DETERMINING THE POTENTIAL AND THE GRADIENT COUPLING OF TWO-STATE QUANTUM SYSTEMS IN AN INFINITE WAVEGUIDE

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ABSTRACT. We consider the inverse coefficient problem of simultaneously determining the space dependent electric potential, the zero-th order coupling term and the first order coupling vector of a two-state Schrödinger equation in an infinite cylindrical domain of $\mathbb{R}^n$, $n \geq 2$, from finitely many partial boundary measurements of the solution. We prove that these $n+3$ unknown scalar coefficients can be Hölder stably retrieved by $(n+1)$-times suitably changing the initial condition attached at the system.

Keywords: Inverse problem, stability estimate, two-state Schrödinger equation.

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1. Introduction

The present article deals with a system of two Schrödinger equations on an infinite cylindrical domain, that are interconnected through a linear gradient coupling. This formalism is used to describe the simple mixing of quantum states, leading to physical phenoma with significant applications such as lasers or quantum computers.

1.1. **Settings.** Throughout this article, ω is a bounded domain of $\mathbb{R}^{n-1}$, $n \geq 2$, with smooth boundary $\gamma := \partial\omega$, and $\Omega := \omega \times \mathbb{R}$. For $T > 0$, we consider the following initial-boundary value problem (IBVP) with initial states $u_0^{\pm}$ and non-homogenous Dirichlet boundary conditions $g^{\pm}$, for the coupled Schrödinger equations in the unknowns $u^{\pm}$,

$$(1.1) \qquad \begin{cases} -i\partial_t u^+ - \Delta u^+ + q^+ u^+ + A \cdot \nabla u^- + pu^- = 0 & \text{in } Q := \Omega \times (0,T) \\ -i\partial_t u^- - \Delta u^- + q^- u^- - A \cdot \nabla u^+ + pu^+ = 0 & \text{in } Q \\ u^+(\cdot,0) = u_0^+,\ u^-(\cdot,0) = u_0^- & \text{in } \Omega \\ u^+ = g^+,\ u^- = g^- & \text{on } \Sigma := \Gamma \times (0,T), \end{cases}$$

where $\Gamma := \gamma \times \mathbb{R}$. Since Γ is unbounded, let us make the above boundary condition more precise. For all $x \in \Omega$, we write $x = (x', x_n)$ where $x' = (x_1, \ldots, x_{n-1}) \in \omega$ and $x_n \in \mathbb{R}$, and using a standard density argument we extend the mapping

$$\begin{array}{ccc} C_0^\infty(\mathbb{R} \times (0,T), H^2(\omega)) & \to & L^2(\mathbb{R} \times (0,T), H^{\frac{3}{2}}(\omega)) \\ w & \mapsto & \left[(x_n,t) \in \mathbb{R} \times (0,T) \mapsto w(\cdot, x_n, t)_{|\gamma}\right], \end{array}$$

to a bounded operator γ_0 acting from $L^2(\mathbb{R} \times (0,T), H^2(\omega))$ into $L^2(\mathbb{R} \times (0,T), H^{\frac{3}{2}}(\gamma))$. Then, for all $u^{\pm} \in L^2(0,T,H^2(\Omega))$, the boundary condition in (1.1) reads $\gamma_0 u^{\pm} = g^{\pm}$.

In the present paper we aim to stably retrieve the electric potentials $q^{\pm} : \Omega \to \mathbb{R}$, the zero-th order coupling term $p : \Omega \to \mathbb{R}$ and the first order coupling vector $A : \Omega \to \mathbb{R}^n$, by finitely many partial boundary measurements over the entire time-span $(0,T)$ of the solution $u^{\pm}$ to (1.1). In contrast with [20] where the spatial domain Ω is bounded, here we consider an infinitely extended cylindrical domain and we address the problem of simultaneous identification of non-compactly supported unknown coefficients p, $q^{\pm}$ and A. This requires a slightly different and technically more demanding approach than the one implemented in [20].

1.2. **Motivations.** Two-state (or two level) quantum systems are the simplest non-trivial quantum systems found in nature. The best-known example of a two-state system is the spin (an intrinsic form of angular momentum) of an electron, where the two levels are represented by spin-up and spin-down states. Electrons can behave as the combination of both states at the same time. This quantum feature called superposition is a fundamental concept in quantum mechanics where quantum systems can exist in multiple states simultaneously. Being in two states at the same time makes electrons good candidates for quantum bits or qubits, which is the fundamental unit of information in quantum computing. Unlike classical bits, which can only be in one state (0 or 1), qubits can exist in a superposition of both states simultaneously. This superposition property is essential for the field of quantum computing, see [32, 36], because it allows quantum computers to process vast amounts of information at once.

Spin qubits need a suitable material to house and control them, as well as read out information in them. Taking that into consideration materials scientists introduced spinning electrons as qubits in a host carbon nanotube. A carbon nanotube is a nomaterial made from carbon atoms only, that has a hollow tubular shape and has thickness of about one nanometer. A great deal of research has been done over the last decade on using carbon nanotubes in complex computing systems, see [16, 28, 31]. This is because carbon-nanotube-based computing promises to start a new era of electronics that are faster and more energy efficient, see [17, 29]. Since carbon nanotubes are highly elongated cylindrical structures with a length-to-diameter ratio of up to 10^8, they enjoy valuable physical properties relevant to electronics, optics, and materials science, see [1, 25, 30, 33]. In a carbon nanotube, electrons are essentially free to propagate along the wire axis (their motion is confined in directions orthogonal to it), resulting in enhanced electrical conductivity, see [24, 37]. As a matter of fact, carbon nanotubes ferry electricity so well that they make better semiconductors than silicon: carbon nanotube processors can run three times faster than silicon ones, and they consume about one-third as much energy as silicon processors, see [29]. Moreover, carbon nanotubes are mostly free of fluctuating nuclear spins that would interfere with the spin of the electron and reduce its coherence time, which is the key property for any practical qubit because it defines the number of quantum operations that can be performed in the lifetime of the qubit, see [12].

It is still a long way from spin qubit in a carbon nanotube to practical technologies, but for all the above mentioned reasons materials scientists are aiming to use carbon nanotubes for spin-based quantum computing, in which the spin of a single electron would represent a bit of data. In view of the foregoing, the IBVP (1.1) may be regarded as a tentative (simplified) mathematical model for carbon-nanotube-based spin-quantum computing, describing the time evolution of a spin-$\frac{1}{2}$ particle such as an electron, confined in an idealized carbon nanotube Ω. The particle's spin can assume values $\pm\frac{\hbar}{2}$, where $\hbar$ represents the reduced Planck constant but, for the sake of notational simplicity, the various physical constants including $\hbar$, the charge and the mass of the particle, are taken equal to one in (1.1). Following [20, 26, 35, 39], the two states $u^{\pm}$ governed by (1.1) are strongly bound together through linear gradient coupling $pu^{\mp} \pm A \cdot \nabla u^{\mp}$, which guarantees that there is a non zero probability for a spin-qubit to go from one state u^+ to the other u^-, or vice versa. We refer the reader to [14, 20, 27] and the references therein for the relevance of gradient coupling in the context of partial differential equations.

1.3. **Bibliography.** The mathematical literature devoted to inverse coefficient problems for the dynamic Schrödinger equation is so extensive that this presentation is not intended to be exhaustive, but we can mention [3, 4, 6, 9, 23] where zero-th or/and first order unknown coefficients of the Schrödinger equation are determined by the Dirichlet-to-Neumann map. These articles assume knowledge of infinitely many boundary data, but in [2, 38] the real-valued electric potential is stably retrieved by one partial lateral observation of the solution. This result was extended to complex-valued electric potentials in [18]. The boundary measurement in [2, 18, 38] is taken on a subpart of the boundary fulfilling a geometric condition related to geometric optics condition insuring observability. This condition was relaxed to arbitrarily small sub-boundaries in [4], provided the potential is known in the vicinity of the boundary. The inverse problem of determining the magnetic vector potential of the autonomous Schrödinger equation is addressed in [18]. The same problem for the space-varying part of the magnetic potential appearing in a non-autonomous Schrödinger equation is treated in [13]. In both cases, the n-th dimensional unknown magnetic vector potential, $n \geq 1$, is retrieved from n partial Neumann data obtained by n-times suitably selecting the initial condition attached at the magnetic Schrödinger equation.

The strategy of [2, 13, 18, 38] relies on a Carleman inequality specifically designed for the Schrödinger equation, see [18, 34, 38] for actual examples of such weighted energy estimates. The idea of using a Carleman estimate for solving inverse problems goes back to 1981 and was introduced by A. L. Bukhgeim and M. V. Klibanov in their seminal article [11]. Since then, the Bukhgeim-Klibanov approach has been successfully applied to parabolic, hyperbolic and Schrödinger systems and even to coupled systems of partial differential equations. We refer the reader to [19] and references therein, for a complete survey of multidimensional inverse problems solved by the Bukhgeim-Klibanov method.

In all the aforementioned papers, the Schrödinger equation under study is stated on a bounded spatial domain. The inverse problem of determining the electric potential of the Schrödinger equation stated in an infinite waveguide is examined in [5, 22]. This is achieved by mean of a specifically designed Carleman estimate for the Schrödinger equation in an unbounded cylindrical domain, which is established in [21]. All the articles listed above are concerned with the regular ("one state") Schrödinger equation. In [26], assuming that the gradient coupling vector is known, the authors show that the zero-th order coupling term of a two state magnetic Schrödinger equation is uniquely determined by one partial Neumann data. Recently in [39], the electric potential of a strongly coupled Schrödinger equations in a bounded spatial domain was Lipschitz stably retrieved by one partial (internal or boundary) measurement of the solution to the system. In [20], the zero-th and first order coefficients of the coupling are Lipschitz stably recovered by finitely many partial boundary observations of the solution.

All the coupled Schrödinger equations under study in [20, 26, 39] were stated on a bounded spatial domain. The main purpose of the present paper is to extend the result of [20] to the case of an unbounded waveguide. Namely, it was proved in [20] that when two states are confined to a bounded spatial domain, the electric potential and the coupling coefficients can be stably determined by a finite number of partial boundary observations of the system. Here, we aim for the same identification result when the motion of the quantum particle is no longer bounded and may escape to infinity in one direction over time.

1.4. **Notations.** Throughout this text $x = (x_1, ..., x_n)$ is a generic point of $\overline{\Omega}$ that is sometimes written $x = (x', x_n)$ where $x' = (x_1, \cdots, x_{n-1}) \in \overline{\omega}$ is the variable of the transverse section of Ω and $x_n \in \mathbb{R}$ is the longitudinal variable. For all $x = (x', x_n) \in \Gamma$, the outward unit normal ν to Γ reads $\nu(x) = \nu(x') = (\nu'(x'), 0)^T$, where $\nu'(x') \in \mathbb{R}^{n-1}$ is the outgoing normal vector to γ at x' and a^T denotes the transpose of the row vector a.

For all $i = 1, \dots, n$ we set $\partial_i := \frac{\partial}{\partial x_i}$ in such a way that $\nabla := (\partial_1, \dots, \partial_n)^T$ (resp., $\nabla' := (\partial_1, \dots, \partial_{n-1})^T$) is the gradient operator with respect to $x = (x_1, \dots, x_n)$ (resp., $x' = (x_1, \dots, x_{n-1})$). Similarly, we write $\partial_t = \frac{\partial}{\partial t}$. For the sake of shortness we write ∂_{ij}^2, $i, j = 1, \dots, n$, instead of $\partial_i \partial_j$ and as usual we denote by Δ the Laplace operator $\partial_1^2 + \dots + \partial_n^2$. Next, for any multi-index $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, we put $|k| := k_1 + \dots + k_n$ and $\partial_x^k = \partial_1^{k_1} \dots \partial_n^{k_n}$. .

Further, the symbol $\cdot$ denotes the scalar product in $\mathbb{C}^m$, $m \in \mathbb{N}$, and we set $|\zeta| := \sqrt{\zeta \cdot \zeta}$ for all $\zeta \in \mathbb{C}^m$. We simply write $\nabla \cdot$ for the divergence operator in $\mathbb{R}^n$ and we set $\partial_\nu u := \nabla u \cdot \nu = \nabla' \cdot \nu'$.

Finally, for all $r > 0$ and $s > 0$, we introduce $H^{r,s}(\Sigma) := L^2(0, T; H^r(\Gamma)) \cap H^s(0, T; L^2(\Gamma))$ where $H^s(\Gamma)$ denotes the usual Sobolev space on Γ of order s.

1.5. **Main results.** Prior to investigating the inverse problem under study in this article, we examine the well-posedness issue for the forward problem associated with (1.1). For this purpose we introduce the Hamiltonian operator acting on $(C_0^\infty(Q)')^2$,

$$\mathcal{H}(A, p, q^\pm) := \begin{pmatrix} -\Delta + q^+ & A \cdot \nabla + p \\ -A \cdot \nabla + p & -\Delta + q^- \end{pmatrix}$$

and state the following existence, uniqueness and regularity result for the solution to the IBVP (1.1).

Proposition 1.1. *Let $m \in \mathbb{N}$ and assume that γ is $\mathcal{C}^{2(m+1)}$. Let $A \in W^{2m+1,\infty}(\Omega, \mathbb{R}^n) \cap C^{2(m-1)}(\overline{\Omega}, \mathbb{R}^n)$ be such that $\nabla \cdot A = 0$ a.e. in Ω, let $p \in W^{2m+1,\infty}(\Omega, \mathbb{R}) \cap C^{2(m-1)}(\overline{\Omega}, \mathbb{R})$ and let $q^\pm \in W^{2m+1,\infty}(\Omega, \mathbb{R}) \cap C^{2(m-1)}(\overline{\Omega}, \mathbb{R})$ satisfy*

$$\|A\|_{W^{2m+1,\infty}(\Omega)} + \|p\|_{W^{2m+1,\infty}(\Omega)} + \|q^+\|_{W^{2m+1,\infty}(\Omega)} + \|q^-\|_{W^{2m+1,\infty}(\Omega)} \leq M,$$

for some a priori fixed positive constant M. Then, for all $g = (g^+, g^-)^T \in H^{2(m+7/4),m+7/4}(\Sigma)^2$ and all $u_0 = (u_0^+, u_0^-)^T \in H^{2m+3}(\Omega)^2$ fulfilling the following compatibility conditions

$$\partial_t^\ell g(\cdot, 0) = (-i)^\ell \mathcal{H}(A, p, q^\pm)^\ell u_0 \text{ on } \Gamma,\ \ell = 0, \cdots, m, \tag{1.2}$$

the IBVP (1.1) *admits a unique solution* $u=(u^+,u^-)^T \in \cap_{\ell=0}^{m+1} H^{m+1-\ell}(0,T;H^{2\ell}(\Omega)^2)$. *Moreover, there exists a positive constant* C*, depending only on* ω*,* T *and* M *such that*

$$\sum_{\ell=0}^{m+1} \|u\|_{H^{m+1-\ell}(0,T;H^{2\ell}(\Omega)^2)} \leq C\left(\|u_0\|_{H^{2m+3}(\Omega)^2} + \|g\|_{H^{2(m+7/4),m+7/4}(\Sigma)^2}\right). \tag{1.3}$$

Notice that the divergence-free condition on A requested by Proposition 1.1 is to guarantee that $\mathcal{H}(A,p,q^{\pm})$ endowed with homogeneous Dirichlet boundary condition on Γ, has a self-adjoint realization $H(A,p,q^{\pm})$ in $L^2(\Omega)^2$, see [20, Lemma 2.1]. As a consequence the operator $-iH(A,p,q^{\pm})$ is m-dissipative in $L^2(\Omega)^2$, and since the IBVP (1.1) is equivalently rewritten as

$$\begin{cases} -i\partial_t u + \mathcal{H}(A,p,q^{\pm})u = 0 & \text{in } Q \\ u(\cdot,0)=u_0 & \text{in } \Omega \\ u=g & \text{on } \Sigma, \end{cases}$$

the statement of Proposition 1.1 follows by arguing in the same way as in the proof of [20, Lemma 2.3].

We point out that the regularity assumptions on the coefficients A, p and $q^{\pm}$, the initial states $u_0^{\pm}$ and the boundary conditions $g^{\pm}$, in Proposition 1.1, are only sufficient conditions ensuring a higher order of regularity of the solution $u^{\pm}$ to (1.1), as requested by the analysis of the inverse problem under study in this article. As a matter of fact the Bukhgeim-Klibanov method requires $\partial_t u^{\pm}$ and $\partial_t \nabla u^{\pm}$ to be bounded in Q, which can be achieved upon taking m in Proposition 1.1, sufficiently large relative to n. Namely, we choose

$$N \in \mathbb{N} \cap \left(\frac{n+2}{4}+1, \frac{n+2}{4}+2 \right], \tag{1.4}$$

pick M, κ, ϱ, $\mathfrak{a}$, $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathbb{R}_+$, and for $A_0 \in W^{2N+1,\infty}(\Omega,\mathbb{R}^n) \cap C^{2(N-1)}(\overline{\Omega},\mathbb{R}^n)$, $p_0 \in W^{2N+1,\infty}(\Omega,\mathbb{R}) \cap C^{2(N-1)}(\overline{\Omega},\mathbb{R})$ and $q_0^{\pm} \in W^{2N+1,\infty}(\Omega,\mathbb{R}) \cap C^{2(N-1)}(\overline{\Omega},\mathbb{R})$, we introduce the set of unknown electric potentials as

$$\begin{aligned} \mathcal{P}_{\mathfrak{p}}(p_0) \quad := \quad & \Big\{ p \in W^{2N+1,\infty}(\Omega,\mathbb{R}) \cap C^{2(N-1)}(\overline{\Omega},\mathbb{R}) \text{ s.t. } \|p\|_{W^{2N+1,\infty}(\Omega)} \leq M, \\ & \partial_x^k p = \partial_x^k p_0 \text{ on } \Gamma,\ k=0,\ldots,2(N-1) \text{ and } |(p-p_0)(\cdot,x_n)| \leq \mathfrak{p} e^{-\kappa \langle x_n \rangle^{\varrho}},\ x_n \in \mathbb{R} \Big\}, \end{aligned} \tag{1.5}$$

the set of unknown zero-th order coupling coefficients as $\mathcal{P}_{\mathfrak{q}}(q_0^{\pm})$, and the set of unknown first order coupling vectors as

$$\begin{aligned} \mathcal{A}_{\mathfrak{a}}(A_0) \quad := \quad & \Big\{ A \in W^{2N+1,\infty}(\Omega,\mathbb{R}^n) \cap C^{2(N-1)}(\overline{\Omega},\mathbb{R}) \text{ s.t. } \|A\|_{W^{2N+1,\infty}(\Omega)^n} \leq M,\ \nabla\cdot A = 0 \text{ in } \Omega, \\ & \partial_x^k A = \partial_x^k A_0 \text{ on } \Gamma,\ |k|=0,\ldots,2(N-1) \text{ and } |(A-A_0)(\cdot,x_n)| \leq \mathfrak{a} e^{-\kappa \langle x_n \rangle^{\varrho}},\ x_n \in \mathbb{R} \Big\}. \end{aligned} \tag{1.6}$$

Here, the notation ∂_x^k for $|k|=m\in\mathbb{N}_0$ is a shorthand for $\partial_1^{k_1}\ldots\partial_n^{k_n}$ where $k=(k_1,\ldots,k_n)\in\mathbb{N}_0^n$ satisfies $|k|=k_1+\ldots+k_n=m$.

Then, the main result of this article can be stated as follows.

Theorem 1.2. *Assume that* γ *is* $\mathcal{C}^{2(N+1)}$. *For* $j=1,2$, *let* $A_j \in \mathcal{A}_{\mathfrak{a}}(A_0)$ *satisfy*

$$\exists y_* \in \mathbb{R}_+,\ a_{1,n}(x',x_n) = a_{2,n}(x',x_n),\ x'\in\omega,\ x_n \in (-y_*,y_*), \tag{1.7}$$

let $p_j \in \mathcal{P}_{\mathfrak{p}}(p_0)$ *and let* $q_j^{\pm} \in \mathcal{P}_{\mathfrak{q}}(q_0^{\pm})$.

Then, there exist a sub-boundary $\gamma_* \subset \partial\omega$ *and a set of* $n+1$ *initial states* $u_0^k=(u_0^{+,k},u_0^{-,k})^T \in H^{2N+3}(\Omega)^2$ *and boundary conditions* $g^k=(g^{+,k},g^{-,k})^T \in H^{2(N+7/4),N+7/4}(\Sigma)^2$, $k=1,\ldots,n+1$, *fulfilling the compatibility conditions*

$$\partial_t^{\ell} g^k(\cdot,0) = (-i)^{\ell}\mathcal{H}(A_0,p_0,q_0^{\pm})^{\ell} u_0^k \text{ on } \Gamma,\ \ell=0,\cdots,N, \tag{1.8}$$

such that for all $\theta \in \left(0, \frac{1}{2}\right)$, the following estimate

$$\begin{aligned}(1.9) \qquad & \|A_1 - A_2\|^2_{L^2(\Omega)} + \|p_1 - p_2\|^2_{L^2(\Omega)} + \left\|q_1^+ - q_2^+\right\|^2_{L^2(\Omega)} + \left\|q_1^- - q_2^-\right\|^2_{L^2(\Omega)} \\ \leq \quad & C \sum_{k=1}^{n+1} \left(\left\|\partial_\nu \partial_t u_1^{-,k} - \partial_\nu \partial_t u_2^{-,k}\right\|^\theta_{L^2(\Sigma_*)} + \left\|\partial_\nu \partial_t u_1^{+,k} - \partial_\nu \partial_t u_2^{+,k}\right\|^\theta_{L^2(\Sigma_*)} \right),\end{aligned}$$

holds for some positive constant C depending only on ω, T, γ^*, M, y_*, θ, κ, ϱ, $\mathfrak{a}$, $\mathfrak{p}$, $\mathfrak{q}$ and $(u_0^{\pm,k}, g^{\pm,k})$, $k = 1, \dots, n+1$. Here, $\Sigma_* := \gamma_* \times \mathbb{R} \times (0,T)$ and $u_j^k = (u_j^{+,k}, u_j^{-,k})^T$, for $j = 1, 2$, is the solution to (1.1) *given by Proposition 1.1, where $(A_j, p_j, q_j^\pm, u_0^{\pm,k}, g^{\pm,k})$ is substituted for $(A, p, q^\pm, u_0^\pm, g^\pm)$.*

1.6. **Brief comments.** Theorem 1.2 claims that $n+1$ Neumann data stably determine $n+2$ unknown scalar coefficients (strictly speaking there are $n+3$ unknown scalar coefficients in the inverse problem that Theorem 1.2 is dealing with, but since the n components of the gradient coupling vector are bound together through the divergence free condition, they only amount for $n-1$ free unknown scalar coefficients). This may seem surprising from the viewpoint of the analysis of inverse problems, but it should be noticed that Assumption (1.7) implies full knowledge of the n-th component of A on a bounded subpart of Ω.

The statement and the strategy of the proof of Theorem 1.2 are very similar to the ones of [20, Theorem 1.2], which holds for a bounded spatial domain Ω. Nevertheless, there are two major differences in the derivation of Theorem 1.2 as compared to the one of [20, Theorem 1.2]. Firstly, the Carleman estimate that is used in the present article is specifically designed for a Schrödinger equation in an unbounded cylindrical domain, and it is quite different from the one used in [20]. Secondly, the construction of the initial states u_0 used for probing the system in the analysis of the inverse problem under examination in this article, is more delicate than in [20]. This can be understood from the fact that it is technically more challenging to design a suitable set of square integrable initial states u_0 in an infinitely extended domain than in a bounded one. As a matter of fact, we shall see in Section 3 that an an additional decay condition with respect to the infinite direction of the waveguide is needed on u_0.

We point out that the method of derivation of Theorem 1.2 presented in this work does not apply to magnetic Schrödinger equations. This is due to the time-symmetrization technique that we use to avoid observation data at $t = 0$ over Ω, that is no longer valid in presence of a non-zero magnetic potential. Actually the magnetic case requires a specific treatment for applying the Bukhgeim-Klibanov method to Schrödinger equations, and we refer the reader to [18] for more details on this peculiar topic.

1.7. **Outline.** The paper is designed as follows: In the following section we collect several technical results needed for the proof of Theorem 1.2, which is given in Section 3.

2. Preliminaries

We first establish that the solution to (1.1) is bounded in Q.

2.1. **Boundedness of the solution.** The result we have in mind is as follows.

Lemma 2.1. *Assume that conditions of Proposition 1.1 are satisfied with $m = N$, where N is the same as in* (1.4)*. Then, the solution u to* (1.1) *lies in $W^{1,\infty}(0,T; W^{1,\infty}(\Omega)^2)$ and satisfies*

$$\|u\|_{W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^2)} \leq C,$$

for some positive constant C depending only on ω, T, M, u_0 and g.

Proof. We have $u \in H^2(0,T, H^{2(N-1)}(\Omega)^2)$ by Proposition 1.1, with $2(N-1) > \frac{n}{2} + 1$ from (1.4). Since $H^k(\Omega)$ is continuously embedded in $L^\infty(\Omega)$ for all $k > \frac{n}{2}$, according to [22, Lemma 2.7] (which extends the corresponding well-known Sobolev embedding theorem in $\mathbb{R}^n$, see e.g. [10, Corollary IX.13] or [15, Section 5.10, Problem 18], to the case of the unbounded cylindrical domain Ω), the result follows from this and (1.3). □

2.2. **Global Carleman estimate for the Schrödinger equation in $\Omega = \omega \times \mathbb{R}$.** In this section we establish a global Carleman estimate specifically designed for the Schrödinger equation in the unbounded cylindrical domain Ω. This estimate, which is our main tool in the proof of Theorem (1.2), is stated in Corollary 2.5, below. For this purpose we pick a function $\alpha \in C^4(\overline{\omega}, \mathbb{R}_+)$ and an open subset $\gamma_* \subset \partial\omega$ satisfying the following conditions:

Assumption 2.2.
(H1) $\exists c \in \mathbb{R}_+$ *s.t.* $|\nabla'\alpha(x')| \geq c$ *for all* $x' \in \omega$.
(H2) $\forall x' \in \gamma \setminus \gamma_*$, $\partial_\nu\alpha(x') = \nabla'\alpha(x') \cdot \nu'(x') < 0$.
(H3) $\exists \lambda_0 \in \mathbb{R}_+$, $\exists c \in \mathbb{R}_+$ *s.t.*

$$\lambda \left|\nabla'\alpha(x') \cdot \zeta\right|^2 + D^2\alpha(x', \zeta) \geq c \left|\zeta\right|^2, \ \zeta \in \mathbb{R}^{n-1}, \ x' \in \omega, \ \lambda \geq \lambda_0,$$

where $D^2\alpha(x') := \left(\partial^2_{i,j}\alpha(x')\right)_{1 \leq i,j \leq n-1}$ *and* $D^2\alpha(x', \zeta)$ *denotes the* $\mathbb{R}^{n-1}$*-scalar product of* $D^2\alpha(x')\zeta$ *with* ζ.

Remark 2.3. *We point out that there exist* α *and* γ_* *fulfilling the above conditions (H1), (H2) and (H3). As a matter of fact, for all* $x'_0 \in \mathbb{R}^{n-1} \setminus \overline{\omega}$ *fixed, this is the case of the function* $\alpha(x') = \left|x' - x'_0\right|^2$ *and any open subset* $\gamma_* \subset \gamma$ *such that* $\{x' \in \gamma; \ (x' - x'_0) \cdot \nu(x') \geq 0\} \subset \gamma_*$.

Next, we choose $r \in (1, +\infty)$, put $K := r \left\|\alpha\right\|_{L^\infty(\omega)}$ and set

$$\beta(x) := \alpha(x') + K, \ x = (x', x_n) \in \Omega. \tag{2.10}$$

Then, for $\lambda > 0$ fixed, we introduce the weight functions,

$$\varphi(t, x) := \frac{e^{2\lambda\beta(x)}}{(T+t)(T-t)} \quad \text{and} \quad \eta(t, x) := \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)}, \quad (t, x) \in \tilde{Q} := (-T, T) \times \Omega, \tag{2.11}$$

and for all $s > 0$, we define two operators M_j, $j = 1, 2$, acting in $(C_0^\infty)'(\tilde{Q})$, by

$$M_1 := i\partial_t + \Delta + s^2|\nabla\eta|^2 \quad \text{and} \quad M_2 := is\eta' + 2s\nabla\eta \cdot \nabla + s(\Delta\eta). \tag{2.12}$$

Evidently, M_1 (respectively, M_2) is the adjoint (respectively, skew-adjoint) part of the operator $e^{-s\eta}Le^{s\eta}$, where $L := -i\partial_t - \Delta$.
Let us notice for further use that

$$\eta(x, t) \geq \eta_0(x) > 0, \ (x, t) \in \tilde{Q}, \tag{2.13}$$

where $\eta_0(x) := \eta(0, x)$ for all $x \in \Omega$. This being said, we may now state the following global Carleman estimate, which is borrowed from [21, Proposition 3.3].

Proposition 2.4. *Suppose that* α *and* γ_* *fulfill Assumption 2.2. Let* β *be as in* (2.10) *and let* φ *and* η *be defined by* (2.11)*. Then, there exist two constants* $s_0 > 0$ *and* $C > 0$*, depending only on* T*,* ω *and* γ_**, such that the estimate*

$$\begin{aligned} s\|e^{-s\eta}\nabla_{x'}w\|^2_{0,\tilde{Q}} + s^3\|e^{-s\eta}w\|^2_{0,\tilde{Q}} + \sum_{j=1,2}\|M_je^{-s\eta}w\|^2_{0,\tilde{Q}} \\ \leq C\big(s\|e^{-s\eta}\varphi^{1/2}|\partial_\nu\beta|^{1/2}\partial_\nu w\|^2_{0,\tilde{\Sigma}_*} + \|e^{-s\eta}Lw\|^2_{0,\tilde{Q}}\big), \end{aligned}$$

holds for all $s \geq s_0$ *and any function* $w \in L^2(-T, T; H^1_0(\Omega))$ *verifying* $Lw \in L^2(\tilde{Q})$ *and* $\partial_\nu w \in L^2(-T, T; L^2(\Gamma_*))$*, where* $\tilde{\Sigma}_* = (-T, T) \times \Gamma_*$.

As a byproduct of Proposition 2.4, we have the following statement. Its proof can be found in [21, Section 4.1.1] but for the sake of self-containedness and the convenience of the reader, we provide it below.

Corollary 2.5. *Under the conditions of Proposition 2.4, we have*

$$\begin{aligned} & s^{-1/2}\left\|e^{-s\eta}\nabla'w\right\|^2_{L^2(\tilde{Q})} + s^{-1/2}\left\|e^{-s\eta}w\right\|^2_{L^2(\tilde{Q})} + \left\|e^{-s\eta_0}w(\cdot, 0)\right\|^2_{L^2(\Omega)} \\ \leq \ & Cs^{-3/2}\left(s\left\|e^{-s\eta}\varphi^{1/2}\left|\partial_\nu\beta\right|^{\frac{1}{2}}\partial_\nu w\right\|^2_{L^2(\tilde{\Sigma}_*)} + \left\|e^{-s\eta}Lw\right\|^2_{L^2(\tilde{Q})}\right) \end{aligned}$$

holds whenever $s \geq s_0$ *and* $w \in L^2(-T, T; H^1_0(\Omega))$ *satisfies* $Lw \in L^2(\tilde{Q})$ *and* $\partial_\nu w \in L^2(\tilde{\Sigma}_*)$*. Here,* $\tilde{\Sigma}_* := (-T, T) \times \Gamma_*$ *and* $\Gamma_* := \gamma_* \times \mathbb{R}$.

Proof. Put $w := e^{-s\eta}z$. Since $\lim_{t\longrightarrow -T}\eta(x,t)=+\infty$ for all $x\in\Omega$ then $\lim_{t\longrightarrow -T} w(\cdot,t)=0$ in $L^2(\Omega)$ and hence

$$\|w(\cdot,0)\|_{0,\Omega}^2=\int_{(-T,0)\times\Omega}\partial_t|w(x,t)|^2dxdt=2\Re\left(\int_{(-T,0)\times\Omega}\partial_t w\overline{w}(x,t)dxdt\right).$$

On the other hand, we have

$$\begin{aligned}
&\Im\left(\int_{(-T,0)\times\Omega}M_1 w\overline{w}(x,t)dxdt\right)\\
=\ &\Re\left(\int_{(-T,0)\times\Omega}\partial_t w\overline{w}(x,t)dxdt\right)+\Im\left(\int_{(-T,0)\times\Omega}(\Delta w\overline{w}+s^2|\nabla\eta|^2 w\overline{w})(x,t)dxdt\right)\\
=\ &\Re\left(\int_{(-T,0)\times\Omega}\partial_t w\overline{w}(x,t)dxdt\right)-\Im\left(\int_{(-T,0)\times\Omega}(|\nabla w|^2-s^2|\nabla\eta|^2|w|^2)(x,t)dxdt\right)\\
=\ &\Re\left(\int_{(-T,0)\times\Omega}\partial_t w\overline{w}(x,t)dxdt\right),
\end{aligned}$$

whence $\|w(\cdot,0)\|_{L^2(\Omega)}^2=2\Im\left(\int_{(-T,0)\times\Omega}M_1 w\overline{w}(x,t)dxdt\right)$. Therefore, we get

$$\|e^{-s\eta(\cdot,0)}z(\cdot,0)\|_{L^2(\Omega)}^2\le 2\|M_1 w\|_{L^2(\tilde{Q})}\|w\|_{L^2(\tilde{Q})}\le s^{-3/2}\left(s^3\|e^{-s\eta}z\|_{L^2(\tilde{Q})}^2+\|M_1 e^{-s\eta}z\|_{L^2(\tilde{Q})}^2\right)$$

with the help of the Cauchy Schwarz and Hölder inequalities. As a consequence, we have

$$\begin{aligned}
&s^{-1/2}\left(\|e^{-s\eta}z\|_{L^2(\tilde{Q})}^2+\|e^{-s\eta}\nabla z\|_{L^2(\tilde{Q})}^2\right)+\|e^{-s\eta(\cdot,0)}z(\cdot,0)\|_{L^2(\Omega)}^2\\
\le\ &s^{-3/2}\left(s\|e^{-s\eta}\nabla z\|_{L^2(\tilde{Q})}^2+s^3\|e^{-s\eta}z\|_{L^2(\tilde{Q})}^2+\|M_1 e^{-s\eta}z\|_{L^2(\tilde{Q})}^2\right)\\
\le\ &Cs^{-3/2}\left(s\|e^{-s\eta}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu\, z\|_{L^2(\tilde{\Sigma}_*)}^2+\|e^{-s\eta}Lz\|_{L^2(\tilde{Q})}^2\right),
\end{aligned}$$

by Proposition 2.4, which is the desired result. □

Armed with Corollary 2.5, we turn now to proving the main result of this article.

3. **Proof of Theorem 1.2**

We follow the strategy of Bukhgeim and Klibanov, which is to linearize the system and then differentiate it with respect to the time variable. This will put the unknowns of the inverse problem in the initial condition of the obtained system, which, in turn will be estimated in terms of the Neumann data with the aid of the Carleman estimate of Corollary 2.5.

3.1. **Linearization, time-differentiation and all that.** We start by linearizing the system (1.1). For this purpose we consider the two solutions $u_j=(u_j^+,u_j^-)^T$, $j=1,2$, to the IBVP (1.1) where $(A_j,p_j,q_j^\pm)$ is substituted for $(A,p,q^\pm)$. Then, $u^\pm:=u_1^\pm-u_2^\pm$ solves

$$\text{(3.14)}\qquad \begin{cases}
-i\partial_t u^+-\Delta u^++q_1^+u^+=-A_1\cdot\nabla u^--A\cdot\nabla u_2^--q^+u_2^+-p_1u^--pu_2^- & \text{in } Q\\
-i\partial_t u^--\Delta u^-+q_1^-u^-=A_1\cdot\nabla u^++A\cdot\nabla u_2^+-q^-u_2^--p_1u^+-pu_2^+ & \text{in } Q\\
u^+(\cdot,0)=0,\ u^-(\cdot,0)=0 & \text{in } \Omega\\
u^+=0,\ u^-=0 & \text{on } \Sigma,
\end{cases}$$

where $A := A_1 - A_2$, $p := p_1 - p_2$ and $q^\pm := q_1^\pm - q_2^\pm$. Further, $u^\pm$ lies in $H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^2(\Omega) \cap H_0^1(\Omega))$, we differentiate (3.14) with respect to the time-variable and find that

$$\begin{cases} -i\partial_t v^+ - \Delta v^+ + q_1^+ v^+ = -A_1 \cdot \nabla v^- - A \cdot \nabla \partial_t u_2^- - q^+ \partial_t u_2^+ - p_1 v^- - p\partial_t u_2^- & \text{in } Q \\ -i\partial_t v^- - \Delta v^- + q_1^- v^- = A_1 \cdot \nabla v^+ + A \cdot \nabla \partial_t u_2^+ - q^- \partial_t u_2^- - p_1 v^+ - p\partial_t u_2^+ & \text{in } Q \\ v^+(\cdot,0) = -i(A\cdot \nabla u_0^- + q^+ u_0^+ + p u_0^-) & \text{in } \Omega \\ v^-(\cdot,0) = -i(-A\cdot \nabla u_0^+ + q^- u_0^- + p u_0^+) & \text{in } \Omega \\ v^+ = 0,\ v^- = 0 & \text{on } \Sigma, \end{cases}$$

where $v^\pm := \partial_t u^\pm$. The next step is to extend $u_2^\pm$ to $\tilde{Q} = \Omega \times (-T,T)$ by setting $u_2^\pm(x,t) := \overline{u_2^\pm(x,-t)}$ for a.e. $(x,t) \in \Omega \times (-T,0)$. Since $u_0^\pm$, A, p and $q^\pm$ are-real valued, it is not hard to see that the function $v^\pm$, extended to $\Omega \times (-T,0)$ as $v^\pm(x,t) := -\overline{v^\pm(x,-t)}$, satisfies

(3.15)

$$\begin{cases} -i\partial_t v^+ - \Delta v^+ + q_1^+ v^+ = -A_1 \cdot \nabla v^- - A \cdot \nabla \partial_t u_2^- - q^+ \partial_t u_2^+ - p_1 v^- - p\partial_t u_2^- & \text{in } \tilde{Q} \\ -i\partial_t v^- - \Delta v^- + q_1^- v^- = A_1 \cdot \nabla v^+ + A \cdot \nabla \partial_t u_2^+ - q^- \partial_t u_2^- - p_1 v^+ - p\partial_t u_2^+ & \text{in } \tilde{Q} \\ v^+(\cdot,0) = -i(A\cdot \nabla u_0^- + q^+ u_0^+ + p u_0^-) & \text{in } \Omega \\ v^-(\cdot,0) = -i(-A\cdot \nabla u_0^+ + q^- u_0^- + p u_0^+) & \text{in } \Omega \\ v^+ = 0,\ v^- = 0 & \text{on } \tilde{\Sigma} := \Gamma \times (-T,T). \end{cases}$$

The main benefit of this time-symmetrization method already used in [2, 13, 20, 21, 22] for Schrödinger systems, is to apply the Carleman inequality of Corollary 2.5 on the extended domain $\tilde{Q}$ in order to avoid observation data at $t = 0$ over Ω, appearing in Carleman estimates on Q.

Put $\mu^\pm := \left\| e^{-s\eta_0} \varphi^{1/2} \left|\partial_\nu \beta\right|^{1/2} \partial_\nu v^\pm \right\|^2_{L^2(\tilde{\Sigma}_*)}$. Then, applying Corollary 2.5 to (3.15), we get for all $s \geq s_0$ that

$$\begin{aligned} & s^{-1/2} \left\| e^{-s\eta} \nabla' v^\pm \right\|^2_{L^2(\tilde{Q})} + s^{-1/2} \left\| e^{-s\eta} v^\pm \right\|^2_{L^2(\tilde{Q})} + \left\| e^{-s\eta_0} v^\pm(\cdot,0) \right\|^2_{L^2(\Omega)} \\ \leq \ & Cs^{-3/2} \left(s\mu^\pm + \left\| e^{-s\eta} \left(\pm A_1 \cdot \nabla v^\mp \pm A \cdot \nabla \partial_t u_2^\mp + q^\pm \partial_t u_2^\pm + p_1 v^\mp + p\partial_t u_2^\mp \right) \right\|^2_{L^2(\tilde{Q})} \right), \end{aligned} \tag{3.16}$$

for some positive constant C depending only on ω, T and γ_*. Taking into account that $\|A_1\|_{L^\infty(\Omega)} \leq M$, $\|p_1\|_{L^\infty(\Omega)} \leq M$, and that the two functions $\partial_t u_2^\pm$ and $\nabla \partial_t u_2^\pm$ are bounded on $\tilde{Q}$ by some positive constant depending only on ω, T, M, u_0 and g according to Lemma 2.1, (2.13) and (3.16) then yield that

$$\begin{aligned} & s^{-1/2} \left\| e^{-s\eta} \nabla' v^\pm \right\|^2_{L^2(\tilde{Q})} + s^{-1/2} \left\| e^{-s\eta} v^\pm \right\|^2_{L^2(\tilde{Q})} + \left\| e^{-s\eta_0} v^\pm(\cdot,0) \right\|^2_{L^2(\Omega)} \\ \leq \ & Cs^{-3/2} \left(s\mu^\pm + \left\| e^{-s\eta} \nabla_{x'} v^\mp \right\|^2_{L^2(\tilde{Q})} + \left\| e^{-s\eta} v^\mp \right\|^2_{L^2(\tilde{Q})} + \left\| e^{-s\eta_0} A \right\|^2_{L^2(\Omega)^n} + \left\| e^{-s\eta_0} q^\pm \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} p \right\|^2_{L^2(\Omega)} \right), \end{aligned}$$

provided $s \geq s_0$. Here and in the remaining part of this proof, C denotes a generic positive constant which may change from line to line. Although the constant C depends only on ω, T, γ_*, M, u_0 and g in the above estimate, in the sequel it might also depend on one or several of the parameters n, y_*, κ, ϱ, $\mathfrak{a}$, $\mathfrak{p}$, $\mathfrak{q}$ and θ of the problem, as well. Nevertheless, we shall not systematically specify the dependence of C with respect to the above mentioned parameters.

As a consequence we have

$$\begin{aligned} & s^{-\frac{1}{2}} \left(1 - Cs^{-1}\right) \sum_{\ell=\pm} \left(\left\| e^{-s\eta} \nabla' v^\ell \right\|^2_{L^2(\tilde{Q})} + \left\| e^{-s\eta} v^\ell \right\|^2_{L^2(\tilde{Q})} \right) + \sum_{\ell=\pm} \left\| e^{-s\eta_0} v^\ell(\cdot,0) \right\|^2_{L^2(\Omega)} \\ \leq \ & Cs^{-\frac{3}{2}} \left(\left\| e^{-s\eta_0} A \right\|^2_{L^2(\Omega)^n} + \left\| e^{-s\eta_0} p \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^+ \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^- \right\|^2_{L^2(\Omega)} + s\left(\mu^+ + \mu^-\right) \right), \end{aligned}$$

provided $s \geq s_0$. Thus, taking $s_1 := \max(s_0, 2C)$ in the above estimate, we infer from (3.15) that

$$\begin{aligned}(3.17)\qquad & \left\| e^{-s\eta_0}\left(q^+u_0^+ + A\cdot\nabla u_0^- + pu_0^-\right)\right\|_{L^2(\Omega)}^2 + \left\| e^{-s\eta_0}\left(q^-u_0^- - A\cdot\nabla u_0^+ + pu_0^+\right)\right\|_{L^2(\Omega)}^2 \\ \leq\ & Cs^{-\frac{3}{2}}\left(\left\|e^{-s\eta_0}A\right\|_{L^2(\Omega)^n}^2 + \left\|e^{-s\eta_0}p\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^+\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^-\right\|_{L^2(\Omega)}^2 + s\left(\mu^+ + \mu^-\right)\right),\end{aligned}$$

whenever $s \geq s_1$.

The rest of the proof is to adequately choose $n+1$ initial states $u_0^k := (u_0^{+,k}, u_0^{-,k})^T$, $k = 1, \ldots, n+1$, in order to estimate each of the four unknown functions A, p and $q^\pm$ separately, in terms of the corresponding boundary data $\mu_k^\pm := \left\| e^{-s\eta_0}\varphi^{1/2}\left|\partial_\nu\beta\right|^{1/2}\partial_\nu v^{\pm,k}\right\|_{L^2(\tilde{\Sigma}_*)}^2$, where $v^{\pm,k}$ is the solution to (3.15) with $u_0^\pm = u_0^{\pm,k}$.

3.2. **Building $n+1$ suitable initial data.** We proceed in two steps.

Step 1: Estimation of p, $q^\pm$ and a_n. We pick $\epsilon \in (0,1)$, put $u_0^{+,1}(x', x_n) := 0$, $u_0^{-,1}(x', x_n) := \langle x_n\rangle^{-\frac{1+\epsilon}{2}}$ for all $(x', x_n) \in \Omega$ and take $u_0^\pm = u_0^{\pm,1}$ in (3.17). For all $s \geq s_1$, we get that

$$\begin{aligned}& \left\| e^{-s\eta_0}\left(2\langle x_n\rangle^{-\frac{1+\epsilon}{2}}p - (1+\epsilon)\langle x_n\rangle^{-\frac{5+\epsilon}{2}}x_n a_n\right)\right\|_{L^2(\Omega)}^2 + 4\left\| e^{-s\eta_0}\langle x_n\rangle^{-\frac{1+\epsilon}{2}}q^-\right\|_{L^2(\Omega)}^2 \\ \leq\ & Cs^{-\frac{3}{2}}\left(\left\|e^{-s\eta_0}A\right\|_{L^2(\Omega)^n}^2 + \left\|e^{-s\eta_0}p\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^+\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^-\right\|_{L^2(\Omega)}^2 + s\left(\mu^{+,1} + \mu^{-,1}\right)\right),\end{aligned}$$

which entails that

$$\begin{aligned}(3.18)\qquad & \left\| e^{-s\eta_0}\langle x_n\rangle^{-\frac{1+\epsilon}{2}}q^-\right\|_{L^2(\Omega)}^2 \\ \leq\ & Cs^{-\frac{3}{2}}\left(\left\|e^{-s\eta_0}A\right\|_{L^2(\Omega)^n}^2 + \left\|e^{-s\eta_0}p\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^+\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^-\right\|_{L^2(\Omega)}^2 + s\left(\mu^{+,1} + \mu^{-,1}\right)\right)\end{aligned}$$

and

$$\begin{aligned}(3.19)\qquad & \left\| e^{-s\eta_0}\left(2\langle x_n\rangle^{-\frac{1+\epsilon}{2}}p - (1+\epsilon)\langle x_n\rangle^{-\frac{5+\epsilon}{2}}x_n a_n\right)\right\|_{L^2(\Omega)}^2 \\ \leq\ & Cs^{-\frac{3}{2}}\left(\left\|e^{-s\eta_0}A\right\|_{L^2(\Omega)^n}^2 + \left\|e^{-s\eta_0}p\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^+\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^-\right\|_{L^2(\Omega)}^2 + s\left(\mu^{+,1} + \mu^{-,1}\right)\right).\end{aligned}$$

Doing the same with $u_0^\pm = u_0^{\pm,2} := u_0^{\mp,1}$, we obtain for all $s \geq s_1$ that

$$\begin{aligned}(3.20)\qquad & \left\| e^{-s\eta_0}\langle x_n\rangle^{-\frac{1+\epsilon}{2}}q^+\right\|_{L^2(\Omega)}^2 \\ \leq\ & Cs^{-\frac{3}{2}}\left(\left\|e^{-s\eta_0}A\right\|_{L^2(\Omega)^n}^2 + \left\|e^{-s\eta_0}p\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^+\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^-\right\|_{L^2(\Omega)}^2 + s\left(\mu^{+,2} + \mu^{-,2}\right)\right),\end{aligned}$$

and

$$\begin{aligned}(3.21)\qquad & \left\| e^{-s\eta_0}\left(2\langle x_n\rangle^{-\frac{1+\epsilon}{2}}p + (1+\epsilon)\langle x_n\rangle^{-\frac{5+\epsilon}{2}}x_n a_n\right)\right\|_{L^2(\Omega)}^2 \\ \leq\ & Cs^{-\frac{3}{2}}\left(\left\|e^{-s\eta_0}A\right\|_{L^2(\Omega)^n}^2 + \left\|e^{-s\eta_0}p\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^+\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^-\right\|_{L^2(\Omega)}^2 + s\left(\mu^{+,2} + \mu^{-,2}\right)\right).\end{aligned}$$

Since $8\left\| e^{-s\eta_0}\langle x_n\rangle^{-\frac{1+\epsilon}{2}}p\right\|_{L^2(\Omega)}^2$ is upper-bounded by the sum of $\left\| e^{-s\eta_0}\left(2\langle x_n\rangle^{-\frac{1+\epsilon}{2}}p + (1+\epsilon)\langle x_n\rangle^{-\frac{5+\epsilon}{2}}x_n a_n\right)\right\|_{L^2(\Omega)}^2$ and $\left\| e^{-s\eta_0}\left(2\langle x_n\rangle^{-\frac{1+\epsilon}{2}}p - (1+\epsilon)\langle x_n\rangle^{-\frac{5+\epsilon}{2}}x_n a_n\right)\right\|_{L^2(\Omega)}^2$, it follows from (3.19) and (3.21) that

$$\begin{aligned}(3.22)\ & \left\| e^{-s\eta_0}\langle x_n\rangle^{-\frac{1+\epsilon}{2}}p\right\|_{L^2(\Omega)}^2 \\ \leq\ & Cs^{-\frac{3}{2}}\left(\left\|e^{-s\eta_0}A\right\|_{L^2(\Omega)^n}^2 + \left\|e^{-s\eta_0}p\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^+\right\|_{L^2(\Omega)}^2 + \left\|e^{-s\eta_0}q^-\right\|_{L^2(\Omega)}^2 + s\sum_{i=1}^{2}\left(\mu^{+,i} + \mu^{-,i}\right)\right),\end{aligned}$$

whenever $s \geq s_1$. Similarly, upon estimating $\left\| e^{-s\eta_0} \left(2\langle x_n \rangle^{-\frac{1+\epsilon}{2}} p + (1+\epsilon)\langle x_n \rangle^{-\frac{5+\epsilon}{2}} x_n a_n \right) \right\|^2_{L^2(\Omega)}$ from below by the difference $\frac{(1+\epsilon)^2}{2} \left\| e^{-s\eta_0} \langle x_n \rangle^{-\frac{5+\epsilon}{2}} x_n a_n \right\|^2_{L^2(\Omega)} - 4 \left\| e^{-s\eta_0} \langle x_n \rangle^{-\frac{1+\epsilon}{2}} p \right\|^2_{L^2(\Omega)}$, we get from (3.21)-(3.22) that

$$\begin{aligned} &\left\| e^{-s\eta_0} \langle x_n \rangle^{-\frac{5+\epsilon}{2}} x_n a_n \right\|^2_{L^2(\Omega)} \\ \leq\ & Cs^{-\frac{3}{2}} \left(\left\| e^{-s\eta_0} A \right\|^2_{L^2(\Omega)^n} + \left\| e^{-s\eta_0} p \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^+ \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^- \right\|^2_{L^2(\Omega)} + s \sum_{i=1}^{2} \left(\mu^{+,i} + \mu^{-,i} \right) \right), \end{aligned} \tag{3.23}$$

for all $s \geq s_1$. Bearing in mind that $|x_n a_n| \geq y_* |a_n|$ in Ω, by virtue of the assumption (1.7), it follows from (3.23) that

$$\begin{aligned} &\left\| e^{-s\eta_0} \langle x_n \rangle^{-\frac{5+\epsilon}{2}} a_n \right\|^2_{L^2(\Omega)} \\ \leq\ & Cs^{-\frac{3}{2}} \left(\left\| e^{-s\eta_0} A \right\|^2_{L^2(\Omega)^n} + \left\| e^{-s\eta_0} p \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^+ \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^- \right\|^2_{L^2(\Omega)} + s \sum_{i=1}^{2} \left(\mu^{+,i} + \mu^{-,i} \right) \right), \end{aligned} \tag{3.24}$$

provided we have $s \geq s_1$.

Step 2: Estimation of the $n-1$ first components a_j, $j = 1, \ldots, n-1$, of A. For all $k = 1, \cdots, n-1$ and all $x = (x_1, \ldots, x_n) \in \Omega$, we put $u_0^{\pm,k+2}(x) := x_k \langle x_n \rangle^{-\frac{1+\epsilon}{2}}$, substitute $u_0^{\pm,k+2}$ for $u_0^{\pm}$ in (1.1) and then apply Corollary 2.5 to (3.15). We get for all $s \geq s_1$ that

$$\begin{aligned} &\left\| e^{-s\eta_0} \left(p u_0^{-,k+2} + A \cdot \nabla u_0^{-,k+2} + q^+ u_0^{+,k+2} \right) \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} \left(p u_0^{+,k+2} - A \cdot \nabla u_0^{+,k+2} + q^- u_0^{-,k+2} \right) \right\|^2_{L^2(\Omega)} \\ \leq\ & Cs^{-3/2} \left(\left\| e^{-s\eta_0} A \right\|^2_{L^2(\Omega)^n} + \left\| e^{-s\eta_0} p \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^+ \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^- \right\|^2_{L^2(\Omega)} + s \left(\mu^{+,k+2} + \mu^{-,k+2} \right) \right). \end{aligned}$$

Since $\left| p u_0^{\mp,k+2} \pm A \cdot \nabla u_0^{\mp,k+2} + q^{\pm} u_0^{\pm,k+2} \right|^2 \geq \frac{\left| A \cdot \nabla u_0^{\mp,k+2} \right|^2}{2} - \left| p u_0^{\mp,k+2} + q^{\pm} u_0^{\pm,k+2} \right|^2$, this entails that

$$\begin{aligned} &\left\| e^{-s\eta_0} A \cdot \nabla u_0^{+,k+2} \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} A \cdot \nabla u_0^{-,k+2} \right\|^2_{L^2(\Omega)} \\ \leq\ & Cs^{-3/2} \left(\left\| e^{-s\eta_0} A \right\|^2_{L^2(\Omega)^n} + \left\| e^{-s\eta_0} p \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^+ \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^- \right\|^2_{L^2(\Omega)} + s(\mu^{+,k+2} + \mu^{-,k+2}) \right) \\ & + \left\| e^{-s\eta_0} \left(p u_0^{+,k+2} + q^- u_0^{-,k+2} \right) \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} \left(p u_0^{-,k+2} + q^+ u_0^{+,k+2} \right) \right\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.25}$$

Moreover, $\left\| e^{-s\eta_0} \left(p u_0^{\pm,k+2} + q^{\mp} u_0^{\mp,k+2} \right) \right\|^2_{L^2(\Omega)} = \left\| e^{-s\eta_0} x_k \langle x_n \rangle^{-\frac{1+\epsilon}{2}} (p + q^{\mp}) \right\|^2_{L^2(\Omega)}$ being upper-bounded by $2|\omega|^2 \left(\left\| e^{-s\eta_0} \langle x_n \rangle^{-\frac{1+\epsilon}{2}} p \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} \langle x_n \rangle^{-\frac{1+\epsilon}{2}} q^{\mp} \right\|^2_{L^2(\Omega)} \right)$, (3.18), (3.20), (3.22) and (3.25) then yield

$$\begin{aligned} &\left\| e^{-s\eta_0} A \cdot \nabla u_0^{+,k+2} \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} A \cdot \nabla u_0^{-,k+2} \right\|^2_{L^2(\Omega)} \\ \leq\ & Cs^{-3/2} \left(\left\| e^{-s\eta_0} A \right\|^2_{L^2(\Omega)^n} + \left\| e^{-s\eta_0} p \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^+ \right\|^2_{L^2(\Omega)} + \left\| e^{-s\eta_0} q^- \right\|^2_{L^2(\Omega)} \right. \\ & \left. + s \left(\sum_{i=1}^{2} \left(\mu^{+,i} + \mu^{-,i} \right) + \mu^{+,k+2} + \mu^{-,k+2} \right) \right),\ s \geq s_1, \end{aligned}$$

From this, (3.23) and the estimates $\left|A.\nabla u_0^{\pm,k+2}\right|^2 \geq \frac{1}{2}\left|\langle x_n\rangle^{-\frac{1+\epsilon}{2}} a_k\right|^2 - \frac{(1+\epsilon)^2}{4}\left|\langle x_n\rangle^{-\frac{5+\epsilon}{2}} x_k x_n a_n\right|^2$ and $\left\|e^{-s\eta_0}\langle x_n\rangle^{-\frac{5+\epsilon}{2}} x_k x_n a_n\right\|_{L^2(\Omega)} \leq |\omega| \left\|e^{-s\eta_0}\langle x_n\rangle^{-\frac{5+\epsilon}{2}} x_n a_n\right\|_{L^2(\Omega)}$, it then follows that

$$\begin{aligned}
&\left\|e^{-s\eta_0}\langle x_n\rangle^{-\frac{1+\epsilon}{2}} a_k\right\|^2_{L^2(\Omega)} \\
\leq\ & Cs^{-3/2}\Big(\left\|e^{-s\eta_0}A\right\|^2_{L^2(\Omega)^n} + \left\|e^{-s\eta_0}p\right\|^2_{L^2(\Omega)} + \left\|e^{-s\eta_0}q^+\right\|^2_{L^2(\Omega)} + \left\|e^{-s\eta_0}q^-\right\|^2_{L^2(\Omega)} \\
& +s\left(\sum_{i=1}^{2}\left(\mu^{+,i}+\mu^{-,i}\right) + \mu^{+,k+2} + \mu^{-,k+2}\right)\Big),\ s \geq s_1.
\end{aligned}$$

Summing up the above inequality over $k = 1, \ldots, n-1$ and remembering (3.24), we obtain

$$\begin{aligned}
&\left\|e^{-s\eta_0}\langle x_n\rangle^{-\frac{5+\epsilon}{2}} A\right\|^2_{L^2(\Omega)^n} \\
\leq\ & Cs^{-3/2}\left(\left\|e^{-s\eta_0}A\right\|^2_{L^2(\Omega)^n} + \left\|e^{-s\eta_0}p\right\|^2_{L^2(\Omega)} + \left\|e^{-s\eta_0}q^+\right\|^2_{L^2(\Omega)} + \left\|e^{-s\eta_0}q^-\right\|^2_{L^2(\Omega)} + s\xi\right),
\end{aligned} \tag{3.26}$$

for $s \geq s_1$, where $\xi := \sum_{i=1}^{n+1}\left(\mu^{+,i}+\mu^{-,i}\right)$.

3.3. **End of the proof.** For all $y > 0$ we have

$$\begin{aligned}
&\left(\langle y\rangle^{-(5+\epsilon)} - Cs^{-\frac{3}{2}}\right)\left(\left\|e^{-s\eta_0}A\right\|^2_{L^2(\Omega)^n} + \left\|e^{-s\eta_0}p\right\|^2_{L^2(\Omega_y)} + \left\|e^{-s\eta_0}q^+\right\|^2_{L^2(\Omega_y)} + \left\|e^{-s\eta_0}q^-\right\|^2_{L^2(\Omega_y)}\right) \\
\leq\ & Cs^{-\frac{3}{2}}\left(\left\|e^{-s\eta_0}A\right\|^2_{L^2(\Omega\setminus\Omega)^n} + \left\|e^{-s\eta_0}p\right\|^2_{L^2(\Omega\setminus\Omega_y)} + \left\|e^{-s\eta_0}q^+\right\|^2_{L^2(\Omega\setminus\Omega_y)} + \left\|e^{-s\eta_0}q^-\right\|^2_{L^2(\Omega\setminus\Omega_y)} + s\xi\right), \\
\leq\ & Cs^{-\frac{3}{2}}\left(\|A\|^2_{L^2(\Omega\setminus\Omega)^n} + \|p\|^2_{L^2(\Omega\setminus\Omega_y)} + \left\|q^+\right\|^2_{L^2(\Omega\setminus\Omega_y)} + \left\|q^-\right\|^2_{L^2(\Omega\setminus\Omega_y)} + s\xi\right),\ s \geq s_1,
\end{aligned} \tag{3.27}$$

by (3.18), (3.20), (3.22) and (3.26), where $\Omega_y := \omega \times (-y, y)$. Notice that in the last line of (3.27), we used that η_0 is non-negative in Ω. Moreover, for all $y \geq y_1 := \left((2C)^{-\frac{2}{3}} s_1\right)^{\frac{3}{2(5+\epsilon)}}$ we have $s_y := (2C)^{\frac{2}{3}}\langle y\rangle^{\frac{2(5+\epsilon)}{3}} \geq s_1$ and $2Cs_y^{-\frac{3}{2}} \leq \langle y\rangle^{-(5+\epsilon)}$. Therefore, applying (3.27) with $s = s_y$ and using that $\eta_0(x) \leq \frac{e^{2K}}{T^2}$ for all $x \in \Omega$, we obtain that

$$\Theta_{\Omega_y} \leq C\left(\Theta_{\Omega\setminus\Omega_y} + \langle y\rangle^{\frac{2(5+\epsilon)}{3}}\xi\right),\ y \geq y_1, \tag{3.28}$$

where we set $\Theta_X := \|A\|^2_{0,X} + \|p\|^2_{0,X} + \|q^+\|^2_{0,X} + \|q^-\|^2_{0,X}$ for any subset $X \subset \Omega$. Next, using that $p_j \in \mathcal{P}_{\mathfrak{p}}(p_0)$ for $j = 1, 2$, we infer from (1.5) upon writing $\|p\|_{L^2(\Omega\setminus\Omega_y)} \leq \sum_{j=1,2}\|p_j - p_0\|_{L^2(\Omega\setminus\Omega_y)}$, that

$$\begin{aligned}
\|p\|^2_{L^2(\Omega\setminus\Omega_y)} &\leq 4\mathfrak{p}^2 \int_{\Omega\setminus\Omega_y} e^{-2\kappa\langle x_n\rangle^{\varrho}} dx' dx_n \\
&\leq 4\mathfrak{p}^2 |\omega| \int_{|x_n|>y} e^{-2\kappa\langle x_n\rangle^{\varrho}} dx_n \\
&\leq 4\mathfrak{p}^2 |\omega| \left(\int_{\mathbb{R}} e^{-\delta\langle x_n\rangle^{\varrho}} dx_n\right) e^{-(2\kappa-\delta)\langle y\rangle^{\varrho}},\ \delta \in (0, 2\kappa).
\end{aligned} \tag{3.29}$$

Similarly, since $q_j^{\pm} \in \mathcal{P}_{\mathfrak{q}}(q_0^{\pm})$ and $A_j \in \mathcal{A}_{\mathfrak{a}}(A_0)$ for $j = 1, 2$, we obtain

$$\Theta_{\Omega\setminus\Omega_y} \leq Ce^{-(2\kappa-\delta)\langle y\rangle^{\varrho}},\ \delta \in (0, 2\kappa), \tag{3.30}$$

from (1.6) and (3.29), where $C = 4|\omega|\left(\mathfrak{a}^2 + \mathfrak{p}^2 + 2\mathfrak{q}^2\right)\int_{\mathbb{R}} e^{-\delta\langle x_n\rangle^{\varrho}} dx_n$. It follows from this and (3.28) that

$$\Theta_{\Omega_y} \leq C\left(e^{-(2\kappa-\delta)\langle y\rangle^{\varrho}} + \langle y\rangle^{\frac{2(5+\epsilon)}{3}}\xi\right),\ y \geq y_1,\ \delta \in (0, 2\kappa). \tag{3.31}$$

Put $\xi_1 := e^{-(2\kappa-\delta)\langle y_1\rangle^{\varrho}}$. We shall examine the two cases $\xi \in (0,\xi_1]$ and $\xi \in (\xi_1,+\infty)$ separately. Let us start with $\xi \in (0,\xi_1]$. In this case, we pick $y \in [y_1,+\infty)$ so large that $e^{-(2\kappa-\delta)\langle y\rangle^{\varrho}} = \xi$, i.e., $y = \left(\left(-\frac{\ln \xi}{2\kappa-\delta} \right)^{\frac{2}{\varrho}} - 1 \right)^{\frac{1}{2}}$. Thus, with reference to (3.30)-(3.31) we get for all $\xi \in (0,\xi_1]$ that $\Theta_{\Omega\setminus\Omega_y} \leq C\xi_1^{1-2\theta}\xi^{2\theta}$ and that $\Theta_{\Omega_y} \leq C\left(\xi_1^{1-2\theta} + C_1(\theta)\right)\xi^{2\theta}$, where $C_1(\theta) := \sup_{\xi\in(0,\xi_1]} \left(\xi^{1-2\theta} \left(\frac{-\ln \xi}{2\kappa-\delta} \right)^{\frac{2(5+\epsilon)}{3\varrho}} \right) < \infty$ from the assumption $\varrho > 0$. As a consequence we have

$$\Theta_\Omega \leq C\left(2\xi_1^{1-2\theta} + C_1(\theta)\right)\xi^{2\theta},\ \xi \in (0,\xi_1], \tag{3.32}$$

and the desired result follows. Now, when $\xi \in (\xi_1,+\infty)$, we infer from (1.5) upon majorizing $\|p\|_{L^2(\Omega)}^2$ by $2\sum_{j=1,2}\|p_j - p_0\|_{L^2(\Omega)}^2$, that $\|p\|_{L^2(\Omega)}^2 \leq 4\mathfrak{p}^2 |\omega| \left(\int_{\mathbb{R}} e^{-2\kappa\langle x_n\rangle^{\varrho}} dx_n\right) \xi_1^{-2\theta}\xi^{2\theta}$. Doing the same with $q^{\pm}$ and A, with the aid of, respectively, (1.5) and (1.6), we find that $\Theta_\Omega \leq \tilde{C}_1(\theta)\xi^{2\theta}$, where the notation $\tilde{C}_1(\theta)$ stands for the constant $4\left(\mathfrak{a}^2 + \mathfrak{p}^2 + 2\mathfrak{q}^2\right)|\omega| \left(\int_{\mathbb{R}} e^{-2\kappa\langle x_n\rangle^{\varrho}} dx_n\right)\xi_1^{-2\theta}$. This, (3.32) and the estimates $\mu_k^{\pm} \leq C\left\|\partial_\nu v^{\pm,k}\right\|_{L^2(\tilde{\Sigma}_*)}^2$ for all $k = 1,\ldots,n+1$, yield (1.9), which completes the proof of Theorem 1.2.

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