EDGE STATES FOR QUANTUM HALL HAMILTONIANS

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Abstract. The study of the quantum motion of a charged particle in a half-plane as well as in an infinite strip submitted to a perpendicular constant magnetic field B reveals eigenstates propagating permanently along the edge, the so-called edge states. Moreover, in the half-plane geometry, current carried by edge states with energy in between the Landau levels persists in the presence of a perturbing potential small relative to B. We show here that edge states carrying current survive in an infinite strip for a long time before tunneling between the two edges has a destructive effect on it. The proof relies on Helffer-Sjöstrand functional calculus and decay properties of quantum Hall Hamiltonian resolvent.

1. Introduction

Since the discovery of quantized Hall conductivity by Von Klitzing et al [1], edge states have been at the center of both theoretical explanations and controversies about this effect (see e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10]). The one-electron model approximation, although certainly insufficient to explain all the aspects of Hall quantization, in particular the fractional quantum Hall effect, is nevertheless a source of interesting spectral problems. Some of them have been rigorously investigated by various authors ([11], [12], [13], [14], [15]). In this paper we will summarize and complement existing results about existence and properties of current carrying edge states and point out some open problems.

2. Models and known results

For any $B > 0$, $H_L = p_x^2 + (p_y - Bx)^2$ (where $p_x = -i\partial_x$ and $p_y = -i\partial_y$) denotes the Landau Hamiltonian on $\mathbb{R}^2$ describing a charged particle in a uniform magnetic field orthogonal to the plane. We consider a confining potential V_0 defined as

$$V_0 = V_0^- + V_0^+$$

where

$$V_0^-(x) = \begin{cases} \mathcal{V}_0 & \text{for } x \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$V_0^+(x) = V_0^-(L_0 - x),$$

with $\mathcal{V}_0 > 0$ and $L_0 \gg B^{-1/2}$. Recall that the classical cyclotron radius of an electron in such a uniform magnetic field is proportional to $B^{-1/2}$. In the following, $\mathcal{V}_0$ will be assumed to be very large with respect to B so

$$H_0 = H_L + V_0$$

is the Hamiltonian of a typical quantum Hall device in the idealized situation where it is infinitely extended in the y-direction. Finite geometries can be considered as well such as Laughlin's cylinder geometry by imposing periodic boundary conditions in y. Impurities in the sample are represented by a potential V of unbounded support, and nondecaying at infinity (e.g. an Anderson potential). Then the dynamics of the charges particle are governed by

$$H = H_0 + V.$$

Spectral properties of H give insight into these dynamical properties and have been investigated by many authors in the recent years. We recall now some of them.

Concerning the Landau Hamiltonian H_L on $L^2(\mathbb{R}^2)$, it is well known that

$$\sigma(H_L) = \{(2n+1)B,\ n \in \mathbb{N}\}.$$

If $V_0 = 0$ and V is a random potential describing impurities then for large disorder or large magnetic fields the random Hamiltonian $H_1 = H_L + V$ has bands of dense pure point spectrum contained in the Landau bands $[(2n+1)B, (2n+3)B]$ (see e.g. [16], [17], [18]).

As for the half-plane case, $L_0 = +\infty$, if $B^{-1}\|V\|_{L^\infty(\mathbb{R}^2)}$ is small enough, then the spectrum of $H^- = H_L + V_0^- + V$ has absolutely continuous components contained in the complement of the Landau bands (see [12], [11], [13]). These authors consider smooth confining potentials ([11], [13]), or Dirichlet boundary conditions at $x = 0$, that correspond to $\mathcal{V}_0 = +\infty$ ([12], [13]). More general geometries with one boundary have been considered in [19].

Finally, for the strip geometry, $L_0 < +\infty$, there are very few results. Recently it was shown in [14] that absolutely continuous spectrum of H_0 survives perturbation by V if V is periodic or decays fast enough in y-direction.

The connection between absolutely continuous spectrum and existence of Hall currents is not straightforward. In fact, in macroscopic finite samples, such currents exist although the spectrum is discrete. This was shown in [15] and is certainly one of the most relevant recent result from a physical point of view. It turns out that in some cases investigated in [12] and [13], for which $L_0 = \infty$, the existence of Hall currents carried by edge states is shown through positive commutator estimates. By Mourre's theory, such estimates imply absolutely continuous spectrum. More precisely, let

$$v_y = p_y - Bx$$

denote the velocity operator in the y-direction along the edge. For an infinite system in y-direction, one has

$$v_y = \frac{i}{2}[H, y].$$

Since y is in this case a bona-fide conjugate operator for H in the sense of Mourre (see [20], [21]), positivity of v_y in spectral subspaces of H implies absolute continuity for the spectrum of H in corresponding energy intervals. Indeed, if $P_H(\Delta)$ denotes the spectral projection for H associated to interval Δ, then Mourre's inequality

$$P_H(\Delta) v_y P_H(\Delta) \geq c P_H(\Delta) \text{ for some } c > 0 \tag{2.1}$$

implies $\sigma(H) \cap \Delta = \sigma_{ac}(H) \cap \Delta$, where $\sigma(H)$ (resp. $\sigma_{ac}(H)$) denotes the spectrum (resp. the absolutely continuous spectrum) of H. Of course, in finite samples, neither Mourre's theory nor the virial theorem apply with y as a conjugate operator and positivity of v_y in spectral subspaces of H is not related to spectral properties of H.

When $L_0 < +\infty$, the Hall current has different signs on opposite edges so one can't expect (2.1) to hold. However we can try to obtain a good conjugate operator replacing v_y by a suitable localized version of it changing sign on both edges. This can be done easily if $V = 0$ or V is periodic in y (see [22]) and again in these cases existence of edge currents is related to absolutely continuous spectrum.

It is very likely that for general V there is no conjugate operator for H related to Hall current operator v_y. In fact existence of such an operator would imply that there are states carrying currents of a given direction for all times; this does not seem realistic from a physical point of view. We will show nevertheless in section 6 that current carrying edge states exists although destructive interference effects due to tunneling between the two edges of the sample might prevent them from surviving after some large tunneling time.

After reviewing some basic spectral properties of H_0 we will present in section 4 perturbative arguments along the lines already used in [12] and [13] showing that current carrying edge states still exist in the half-plane or semi-infinite cylinder geometry if $B^{-1}\|V\|_{L^\infty(\mathbb{R}^2)}$ is small enough. The two last sections will be devoted to the strip geometry. We will present there some efficient techniques to deal with the above mentioned tunneling effect.

3. The one-edge geometry

We consider first the case $L_0 = +\infty$, so that the unperturbed Hamiltonian operator is $H_0^- = H_L + V_0^-$ on the Hilbert space $L^2(\mathbb{R}^2)$. Most spectral properties of H_0^- described below remain valid if one adds to V_0^- some potential V_1 representing some mean field interaction due to the distribution of electrons and holes in the system. We can also treat on the same way cylindrical geometries which amounts to imposing periodic boundary conditions in the edge direction.

Partial Fourier transform in the y-direction shows that H_0^- is unitarily equivalent to the direct integral over $\mathbb{R}$,

$$H_0^- \simeq \int_{\mathbb{R}}^{\oplus} h_0^-(k)dk. \tag{3.1}$$

The fiber operators on $L^2(\mathbb{R})$ are the selfadjoint operators $h_0^-(k) = p_x^2 + (Bx - k)^2 + V_0^-(x)$, with the k-independent domain

$$D(h_0^-) = \{v \in H_1^1(\mathbb{R}),\ (p_x^2 + x^2)v \in L^2(\mathbb{R})\},$$

where $H_1^1(\mathbb{R})$ is the space $\{v \in H^1(\mathbb{R}),\ (1+x^2)^{\frac{1}{2}} v \in L^2(\mathbb{R})\}$.

Since the embedding $H_1^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is compact (see corollary IX.B.4 of [23]), every $h_0^-(k)$ has compact resolvent and also a discrete spectrum

$$\sigma(h_0^-(k)) = \{\omega_n^-(k), n \geq 0\},$$

with $\omega_n^-(k) \leq \omega_{n+1}^-(k)$ and $\lim_{n\to+\infty} \omega_n^-(k) = +\infty$. In the following, we denote by $\{\varphi_n^-(x,k),\ n \geq 0\}$ an orthonormal basis of $L^2(\mathbb{R})$ of associated eigenfunctions.

The main properties of the dispersion curves $\omega_n^-(k)$, $n \in \mathbb{N}$, that will be needed are collected in the following lemma :

Lemma 3.1. *For any integer n,*

1. *The multiplicity of the eigenvalue $\omega_n^-(k)$ is equal to one;*
2. *The mapping $k \mapsto \omega_n^-(k)$ is real analytic and decreases from $(2n+1)B + \mathcal{V}_0$ to $(2n+1)B$ as $k \to +\infty$;*

(3) *The derivative of* ω_n^- *is* $(\omega_n^-)'(k) = -\frac{\mathcal{V}_0}{B}(\varphi_n^-)^2(0,k)$ *for any* $k \in \mathbb{R}$*;*
(4) $\omega_0^-(k) \leq 3B$ *for any* $k \geq 0$.

The proof of the two first assertions is standard. The proof of point (3.) follows from the Feynman-Hellman theorem and elementary integration by parts. Noticing that $\omega_0^-(0) = 3B$ in the Dirichlet limit $\mathcal{V}_0 = +\infty$, point (4.) follows from (3.) and the monotonicity of $\omega_0^-(k)$ in $\mathcal{V}_0$. The details will appear in [22]. It follows that H_0^- has absolutely continuous spectrum :

$$\sigma(H_0^-) = [B, +\infty[.$$

Ground state eigenfunctions $\varphi_0^-(.,k)$ are well-known to be positive. Furthermore, they satisfy the following decay estimate in the barrier region:

Lemma 3.2. *For any* $x \leq 0$ *and* $k \in (\omega_0^-)^{-1}([B, 3B])$ *we have :*

$$\varphi_0^-(x,k) \leq \varphi_0^-(0,k)\mathrm{e}^{\left[\mathcal{V}_0 - \omega_0^-(k)\right]^{1/2} x},$$

provided $\mathcal{V}_0$ *is large enough, that is, provided* $2\pi^{1/2}(1 - \omega_0^-(k)/\mathcal{V}_0) < 1$*, for all* $k \in (\omega_0^-)^{-1}([B, 3B])$.

This behavior provides a useful estimate on the derivative of ω_0^-.

Proposition 3.3. *For all* $k \in (\omega_0^-)^{-1}([B, 3B])$ *we have :*

$$(\omega_0^-)'(k) \leq -\frac{1}{4B^{5/2}}(3B - \omega_0^-(k))(\omega_0^-(k) - B)^2 \mathrm{e}^{\frac{k^2}{B}},$$

provided $\mathcal{V}_0$ *is large enough.*

Proof. Let $\psi_n(.,k)$, $n \in \mathbb{N}$, denote the normalized eigenfunctions of $h_L(k) = p_x^2 + (Bx - k)^2$ associated to eigenvalue $(2n+1)B$. In particular :

$$\psi_0(x,k) = \left(\frac{B}{\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{B}{2}\left(x - \frac{k}{B}\right)^2}. \tag{3.2}$$

Then $\varphi_0^-(.,k) = \sum_{n \geq 0} \alpha_n(k)\psi_n(.,k)$ and $h_L(k)\varphi_0^-(.,k) = \sum_{n\geq 0}(2n+1)B\alpha_n(k)\psi_n(.,k)$. Since $V_0^- = h_0^-(k) - h_L(k)$, we have

$$\left(\varphi_0^-(.,k), V_0^-\varphi_0^-(.,k)\right)_{L^2(\mathbb{R})} = \omega_0^-(k) - \sum_{n\geq 0}(2n+1)B|\alpha_n(k)|^2,$$

and also :

$$\begin{array}{rcl}(\omega_0^-(k) - B)|\alpha_0(k)|^2 & = & \left(\varphi_0^-(.,k), V_0^-\varphi_0^-(.,k)\right)_{L^2(\mathbb{R})} \\ & + & \displaystyle\sum_{n\geq 1}\left[(2n+1)B - \omega_0^-(k)\right]|\alpha_n(k)|^2 \\ & \geq & \left[3B - \omega_0^-(k)\right](1 - |\alpha_0(k)|^2).\end{array}$$

Therefore, we finally obtain

$$|\alpha_0(k)| \geq \left(\frac{3B - \omega_0^-(k)}{2B}\right)^{\frac{1}{2}}, \tag{3.3}$$

for any $k \in (\omega_0^-)^{-1}([B, 3B])$. Thus, $\left(\varphi_0^-(.,k), V_0^-\psi_0(.,k)\right)_{L^2(\mathbb{R})} = (\omega_0^-(k) - B)\alpha_0(k)$ and (3.3) imply :

$$\left|\left(\varphi_0^-(.,k), V_0^-\psi_0(.,k)\right)_{L^2(\mathbb{R})}\right| \geq (\omega_0^-(k) - B)\left(\frac{3B - \omega_0^-(k)}{2B}\right)^{\frac{1}{2}}. \tag{3.4}$$

Then, by lemma 3.2 one obtains :

$$\begin{aligned}\left(\varphi_0^-(.,k), V_0^-\psi_0(.,k)\right)_{L^2(\mathbb{R})} &= \left(\frac{B}{\pi}\right)^{\frac{1}{4}}\mathcal{V}_0\int_{-\infty}^0 \mathrm{e}^{-\frac{B}{2}\left(t-\frac{k}{B}\right)^2}\varphi_0^-(t,k)dt\\ &\leq \left(\frac{B}{\pi}\right)^{\frac{1}{4}}\mathrm{e}^{-\frac{k^2}{2B}}\mathcal{V}_0\varphi_0^-(0,k)\int_{-\infty}^0 \mathrm{e}^{\left[\mathcal{V}_0-\omega_0^-(k)\right]^{\frac{1}{2}}t}dt\\ &\leq \left(\frac{B}{\pi}\right)^{\frac{1}{4}}\mathrm{e}^{-\frac{k^2}{2B}}\left[\mathcal{V}_0-\omega_0^-(k)\right]^{-\frac{1}{2}}\mathcal{V}_0\varphi_0^-(0,k).\end{aligned}$$

Inserting this inequality in (3.4) we get :

$$\mathcal{V}_0(\varphi_0^-)^2(0,k) \geq \mathrm{e}^{\frac{k^2}{B}}\left(\frac{\pi}{B}\right)^{\frac{1}{2}}\left(1-\frac{\omega_0^-(k)}{\mathcal{V}_0}\right)\left(\frac{3B-\omega_0^-(k)}{2B}\right)(\omega_0^-(k)-B)^2. \tag{3.5}$$

This, together with the third point of lemma 3.1 completes the proof. □

Remark 3.4.

(1) Let $\varphi \in P_{H_0^-}(\Delta_0)L^2(\mathbb{R}^2)$ where $P_{H_0^-}$ denotes the spectral projection associated to H_0^- and $\Delta_0 = [a_1B, a_2B]$, $1 < a_1 < a_2 < 3$. The partial Fourier transform with respect to y of φ decomposes on $\{\varphi_n^-,\ n \geq 0\}$:

$$\hat{\varphi}(x,k) = \beta_0(k)\mathbf{1}_{(\omega_0^-)^{-1}(\Delta_0)}(k)\varphi_0^-(x,k),$$

where $\beta_0(k) = \left(\hat{\varphi}(.,k), \varphi_0^-(.,k)\right)_{L^2(\mathbb{R})}$. Thus, by the Feynman-Hellman theorem we have

$$\begin{aligned}&(\hat{\varphi}, (k-Bx)\hat{\varphi})_{L^2(\mathbb{R}^2)}\\ =\ &\int_{(\omega_0^-)^{-1}(\Delta_0)}|\beta_0(k)|^2\left(\varphi_0^-(.,k), (k-Bx)\varphi_0^-(.,k)\right)_{L^2(\mathbb{R})}dk\\ =\ &\frac{1}{2}\int_{(\omega_0^-)^{-1}(\Delta_0)}|\beta_0(k)|^2(\omega_0^-)'(k)dk,\end{aligned}$$

and proposition 3.3 implies :

$$(\varphi, v_y\varphi)_{L^2(\mathbb{R}^2)} \leq -\frac{(3-a_2)}{8B^{3/2}}\int_{(\omega_0^-)^{-1}(\Delta_0)}|\beta_0(k)|^2(\omega_0^-(k)-B)^2dk.$$

Then, by noticing that

$$\int_{(\omega_0^-)^{-1}(\Delta_0)}|\beta_0(k)|^2(\omega_0^-(k)-B)^2dk = \|(H_0^- - B)\varphi\|^2_{L^2(\mathbb{R}^2)},$$

we finally get :

$$(\varphi, v_y\varphi)_{L^2(\mathbb{R}^2)} \leq -\frac{(3-a_2)(a_1-1)^2}{8}B^{1/2}\|\varphi\|^2_{L^2(\mathbb{R}^2)}. \tag{3.6}$$

So states in $P_{H_0^-}(\Delta_0)L^2(\mathbb{R}^2)$ carry a current which is $O(B^{1/2})$.

(2) Such φ live within a strip of size $O(B^{-1/2})$ along the edge. This follows from the above decomposition of $\hat{\varphi}$ in terms of φ_0^- and from the fact that if $k \in (\omega_0^-)^{-1}(\Delta_0)$ then the exterior of such a strip is in the classically forbidden region for the harmonic potential $(k-Bx)^2$. This justifies the terminology "edges states" for any $\varphi \in P_{H_0^-}(\Delta_0)L^2(\mathbb{R}^2)$ as above.

4. Perturbation theory for the one-edge Hamiltonian

Let $H^- = H_0^- + V$ and note P_{H^-} the spectral projection associated to H^-. Let $\psi \in P_{H^-}(\Delta)L^2(\mathbb{R}^2)$ where for simplicity Δ is assumed to be an interval of size $O(1)$ centered in $E \in]3B/2, 2B[$. We want to show that ψ still carries a current, that is :

$$(\psi, v_y\psi)_{L^2(\mathbb{R}^2)} \leq -C(B)\|\psi\|^2_{L^2(\mathbb{R}^2)},$$

for some constant $C(B) > 0$ depending on B, provided the impurity potential V is not too strong. Noticing that $(\psi, v_y\psi)_{L^2(\mathbb{R}^2)} = \left(\hat{\psi}, (k - Bx)\hat{\psi}\right)_{L^2(\mathbb{R}^2)}$ where $\hat{\psi}$ denotes the partial Fourier transform with respect to y of ψ, and integrating by parts we get

$$2B\,(\psi, v_y\psi)_{L^2(\mathbb{R}^2)} = (\psi, \partial_x V\psi)_{L^2(\mathbb{R}^2)} - \mathcal{V}_0 \int_{\mathbb{R}} |\psi(0,y)|^2 dy, \tag{4.1}$$

so existence of current carrying edge states should hold in particular if states $\psi \in P_{H^-}(\Delta)L^2(\mathbb{R}^2)$ have support satisfying this condition up to controllable corrections. One can think that this holds in particular if the impurity potential V is not strong enough to create states with energy in Δ living in the bulk region $x \gg B^{-1/2}$. This type of result is shown e.g. in [12] and [13]. In the case considered here one has :

Proposition 4.1. *For $\psi \in P_{H^-}(\Delta)L^2(\mathbb{R}^2)$, we have*

$$(\psi, v_y\psi)_{L^2(\mathbb{R}^2)} \leq -C(\Delta)\|\psi\|^2_{L^2(\mathbb{R}^2)}$$

for some constant $C(\Delta) > 0$, depending on Δ, provided $\|V\|_{L^\infty(\mathbb{R}^2)} = O(B)$.

For this we decompose ψ in

$$\psi = \varphi + \xi \text{ with } \varphi = P_{H_0^-}(\Delta_0)\psi \text{ and } \xi = P_{H_0^-}(\Delta_0^c)\psi,$$

where Δ_0 is an open interval $]a_1B, a_2B[$, $1 < a_1 < a_2 < 3$, containing Δ. One easily obtains :

$$(\psi, v_y\psi)_{L^2(\mathbb{R}^2)} \leq (\varphi, v_y\varphi)_{L^2(\mathbb{R}^2)} + 2\|v_y\xi\|_{L^2(\mathbb{R}^2)}. \tag{4.2}$$

According to formula (3.6) in remark 3.4 the first term has an upper bound which is $O(B^{1/2})$. To estimate the second term we observe by an easy perturbation argument that for any $u \in P_{H^-}(\Delta)L^2(\mathbb{R}^2)$,

$$\|P_{H_0^-}(\Delta_0^c)u\|_{L^2(\mathbb{R}^2)} \leq \operatorname{dist}^{-1}(E, \Delta_0^c)\left(\frac{|\Delta|}{2} + \|V\|_{L^\infty(\mathbb{R}^2)}\right)\|u\|_{L^2(\mathbb{R}^2)},$$

so we get

$$\begin{aligned}
\|v_y\xi\|^2_{L^2(\mathbb{R}^2)} &\leq \left((p_y - Bx)^2\xi, \xi\right)_{L^2(\mathbb{R}^2)} \\
&\leq \left(H_0^-\xi, \xi\right)_{L^2(\mathbb{R}^2)} \\
&\leq \left(H_0^-\psi, \xi\right)_{L^2(\mathbb{R}^2)} \\
&\leq \left((H^- - V)\psi, \xi\right)_{L^2(\mathbb{R}^2)} \\
&\leq \|(H^- - V)\psi\|_{L^2(\mathbb{R}^2)}\|\xi\|_{L^2(\mathbb{R}^2)} \\
&\leq \left(E + \frac{|\Delta|}{2}\right)\|\psi\|_{L^2(\mathbb{R}^2)}\|\xi\|_{L^2(\mathbb{R}^2)} \\
&\leq \frac{E + \frac{|\Delta|}{2}}{\operatorname{dist}(E, \Delta_0^c)}\left(\frac{|\Delta|}{2} + \|V\|_{L^\infty(\mathbb{R}^2)}\right)\|\psi\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

It just remains to choose a_1 and a_2 in order that $\frac{E+\frac{|\Delta|}{2}}{\mathrm{dist}(E,\Delta_0^c)}$ is of order $O(1)$ and the result follows.

Remark 4.2.

(1) Notice that one obtains here a Mourre's inequality

$$iP_{H^-}(\Delta)[H^-,y]P_{H^-}(\Delta) \geq 2C(\Delta)P_{H^-}(\Delta),$$

so that $\sigma(H^-)\cap\Delta = \sigma_{ac}(H^-)\cap\Delta$.

(2) Under conditions involving only $\|V\|_{L^\infty(\mathbb{R}^2)}$ this type of result is quite optimal. To see this assume for example that Δ is a small interval centered around $E_0 \in [a_1B, a_2B]$, $1 < a_1 < a_2 < 3$. Next, consider the following potential

$$V = E_0 + W_\omega,$$

where W_ω is a random potential of Anderson type. Then, general results on band-edge localization (see [24]) ensure that $\sigma(H^-)$ is pure point in the vicinity of E_0, so that $(\psi, v_y\psi)_{L^2(\mathbb{R}^2)} = 0$ for corresponding eigenfunctions by the virial theorem.

(3) As we have seen in section 3, unperturbed edge states live in a strip of width $O(B^{-1/2})$ along the edge. So, if conditions on the support of V are added, current carrying edge states should survive under much weaker conditions on $\|V\|_{L^\infty(\mathbb{R}^2)}$. We will prove this fact in the more general setting of the strip geometry in section 5.

5. The strip geometry

We now consider the case $L_0 < +\infty$. One still has a decomposition (3.1) for H_0 but it is clear that the dispersion curves ω_n are no more monotonic because $w_n(k) = w_n(BL_0 - k)$. As a consequence the current carried by states in $P_{H_0}(\Delta)$ with $\Delta \subset]B, 3B[$ for example, will not have a definite sign unless one specifies their location on one of the edges of the strip. This can be seen from the following properties of ω_n's :

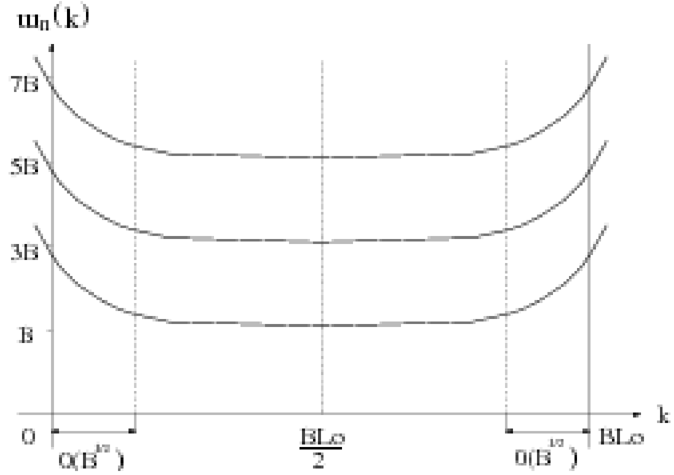
Lemma 5.1. *For any $r \geq 0$ and any $k \leq [r+1+(2n+1)^{1/2}]B^{1/2}$ such that $\omega_n^-(k) \leq (2n+3)B$, one has*

$$\sigma(h_0(k)) \cap [w_n^-(k), w_n^-(k) + C_n(r)\mathrm{e}^{-\frac{B}{4}L_0^2}] = \{w_n(k)\},$$

for some constant $C_n(r) > 0$ and large enough B.
Under the same assumptions on k one gets in addition

$$\|\varphi_n^-(.,k) - \varphi_n(.,k)\|_{L^2(\mathbb{R}^2)} \leq C_n(r)\mathrm{e}^{-\frac{B}{4}L_0^2}.$$

A consequence of lemma 5.1 is the following picture of dispersion curves near the edges:



Notice that the monotonicity of ω_n, $n \in \mathbb{N}$, in a neighbourhood of size $B^{\frac{1}{2}}$ of 0 or BL_0 follows easily from lemma 5.1. For the ground state, monotonicity of w_0 on $]-\infty, BL_0/2]$ follows from theorem 5 of [25]. So one can construct edge states carrying currents of negative sign as follows :

$$\hat{\varphi}(x,k) = \beta_0(k)\mathbf{1}_{\omega_0^{-1}(\Delta_0)\cap]-\infty,BL_0/2[}(k)\varphi_0(x,k),$$

where $\Delta_0 \subset]B, 3B[$, and similar construction for higher levels. Replacing interval $]-\infty, BL_0/2[$ by $]BL_0/2, +\infty[$ one obtains edge states localized near $x = L_0$ carrying positive current. Such states will carry current for all times and don't interfere with each other. In other words one has a decomposition

$$H_0 = \tilde{H}_0^- \oplus \tilde{H}_0^+$$

into time invariant subspaces of H_0 where $\tilde{H}_0^\pm = \int_{\pm(k-BL_0/2)>0}^\oplus h_0(k)dk$.

6. Perturbation theory for the strip geometry

Adding now the impurity potential V, the strategy developed for the one-edge geometry obviously no longer works for the strip. In fact, V induces tunneling between the two edges and it is a natural question to ask for how long edge states will carry a current of definite sign before tunneling has a destructive effect on it. One way to provide answer to this question is given by Helffer-Sjöstrand functional calculus (see [26]).

Let $f \in C^{n+1}(\mathbb{R})$, $n \geq 1$, and define an almost analytic extension of f as

$$\tilde{f}(z) = \left[\sum_{r=0}^{n} f^{(r)}\frac{(i\gamma)^r}{r!}\right]\sigma(E,\gamma), \; z = E + i\gamma, \tag{6.1}$$

where $\sigma(E,\gamma) = \tau\left(\frac{\gamma}{E}\right)$ for some real-valued function $\tau \in C^1(\mathbb{R})$ such that

$$\tau(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{if } |t| < 2. \end{cases}$$

Then Helffer-Sjöstrand formula states that

$$f(H)=\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\tilde f}{\partial\bar z}(H-z)^{-1}dEd\gamma, \tag{6.2}$$

when $H=H^*$. Notice that

$$\frac{\partial\tilde f}{\partial\bar z}(z)=\frac{1}{2}\left[\sum_{r=0}^{n}f^{(r)}(E)\frac{(i\gamma)^r}{2!}\right]\left(\frac{\partial\sigma}{\partial E}+i\frac{\partial\sigma}{\partial\gamma}\right)+\frac{1}{2}f^{(n+1)}(E)\frac{(i\gamma)^n}{n!}\sigma(E,\gamma)$$

and that $\frac{\partial\sigma}{\partial E}$ and $\frac{\partial\sigma}{\partial\gamma}$ vanish if $\gamma<E$ or $\gamma>2E$. These properties of $\frac{\partial\tilde f}{\partial\bar z}$ allow a good control of the integral near the spectrum of H and are very useful for perturbation theory in regions where the spectrum is dense.

A useful tool for the analysis of the tunneling effects is the geometric resolvent equation which allows to take into account local properties of a given Hamiltonian in the analysis of its Green's functions (see e.g. [27]). Given the Hamiltonian H and a C^2-function J, let H_J be any self-adjoint operator such that $H_JJ=HJ$. Then if $z\in\rho(H_J)\cap\rho(H)$ (where $\rho(M)$ denotes the resolvent set of operator M) one has :

$$J(H-z)^{-1}=(H_J-z)^{-1}\left[J+W(J)(H-z)^{-1}\right] \tag{6.3}$$

where $W(J)=[H,J]$.

We apply this in two situations. First, as a warm-up, we assume that the impurity potential V has a support away from the edges $x=0$ and $x=L_0$ and centered around $x=L_0/2$, with

$$\text{dist }(\text{supp}(V),\text{edges})=d,\ 0<d<L_0/2.$$

Let $\psi_0\in P_{H_0}(\Delta)L^2(\mathbb{R}^2)$, $\|\psi_0\|_{L^2(\mathbb{R}^2)}=1$, for some $\Delta\in[a_1B,a_2B]$ with $1<a_1<a_2<3$, where $H_0=H_L+V_0$, as defined in section 2. From lemma 5.1, it is easy to see, as we did in the half-plane case, that the restriction of ψ_0 to supp(V) has norm $O(\mathrm{e}^{-\alpha Bd^2})$, for some finite constant $0<\alpha<\infty$, independent of B. Let f be a smooth function which is one on Δ and zero outside some bigger interval $\tilde\Delta\supset\Delta$, of the same type as Δ. Then, $f(H_0)\psi_0=\psi_0$, and if $\tilde f$ denotes an almost analytic extension of f as given by (6.1) with $n=1$, the Helffer-Sjöstrand formula and the second resolvent equation for H_0 and H imply that :

$$\psi\equiv f(H)\psi_0=\psi_0-\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial\tilde f}{\partial\bar z}(H-z)^{-1}V(H_0-z)^{-1}\psi_0dz\wedge d\bar z. \tag{6.4}$$

Let J_i be a C^2-function supported near V, that is

$$J_i(x)=\begin{cases}1 & \text{if } d/2<x<L_0-d/2\\ 0 & \text{if } x<d/4 \text{ or } x>L_0-d/4,\end{cases}$$

so that $VJ_i=V$ and $V_0J_i=0$. Applying the geometric resolvent equation (6.3) for $J=J_i$, H_L and H_0, we have

$$J_i(H_0-z)^{-1}=(H_L-z)^{-1}\left[J_i+W(J_i)(H_0-z)^{-1}\right], \tag{6.5}$$

where $W(J_i)=[H_L,J_i]$. Furthermore, $\|J_i\psi_0\|_{L^2(\mathbb{R}^2)}=O(\mathrm{e}^{-\alpha Bd^2})$, and from the decay properties of $(H_L-z)^{-1}$, it follows that :

$$\begin{aligned}&\|V(H_L-z)^{-1}W(J_i)(H_0-z)^{-1}\psi_0\|_{L^2(\mathbb{R}^2)}\\ \le\ &\|V\|_{L^\infty(\mathbb{R}^2)}\mathrm{e}^{-\alpha Bd^2}\mathrm{d}^{-1}(z,\sigma(H_L))|\mathrm{Im}(z)|^{-1}\left(\|J_i'\|_{L^\infty(\mathbb{R})}+\|J_i''\|_{L^\infty(\mathbb{R})}\right).\end{aligned}$$

Inserting (6.5) and this inequality in (6.4) gives :

$$\psi = \psi_0 + \xi \text{ with } \|\xi\|_{L^2(\mathbb{R}^2)} \leq C_1 \|V\|_{L^\infty(\mathbb{R}^2)} \mathrm{e}^{-C_2 B d^2} \tag{6.6}$$

for some constants C_1 and C_2 depending only on f, J_i and Δ. It is easy to show that the same type of estimate holds for $\|v_y \xi\|_{L^2(\mathbb{R}^2)}$, so that $(\psi, v_y \psi)_{L^2(\mathbb{R}^2)} = O(B^{1/2})$, as in (3.6) provided $\|V\|_{L^\infty(\mathbb{R}^2)} \ll \mathrm{e}^{C_2 B d^2}$.

Remark 6.1. There is a major difference between this result and the one obtained in section 4. Here estimates on $(\psi, v_y \psi)_{L^2(\mathbb{R}^2)}$ are not uniform in $\psi \in P_H(\tilde{\Delta}) L^2(\mathbb{R}^2)$ for some $\tilde{\Delta}$ containing Δ. In fact, constant C_1 in (6.6) depends on $\|f'\|_{L^\infty(\mathbb{R})}$. In particular, if f is changed into f_t where $f_t(E) = \mathrm{e}^{-itE} f(E)$ so that $\psi_t = f_t(H)\psi_0$ is the solution of the Schrödinger equation with initial data ψ , then C_1 in (6.6) grows like t^2. In other words we cannot conclude that ψ_t carries a current of definite sign beyond a time $T(B) = O(\mathrm{e}^{C_2 \frac{Bd^2}{2}})$, that is the quantum tunneling time between the two edges. The same remark applies to the other situations considered below.

Now we don't make any assumption on the support of V but rather investigate a situation closer to the one considered in the perturbed one-edge case in section 4. More precisely define the "bulk Hamiltonian"

$$H_1 = H_L + V$$

(i.e. H_1 is obtained from H by removing the two edges); let $\Delta \subset \tilde{\Delta}$ as above and such that $\tilde{\Delta} \cap \sigma(H_1) = \emptyset$ (this is satisfied for example if $\|V\|_{L^\infty(\mathbb{R}^2)} \leq CB$ for some constant C smaller than $\mathrm{dist}(\tilde{\Delta}, \sigma(H_L))$. Let J_0 satisfy :

$$J_0(x) = \begin{cases} 1 & \text{if } x < L_0/2 \\ 0 & \text{if } x > L_0/2 + 1. \end{cases}$$

One has $HJ_0 = H^- J_0$, where we recall that $H^- = H_0^- + V$ is obtained from H by removing the confining potential on the right edge. Then let $\psi_0 \in P_{H_0^-}(\Delta) L^2(\mathbb{R}^2)$ and define $\psi \equiv f(H)\psi_0$ where $f \in C^2(\mathbb{R})$ has support in $\tilde{\Delta}$; let also $\psi_1 = f(H^-)\psi_0$. Then if E_0 denotes the middle of Δ one has

$$\|(H^- - E_0)\psi_0\|_{L^2(\mathbb{R}^2)} \leq \left(\frac{|\tilde{\Delta}|}{2} + \|V\|_{L^\infty(\mathbb{R}^2)} \right) \|\psi_0\|_{L^2(\mathbb{R}^2)},$$

so that $\|\psi_1\|_{L^2(\mathbb{R}^2)} > 1/2$, if $|\tilde{\Delta}|/2 + \|V\|_{L^\infty(\mathbb{R}^2)} < 1/2 \mathrm{dist}(E_0, \Delta_f)$, where f is one on $\Delta_f \subset \tilde{\Delta}$. Since ψ_0 is an edge state, one also has $\psi = f(H) J_0 \psi_0 + O(\mathrm{e}^{-\alpha B L_0^2})$. By (6.1) and (6.3), one has:

$$\psi = J_0 \psi_1 - \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (H - z)^{-1} W(J_0) (H^- - z)^{-1} \psi_0 dz \wedge d\bar{z} + O(\mathrm{e}^{-\alpha B L_0^2}). \tag{6.7}$$

Now construct a function J_1 which is one on the support of J_0' as follows :

$$J_1(x) = \begin{cases} 1 & \text{if } L_0/4 < x < 3L_0/4 \\ 0 & \text{if } x > 3L_0/4 + 1 \text{ or } x < L_0/4 - 1 \end{cases}$$

Then $W(J_0) = J_1 W(J_0)$ and (6.3) gives

$$(H - z)^{-1} J_1 = J_1 (H_1 - z)^{-1} + (H - z)^{-1} W(J_1) (H_1 - z)^{-1}. \tag{6.8}$$

Inserting (6.8) in (6.7) gives

$$\begin{aligned}\psi &= J_0\psi_1 + \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} J_1(H_1-z)^{-1}W(J_0)(H^- - z)^{-1}\psi_0 dz\wedge d\bar{z} \\ &\quad + \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (H-z)^{-1}W(J_1)(H_1-z)^{-1}W(J_0)(H^- - z)^{-1}\psi_0 dz\wedge d\bar{z} \\ &\quad + O(\mathrm{e}^{-\alpha B L_0^2}). \end{aligned} \tag{6.9}$$

Notice that by the geometric resolvent equation (6.3) and using $W(J_0) = -W(1-J_0)$ one has

$$(H_1-z)^{-1}W(J_0)(H^- - z)^{-1} = (1-J_0)(H^- - z)^{-1} - (H_1-z)^{-1}(1-J_0).$$

By assumption $\mathrm{d}(z,\sigma(H_1)) = d_0$ for some $d_0 > 0$ when $z \in \mathrm{supp}(\tilde{f})$ so that the first two terms on the right hand side of (6.9) give $[J_0 + (1-J_0)J_1]\psi_1$. Obviously the expectation of v_y in this state is $O(B^{1/2})$. It remains to show from the results of section 3 that the last term on the right hand side of (6.9) is negligible provided $\|V\|_{L^\infty(\mathbb{R}^2)}/B$ is small enough; this follows from the fact that the kernel of $(H_1-z)^{-1}$ decreases exponentially when $z \in \mathrm{supp}\left(\frac{\partial \tilde{f}}{\partial \bar{z}}\right)$. This can be seen e.g. from a Combes-Thomas argument since the support of J_0' and J_1' are at a distance $O(\frac{L_0}{4})$. One finally obtains

$$\|\psi - [J_0 + (1-J_0)J_1]\psi_1\|_{L^2(\mathbb{R}^2)} \le C_1 \mathrm{e}^{-BL_0^2} + C_2 \mathrm{e}^{-\frac{d_0 L_0 B^{-1/2}}{2}},$$

where C_2 depends on $\|f'\|_{L^1(\mathbb{R})}$ and $\|f''\|_{L^1(\mathbb{R})}$, and similar estimates for the expectations of v_y. Typically, d_0 is $O(B)$ so that if ψ_t is the solution of the Schrödinger equation with initial data ψ then $(\psi, v_y\psi)_{L^2(\mathbb{R}^2)}$ will be of order $O(B^{1/2})$ for $t < T(B)$ where $T(B) < C\inf(\mathrm{e}^{BL_0^2}, \mathrm{e}^{\frac{d_0 L_0 B^{-1/2}}{2}})$.

7. Concluding remarks

Results presented here leave open questions which are not only of purely mathematical interest. Among them is the optimality of assumptions made on the impurity potential V. We have seen that global conditions on the smallness of $\|V\|_{L^\infty(\mathbb{R}^2)}/B$ can be considerably weakened if extra assumptions are made in the support of the impurity potential. In view of formula (4.1) it seems that a natural borderline for the size of V to preserve current carrying edge states should be $O(B^{3/2})$ and not $O(B)$ since the first term of the right hand side of this equality can be expected to be $O(B^{3/2})$ from the estimates of section 3, provided a sufficiently large fraction of the edges is free of impurities. On the other hand this also seems to be the correct borderline for the disorder in order that Anderson localization holds in the middle of Landau bands (this question is under current investigation).

Another open problem concerns effectiveness of tunneling in the strip problem and whether it is possible to go beyond the results shown in section 5; certainly in finite samples the tunneling time $T(B)$ is much larger than the time required for an electron moving along the edges to reach the measuring devices at the ends of the sample. In this respect this result is consistent with the analysis of C. Ferrari and N. Macris in [28]. Nevertheless it is an interesting mathematical question to see if edges current survive for all times in an infinite sample.

We have not discussed here some important related problem concerning properties of edge conductivity, in particular its quantization. An important step has been achieved in [10] and rederived very recently in [29]. These authors show that for discrete models edge conductivity equals bulk conductivity as defined using Kubo-Chern formula (see [30]). But quantization of edge conductivity for the continuous models considered here remains open.

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