Soccorsi, *Hölder stable determination of a quantum scalar potential in unbounded cylindrical domains*, Jour. Math. Anal. Appl., **426** (1) (2015), 194–210.

[KLU] K. Krupchyk, M. Lassas and G. Uhlmann, *Inverse Problems with Partial Data for a Magnetic Schrödinger Operator in an Infinite Slab or Bounded Domain*, Comm. Math. Phys., **312** (2012), 87–126.

[LU] X. Li and G. Uhlmann, Inverse Problems on a Slab, Inverse Problems and Imaging, **4** (2010), 449–462.

[LM1] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, vol. 1, Dunod, Paris, 1968.

[LM2] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, vol. 2, Dunod, Paris, 1968.

[Na] S. I. Nakamura, *Uniqueness for an Inverse Problem for the Wave Equation in the Half Space*, Tokyo J. of Math., **19** (1) (1996), 187–195.

[Ra] Rakesh, *An inverse problem for the wave equation in the half plane*, Inverse Problems, **9** (1993), 433–441.

[RR] Rakesh and A. G. Ramm, *Property C and an Inverse Problem for a Hyperbolic Equation*, J. Math. Anal. Appl., **156** (1991), 209–219.

[RS] A. G. Ramm and J. Sjöstrand, *An inverse problem of the wave equation*, Math. Z., **206** (1991), 119–130.

[RS1] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Sel-adjointness*, Academic Press, 1978.

[RS2] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, 1978.

[Sa] R. Salazar, *Determination of time-dependent coefficients for a hyperbolic inverse problem*, Inverse Problems, **29** (9) (2013), 095015.

[Sc] L. Schwartz, *Topologie générale et analyse fonctionnelle*, Hermann, Paris, 1970.

[St] P. Stefanov, *Uniqueness of the multi-dimensional inverse scattering problem for time dependent potentials*, Math. Z., **201** (4) (1989), 541–559.

[SW] M. Salo and J. N. Wang, *Complex spherical waves and inverse problems in unbounded domains*, Inverse Problems, **22** (2006), 2299–2309.

[S] Z. Sun, *On continous dependence for an inverse initial boundary value problem for the wave equation*, J. Math. Anal. Appl., **150** (1990), 188–204.

[SU] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), 153–169.

[T] D. Tataru, *Unique continuation for solutions to PDE; between Hörmander's theorem and Holmgren's theorem*, Commun. Partial Diff. Eqns., **20** (1995), 855–884.

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