Edge Currents for Quantum Hall Systems, II. Two-Edge, Bounded and Unbounded Geometries

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Abstract

Devices exhibiting the integer quantum Hall effect can be modeled by one-electron Schrödinger operators describing the planar motion of an electron in a perpendicular, constant magnetic field, and under the influence of an electrostatic potential. The electron motion is confined to bounded or unbounded subsets of the plane by confining potential barriers. The edges of the confining potential barriers create edge currents. This is the second of two papers in which we review recent progress and prove explicit lower bounds on the edge currents associated with one- and two-edge geometries. In this paper, we study various unbounded and bounded, two-edge geometries with soft and hard confining potentials. These two-edge geometries describe the electron confined to unbounded regions in the plane, such as a strip, or to bounded regions, such as a finite length cylinder. We prove that the edge currents are stable under various perturbations, provided they are suitably small relative to the magnetic field strength, including perturbations by random potentials. The existence of, and the estimates on, the edge currents are independent of the spectral type of the operator.

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Contents

1 Introduction

This is the second of two papers dealing with lower bound estimates on edge currents associated with quantum Hall devices. The integer quantum Hall effect (IQHE) refers to the quantization of the Hall conductivity in integer multiples of $2\pi e^2/h$. The IQHE is observed in planar quantum devices at zero temperature and can be described by a Fermi gas of noninteracting electrons. This simplification reduces the study of the dynamics to the one-electron approximation. Typically, experimental devices consist of finitely-extended, planar samples subject to a constant perpendicular magnetic field B. An applied electric field in the x-direction induces a current in the y-direction, the Hall current, and the Hall conductivity σ_{xy} is observed to be quantized. Furthermore, the Hall conductivity is a function of the electron Fermi energy, or, equivalently, the electron filling factor, and plateaus of the Hall conductivity are observed as the filling factor is increased. It is now accepted that the occurrence of the plateaus is due to the existence of localized states near the Landau levels that are created by the random distribution of impurities in the sample. We refer to [10] and references mentioned there for a more detailed discussion. Since the earliest theoretical discussions, the existence of edge currents has played a major role in the explanation of the quantum Hall effect.

To describe the two-edge geometries dealt with in the paper, we first recall the theory for the plane. The Landau Hamiltonian H_L describes a particle constrained to $\mathbb{R}^2$, and moving in a constant, transverse magnetic field with strength $B \geq 0$. Let $p_x = -i\partial_x$ and $p_y = -i\partial_y$ be the two momentum operators. The operator H_L is defined on the dense domain $C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ by

$$H = (-i\nabla - A)^2 = p_x^2 + (p_y - Bx)^2, \tag{1.1}$$

in the Landau gauge for which the vector potential is $A(x, y) = B(0, x)$. This extends to a selfadjoint operator with point spectrum given by $\{E_n(B) = (2n + 1)B \mid n = 0, 1, 2, \ldots\}$, and each eigenvalue is infinitely degenerate.

As in [10], we define the *edge current* as the expectation of the y-component of the velocity operator $V_y \equiv (p_y - Bx)$ in certain states that will be specified below. These are states with energy concentration between two successive Landau levels $E_n(B)$ and $E_{n+1}(B)$.

1.1 Edge Currents in Two-Edge Geometries

Our main results in this paper can be grouped together as follows.

1. Two-Edge, Unbounded Geometries: We study the strip case for which the electron is constrained to the region $-\ell/2 < x < \ell/2$, a strip of width $\ell > 0$, by the sharp confining potential

$$V_0(x) = \mathcal{V}_0 \chi_{\{|x| > \ell/2\}}(x), \;\; \mathcal{V}_0 > 0, \tag{1.2}$$

where χ_J denotes the characteristic function of the set J.

2. Two-Edge, Bounded Geometries: We study models for which the electron on a cylinder $C_D = \mathbb{R} \times [-D/2, D/2]$, for $D > 0$, is confined to the bounded region $[-\ell/2, \ell/2] \times [-D/2, D/2]$ by the sharp confining potential (1.2).

In all cases, the unperturbed Hamiltonian has the form

$$H_0 = H_L + V_0, \tag{1.3}$$

acting on the Hilbert space $L^2(\mathbb{R}^2)$. This is a nonnegative, self-adjoint operator. Our strategy is to analyze the unperturbed operator via the partial Fourier transform in the y-variable. We write $\hat{f}(x, k)$ for the partial Fourier transform of the function $f(x, y)$. For the case of unbounded geometry, we have $k \in \mathbb{R}$, whereas for the case of bounded geometry, the allowable k values are discrete. In either case, this decomposition reduces the problem to a study of the fibered operators of the form

$$h_0(k) = p_x^2 + (k - Bx)^2 + V_0(x), \tag{1.4}$$

acting on $L^2(\mathbb{R})$. Since the effective, nonnegative, potential $(k - Bx)^2 + V_0(x)$ is unbounded as $x \to \pm\infty$, the resolvent of $h_0(k)$ is compact and the spectrum is discrete. We denote the eigenvalues of $h_0(k)$ by $\omega_j(k)$, with corresponding normalized real eigenfunctions $\varphi_j(x; k)$, so that

$$h_0(k)\varphi_j(x; k) = \omega_j(k)\varphi_j(x; k), \quad \|\varphi_j(\cdot; k)\| = 1. \tag{1.5}$$

As in [2] and [10], the properties of the curves $k \in \mathbb{R} \to \omega_j(k)$ play an important role in the proofs. These curves are called the *dispersion curves* for the unperturbed Hamiltonian (1.3). The importance of the properties of the dispersion curves comes from an application of the Feynman-Hellmann formula. To illustrate this, let us first consider the two-edge geometry of a half-plane with the sharp confining potential. We note that unlike for the case of one-edge geometries, the dispersion curves are no longer monotonic in k.

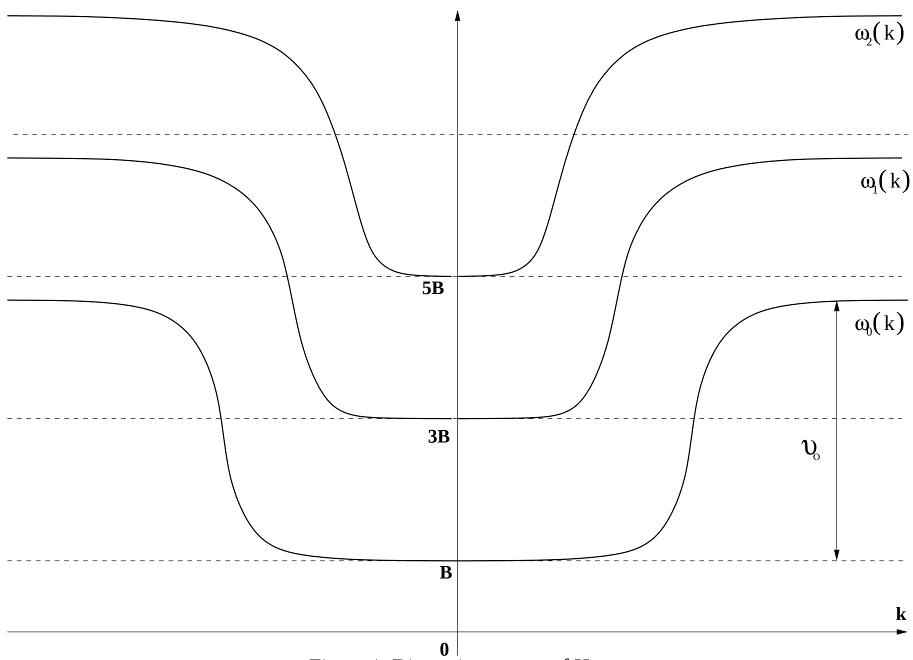

Figure 1: Dispersion curves of H_0

For simplicity, we consider in this introduction a closed interval $\Delta_0 \subset (B, 3B)$ and a normalized wave function ψ satisfying $\psi = E_0(\Delta_0)\psi$, where $E_0(\Delta_0)$ denotes the spectral projection of H_0 associated with Δ_0. Such a function admits a decomposition of the form

$$\psi(x, y) = \frac{1}{\sqrt{2\pi}} \int_{\omega_0^{-1}(\Delta_0)} e^{iky} \beta_0(k) \varphi_0(x; k)\, dk,$$

where the coefficient $\beta_0(k)$ is defined by

$$\beta_0(k) \equiv \langle \hat{\psi}(\cdot; k), \varphi_0(\cdot; k) \rangle. \tag{1.6}$$

The matrix element of the current operator V_y in such a state is

$$\langle \psi, V_y \psi \rangle = \int_{\mathbb{R}} dx \int_{\omega_0^{-1}(\Delta_0)} dk |\beta_0(k)|^2 (k - Bx) \varphi_0(x; k)^2.$$

From (1.5) and the Feynman-Hellmann Theorem, we find that

$$\omega_0'(k) = 2\int_{\mathbb{R}} dx\ (k - Bx)\varphi_0(x;k)^2, \tag{1.7}$$

so that we get

$$\langle \psi, V_y \psi \rangle = \frac{1}{2}\int_{\mathbb{R}} |\beta_0(k)|^2\ \omega_0'(k)\ dk. \tag{1.8}$$

It follows from (1.8) that in order to obtain a lower bound on the expectation of the current operator in the state ψ we need to bound the derivative $\omega_0'(k)$ from below for $k \in \omega_0^{-1}(\Delta_0)$. The next step of the proof involves relating the derivative $\omega_0'(k)$ to the trace of the eigenfunction $\varphi_0(x;k)$ on the boundary of the strip. Taking into account the eigenvalue equation that $\frac{\partial(k-Bx)^2}{\partial k} = -\frac{1}{B}\frac{\partial(k-Bx)^2}{\partial x}$ and $(k-Bx)^2\varphi_0(x;k) = \varphi_0''(x;k) + (\omega_0(k) - V_0(x))\varphi_0(x;k)$, we integrate by parts in (1.7), and find that

$$\omega_0'(k) = \frac{\mathcal{V}_0}{B}(\varphi_0(\ell/2;k)^2 - \varphi_0(-\ell/2,k)^2). \tag{1.9}$$

Consequently, we are left with the task of estimating the trace of the eigenfunction along the two boundary components at $x = \pm\ell/2$.

The key point that allows us to distinguish these two traces is the following. The dispersion curves are symmetric about $k = 0$ if $V_0(x)$ is an even function. Consequently, if a wave function ψ satisfies $\psi = E_0(\Delta_0)\psi$, we have to study the decomposition of ψ in k-space according to the decomposition $\omega_0^{-1}(\Delta_0) = \omega_0^{-1}(\Delta_0)_- \cup \omega_0^{-1}(\Delta_0)_+$, where $\omega_0^{-1}(\Delta_0)_\pm \equiv \omega_0^{-1}(\Delta_0) \cap \mathbb{R}_\pm$. These two components correspond to currents propagating in opposite directions along the left and right edges of the band, respectively. To construct a left-edge current, we construct states ψ so that the coefficients $\beta_0(k)$ in (1.6) satisfy supp $\beta_0(k) \subset \omega_0^{-1}(\Delta_0)_-$. Such a state is spatially concentrated near the left edge $x = -\ell/2$. Hence, the contribution to the left-edge current coming from $\varphi_0(\ell/2;k)$ will be exponentially small since the domain $x \approx \ell/2$ is in the classically forbidden region for energies $\omega_0(k)$, for $k \in \omega_0^{-1}(\Delta_0)_-$. Consequently, the contribution to the integral in (1.8) will be exponentially small. Thus, we prove that if $\psi = E_0(\Delta_0)\psi$ is spectrally concentrated in the set $\omega_0^{-1}(\Delta_0)_-$, then the matrix element $\langle \psi, V_y\psi\rangle$ is bounded from below by a constant times $B^{1/2}\|\psi\|^2$. Much of our technical work, therefore, is devoted to obtaining lower bounds on quantities of the form $\mathcal{V}_0\varphi_0(\pm\ell/2;k)^2$ for such left-edge current states. We also mention that similar results hold for the right-edge current. Of course, in the unperturbed case with a symmetric confining potential, we expect that the net current across any line $y = C$ is zero for the unperturbed problem. We will prove this in Proposition 2.1 below.

1.2 Contents

This paper is organized as follows. Section 2 is devoted to the estimation of edge currents. In section 3, the spectral properties of the model are investigated. Using the Mourre commutator method, we exhibit a class of potentials V_1 (periodic or decreasing in the y-direction) preserving nonempty absolutely continuous spectrum in intervals lying between two consecutive Landau levels for the perturbed Hamiltonian $H_0 + V_1$. In section 4, we address cylinder geometries models and prove the existence of edge currents for Hamiltonians with pure point spectrum in this framework. Appendix 1 in section 5 presents basic properties of the dispersion curves needed in the proofs.

1.3 Acknowledgments

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2 Edge Currents for Two-Edge Geometries

Many quantum devices can be modeled by a confining potential forcing the electrons into a strip of infinite extent in one direction. The dynamics of electrons in an infinite-strip are different from the half-plane cases treated in [10]. We study an electron in a strip of width $\ell > 0$ in the x-direction, and unbounded in the y-direction. We consider confining potential $V_0(x)$ that are step functions, or parabolic functions. After some basic analysis of these models that is independent of the precise form of the confining potential, we study edge currents for parabolic confining potential and sharp confining potential.

2.1 Basic Analysis of Two-Edge Geometries

As in [10], we study the existence of edge currents for a general confining potential $V_0(x)$. We obtain lower bounds on the appropriately localized velocity along the y-direction V_y. The strip geometry is a two-edge geometry. Thus, we expect that there is a current associated with each edge. Classically, these currents propagate along the edges in opposite directions. For the unperturbed system, one expects that the net current flow across the line $y = C$, for any $C \in \mathbb{R}$, to be zero, and we prove this in Proposition 2.2. Once a perturbation V_1 is added, this may no longer be true, and the persistence of edge currents may depend upon a relationship between B and ℓ.

We continue to use the same notation as in [10]. That is, we write $H_0 = H_L + V_0$ for the unperturbed operator. Since we have translational invariance in the y-direction, this operator admits a direct sum decomposition

$$H_0 = \int_{\mathbb{R}}^{\oplus} dk h_0(k). \tag{2.1}$$

We write $h_0(k)$ for the fibered operator acting on $L^2(\mathbb{R})$, where

$$h_0(k) = p_x^2 + (k - Bx)^2 + V_0(x), \tag{2.2}$$

with an even, two-edge confining potential V_0. Although some of our arguments hold for a general confining potential that is monotone on the left and the right, we will explicitly treat the case of the sharp confining potential given in (1.2). We first prove that the total edge current carried by certain *symmetric states* of finite energy vanishes. For this, it is essential that the confining potential be an *even* function. We consider states of finite energy ψ, with $\psi \in E_0(\Delta_n)L^2(\mathbb{R}^2)$, for an interval $\Delta_n \subset (E_n(B), E_{n+1}(B))$, for any $n \geq 0$. The partial Fourier transform $\hat{\psi}$ of ψ in the y-variable can be expressed in terms of the eigenfunctions $\varphi_j(x;k)$ as

$$\hat{\psi}(x,k) = \sum_{j=0}^{n} \chi_{\omega_j^{-1}(\Delta_n)}(k)\beta_j(k)\varphi_j(x;k), \tag{2.3}$$

or equivalently as

$$\psi(x,y) = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} \int_{\mathbb{R}} e^{iky} \ \chi_{\omega_j^{-1}(\Delta_n)}(k)\beta_j(k)\varphi_j(x;k) \ dk, \tag{2.4}$$

where the coefficients $\beta_j(k)$ are defined by

$$\beta_j(k) \equiv \langle \hat{\psi}(\cdot;k), \varphi_j(\cdot;k) \rangle. \tag{2.5}$$

and the normalization condition

$$\|\psi\|^2_{L^2(\mathbb{R}^2)} = \sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)} |\beta_j(k)|^2 \ dk. \tag{2.6}$$

We recall that the properties of the dispersion curves $\omega_j(k)$ result in the disjoint decomposition $\omega_j^{-1}(\Delta_n) = \omega_j^{-1}(\Delta_n)_- \cup \omega_j^{-1}(\Delta_n)_+$ with $\omega_j^{-1}(\Delta_n)_\pm \equiv \omega_j^{-1}(\Delta_n) \cap \mathbb{R}_\pm$.

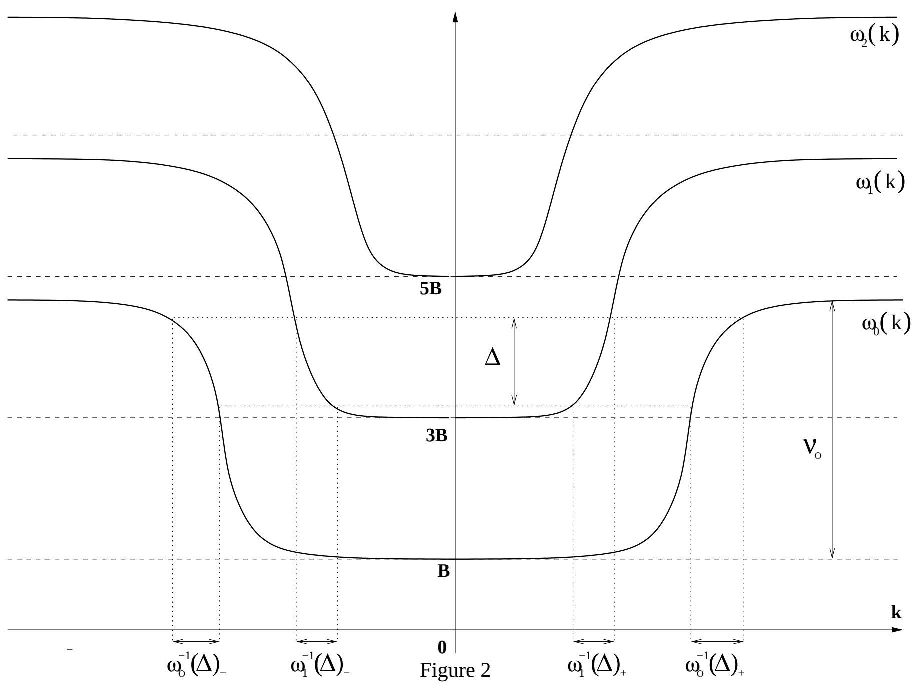

Figure 2

Lemma 2.1 *Let $n \in \mathbb{N}$ and $\Delta_n \subset (E_n(B), E_{n+1}(B))$. Then there is $\delta_n = \delta_n(B, \ell, \mathcal{V}_0) > 0$ such that if $|\Delta_n| < \delta_n$ we have*

$$\omega_j^{-1}(\Delta_n) = \emptyset, \;\; j \geq n+1, \tag{2.7}$$

and, if $n \geq 1$,

$$\omega_j^{-1}(\Delta_n) \cap \omega_l^{-1}(\Delta_n) = \emptyset, \;\; j \neq l, \;\; j, l = 0, 1, \ldots, n. \tag{2.8}$$

Proof.
First (2.7) is evident since $\omega_{n+1}(k) \geq E_{n+1}(B)$ for all $k \in \mathbb{R}$. Next set $\delta_n = \delta_n(B, \ell, \mathcal{V}_0) = \min_{0 \leq j \leq n-1} \inf_{k \in \mathbb{R}} (\omega_{j+1}(k) - \omega_j(k))$. Due to Lemma 5.2(ii), we have $\delta_n > 0$. For all $l, j = 0, 1, \ldots, n$ and $k \in \omega_j^{-1}(\Delta_n) \cap \omega_l^{-1}(\Delta_n)$, we have $|\omega_l(k) - \omega_j(k)| \leq |\Delta_n|$, which leads to a contradiction if $|\Delta_n| < \delta_n$ and $j \neq l$. Henceforth $\omega_j^{-1}(\Delta_n) \cap \omega_l^{-1}(\Delta_n) = \emptyset$ if $j \neq l$. ∎

It is clear from the fact the potential in $h_0(k)$ is centered at $x_0 = k/B$ that the wave function ψ may be more localized near one edge or another depending upon the properties of the weights $\beta_j(k)$. For example, if the $\beta_j(k)$ are supported only by negative wave numbers k, then the wave function will be localized near the left edge.

Proposition 2.1 *Let $n \in \mathbb{N}$ and Δ_n be given by*

$$\Delta_n \equiv [(2n+a)B, (2n+c)B], \;\; \text{for } 1 < a < c < 3. \tag{2.9}$$

Let $\psi \in E_0(\Delta_n) L^2(\mathbb{R}^2)$, as in (2.4), be a finite energy state, such that

$$\beta_j(k) = 0, \;\; k \in \omega_j^{-1}(\Delta_n)_+, \;\; j = 0, 1, \ldots, n.$$

Then there are two constants $\alpha_n = \alpha_n(a) > 0$, depending only on n and a, and $\theta_n = \theta_n(a, c) > 0$, depending only on n, a and c, such that for all $\mathcal{V}_0 \geq E_{n+1}(B)$, all $B\ell^2 \geq \theta_n$ and all $\Delta x_\pm \geq 0$, we have

$$\int_{I(\Delta x_\pm) \times \mathbb{R}} |\psi(x,y)|^2 dx dy \geq (1 - 8 \mathrm{e}^{-(\alpha_n/4) B^{1/2} \Delta x_+} - \mathrm{e}^{-(3-c)^{1/2} B^{1/2} \Delta x_-}) \|\psi\|^2,$$

where

$$I(\Delta x_\pm) = [-\ell/2 - \Delta x_-, -\ell/2 + \alpha_n B^{-1/2} + \Delta x_+].$$

Proof.
In light of (2.3) and the Parseval's Theorem we have

$$\int_{I(\Delta x_\pm)\times\mathbb{R}} \psi^2(x,y)dxdy = \frac{1}{2\pi}\sum_{j=0}^{n}\int_{\omega_j^{-1}(\Delta_n)} \beta_j(k)^2 \left(\int_{I(\Delta x_\pm)} \varphi_j(x;k)^2 dx\right) dk;$$

Hence the result follows from this, (2.6) and Lemma 5.7. ■
Such a wave function should carry a net left-edge current. We will prove this below. We will first prove that if the Fourier Transform of a wave function symmetrically localized with respect to the Fourier variable k, then it carries no net edge current: The left-edge current cancels the right-edge current.

Proposition 2.2 *Let $n \in \mathbb{N}$ and $\Delta_n \subset (E_n(B), E_{n+1}(B))$ be small enough so Lemma 2.1 holds true. Let $\psi \in E_0(\Delta_n)L^2(\mathbb{R}^2)$, as in (2.3), be a finite energy state. Then, the current carried by such a state has the following expression:*

$$\langle \psi, V_y \psi\rangle = \frac{1}{2}\sum_{j=0}^{n}\int_{\omega_j^{-1}(\Delta_n)_-} (|\beta_j(k)|^2 - |\beta_j(-k)|^2)\omega_j'(k)dk. \tag{2.10}$$

Henceforth, if ψ is such that

$$|\beta_j(k)| = |\beta_j(-k)|, \;\; j = 0, 1, \cdots, n, \tag{2.11}$$

then the current carried by ψ vanishes:

$$\langle \psi, V_y \psi\rangle = 0. \tag{2.12}$$

Proof.
The velocity $V_y = p_y - Bx$ has a Fourier transform that we write as $\hat{V}_y = \hat{V}_y(k) = k - Bx$. Using the Fourier decomposition (2.3), the matrix element of the velocity operator V_y is

$$\langle \psi, V_y \psi\rangle \tag{2.13}$$
$$= \sum_{j,l=0}^{n}\int_{\mathbb{R}} \chi_{\omega_j^{-1}(\Delta_n)}(k)\chi_{\omega_l^{-1}(\Delta_n)}(k)\overline{\beta}_j(k)\beta_l(k)\langle \varphi_j(\cdot;k), \hat{V}_y(k)\varphi_l(\cdot;k)\rangle \, dk.$$

As a consequence of the result of Lemma 2.1 below, the cross-terms in (2.13) vanish, at least for $|\Delta_n|$ sufficiently small, giving

$$\begin{aligned}\langle \psi, V_y \psi \rangle &= \sum_{j=0}^{n} \int_{\mathbb{R}} \chi_{\omega_j^{-1}(\Delta_n)}(k) |\beta_j(k)|^2 \langle \varphi_j(\cdot;k), \hat{V}_y(k) \varphi_j(\cdot;k)\rangle \ dk \\ &= \sum_{j=0}^{n} \int_{-\infty}^{0} \chi_{\omega_j^{-1}(\Delta_n)}(k) \Big\{ |\beta_j(k)|^2 \langle \varphi_j(\cdot;k), \hat{V}_y(k) \varphi_j(\cdot;k)\rangle \\ &\quad + |\beta_j(-k)|^2 \langle \varphi_j(\cdot;-k), \hat{V}_y(-k) \varphi_j(\cdot;-k)\rangle \Big\} \, dk, \end{aligned} \tag{2.14}$$

where we used the fact, proved in Lemma 5.1 in Appendix, that the dispersion curves are even functions of k, that is, $\omega_j(k) = \omega_j(-k)$. We also note that the Hamiltonian $h_0(k)$ commutes with the operation P that implements $(x,k) \to (-x,-k)$. The simplicity of the eigenfunctions then implies (this is shown in Lemma 5.1) that $P\varphi_j = \theta_j \varphi_j$ with $\theta_j = \pm 1$. Hence the last term in the r.h.s. of (2.14), $\langle \varphi_j(\cdot;-k), \hat{V}_y(-k)\varphi_j(\cdot;-k)\rangle$ becomes

$$\begin{aligned}\int_{\mathbb{R}} \varphi_j(x;-k)^2(-k-Bx)dx &= \int_{\mathbb{R}} \varphi_j(-x;-k)^2(-k+Bx)dx \\ &= -\int_{\mathbb{R}} \varphi_j(x;k)^2(k-Bx)dx \\ &= -\langle \varphi_j(\cdot;k), \hat{V}_y(k)\varphi_j(\cdot;k)\rangle,\end{aligned}$$

and the result follows from this, (2.14) and the Feynman-Hellmann formula

$$\omega_j'(k) = 2\langle \varphi_j(\cdot;k), (k-Bx)\varphi_j(\cdot;k)\rangle \tag{2.15}$$

■

It can be seen from Lemma 5.1 that a state ψ defined by (2.4)-(2.5) and symmetric about O (i.e. $\psi(-x,-y) = \psi(x,y)$ for $(x,y) \in \mathbb{R}^2$) is characterized by Fourier coefficients β_j, $j = 0,1,\ldots,n$, satisfying the condition

$$\beta_j(-k) = \theta_j \beta_j(k), \ k \in \omega_j^{-1}(\Delta_n)_-, \ j = 0,1,\ldots,n.$$

Consequently symmetric states about O are among states satisfying (2.11), though a state satisfying (2.11) is not necessarily symmetric about O. Moreover it should be noticed that states satisfying the condition (2.11) are not necessarily symmetric about the y-axis either, since the condition $\psi(-x,y) = \psi(x,y)$ for $(x,y) \in \mathbb{R}^2$ is equivalent to

$$\beta_j(k) = 0 \text{ if } \theta_j = -1, \ k \in \omega_j^{-1}(\Delta_n)_-, \ j = 0,1,\ldots,n.$$

2.2 Estimation of the Edge Current for a Strip

We turn now to the estimation of the left-edge current for a strip of width $\ell > 0$. We want to estimate the total current along both edges, carried by appropriately chosen states ψ. That is, for all $n \in \mathbb{N}$ we want to obtain a lower bound on the matrix element of the localized velocity operator (2.10), carried by a state $\psi \in E_0(\Delta_n)L^2(\mathbb{R}^2)$ associated to the energy interval $\Delta_n \subset (E_n(B), E_{n+1}(B))$. Much of the technical work in this paper is devoted to bounding $(-\omega_j'(k))$, $j = 0, 1, \ldots, n$, from below, uniformly for k in $\omega_j^{-1}(\Delta_n)_-$.

Lemma 2.2 *Let $n \in \mathbb{N}$ and Δ_n be given by (2.9). Then, there are two constants $\beta_n = \beta_n(a) > 0$, depending only on n and a, and $C_n > 0$, depending only on n, such that we have*

$$-\omega_j'(k) \geq C_n(a-1)^2(3-c)^3 B^{1/2},\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0, 1, \ldots, n, \quad (2.16)$$

provided $B\ell^2 \geq \beta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$.

The proof of Lemma 2.2 being rather technical, it is postponed to section 2.3.

In light of (2.10) and Lemma 2.2, let us see now the current carried by a state ψ, whose coefficients $\beta_j(k)$, $j = 0, 1, \cdots, n$, are mostly supported on the set of negative wave numbers k, is of size $B^{1/2}$.

More precisely, Δ being a subinterval of $(E_n(B), E_{n+1}(B))$ we consider states of finite energy $\psi \in E_0(\Delta)$, whose Fourier coefficients $\beta_j(k)$, defined by (2.5), satisfy the condition

$$|\beta_j(k)|^2 \geq (1+\gamma^2)|\beta_j(-k)|^2,\ k \in \omega_j^{-1}(\Delta)_-, j = 0, 1, \cdots, n, \quad (2.17)$$

for some $\gamma > 0$.

If γ goes to infinity we find that $\beta_j(-k) = 0$, for $k \in \omega_j^{-1}(\Delta)_-$ and $j = 0, 1, \cdots, n$, whence ψ is localized in a strip of width $\mathcal{O}(B^{-1/2})$ along the left edge $x = -(\ell/2)$ according to Proposition 2.1. Analogously we may expect that all states satisfying (2.17) for some $\gamma > 0$ are mostly supported in the left side of the strip $[-\ell/2, \ell/2] \times \mathbb{R}$.

Theorem 2.1 *Let $n \in \mathbb{N}$ and Δ_n be given by (2.9). Let $\beta_n = \beta_n(a)$ and C_n be defined as in Lemma 2.2. Then for all $B\ell^2 \geq \beta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$, there is a constant $\delta_n = \delta_n(B, \ell, \mathcal{V}_0) > 0$ such that for all interval $\Delta \subset \Delta_n$ with* *size $|\Delta| < \delta_n$ and all states $\psi \in E_0(\Delta)L^2(\mathbb{R}^2)$ satisfying the condition (2.17) for the interval Δ, we have*

$$-\langle\psi, V_y\psi\rangle \geq \frac{\gamma^2}{2+\gamma^2} C_n(a-1)^2(3-c)^3 B^{1/2}\|\psi\|^2. \tag{2.18}$$

Proof.
Due to (2.17) we have $|\beta_j(k)|^2 - |\beta_j(-k)|^2 \geq \gamma^2/(1+\gamma^2)|\beta_j(k)|^2$ for all $j = 0, 1, \ldots, n$ and $k \in \omega_j^{-1}(\Delta)_-$, whence

$$\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta)_-} |\beta_j(k)|^2 dk \geq \frac{1+\gamma^2}{2+\gamma^2}\|\psi\|^2, \tag{2.19}$$

from the normalization condition (2.6). Further, the size of Δ being sufficiently small so Lemma 2.1 holds true, the total current carried by the state ψ is

$$\langle\psi, V_y\psi\rangle = \frac{1}{2}\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta)_-} (|\beta_j(k)|^2 - |\beta_j(-k)|^2)\omega_j'(k) dk,$$

so the result follows from this, (2.19) and Lemma 2.2 since $\omega_j^{-1}(\Delta)_- \subset \omega_j^{-1}(\Delta_n)_-$ for $j = 0, 1, \ldots, n$. ∎

2.3 Proof of Lemma 2.2 : Estimation of the Speed of the Dispersion Relations

For all $n \in \mathbb{N}$, Δ_n defined by (2.9) and $j = 0, 1, \ldots, n$, it is clear from the definition of $\omega_j^{-1}(\Delta_n)_-$ that $\sup \omega_j^{-1}(\Delta_n)_- \leq 0$. Actually, Lemma 5.4 tells us this supremum can be bounded by any number greater than $(-B\ell)/2$ upon taking $B\ell^2$ sufficiently large. Consequently, the region $x \geq 0$ is in the classically forbidden zone for energies $\omega_j(k)$, $k \in \omega_j^{-1}(\Delta_n)_-$, at least in the intense magnetic field regime. This is because the parabolic part of the effective potential

$$W_j(x;k) \equiv (k - Bx)^2 + V_0(x) - \omega_j(k), \tag{2.20}$$

is centered at the coordinate k/B.

Henceforth the eigenfunctions $\varphi_j(.;k)$ of $h_0(k)$ are exponentially decaying in the region $x \geq 0$ for all $k \in \omega_j^{-1}(\Delta_n)_-$. This is not true in the region $x \leq 0$, and $\varphi_j(\ell/2;k)^2$ is also expected to be small relative to $\varphi_j(-\ell/2;k)^2$. Since

$$\omega_j'(k) = \frac{\mathcal{V}_0}{B}\left(\varphi_j(\ell/2;k)^2 - \varphi_j(-\ell/2;k)^2\right), \tag{2.21}$$

by arguing as in the derivation of (1.9), we then get that

$$\omega_j'(k) \approx -\frac{\mathcal{V}_0}{B}\varphi_j(-\ell/2;k)^2.$$

This remark is made precise below. Namely we know from Lemma 5.6 in Appendix that upon choosing $B\ell^2 \geq \zeta_n$ we have

$$\mathcal{V}_0\varphi_j(\ell/2;k)^2 \leq (\nu_n(B\ell^2)^{-1/2})B^{3/2},\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0, 1, \ldots, n, \tag{2.22}$$

both constant $\zeta_n > 0$ and $\nu_n > 0$ depending only on n and a. Hence the remaining term $\mathcal{V}_0\varphi_j(\ell/2;k)^2$ is uniformly bounded by a constant times $B\ell^{-1}$. We turn now to computing a lower bound on the main term $\mathcal{V}_0\varphi_j(-\ell/2;k)^2$. We shall show that it is of size $B^{3/2}$. This will require several steps.

Step 1 : Harmonic Oscillator Eigenfunction Comparison Revisited
The proof of Lemma 2.2 in [10] (based on the properties of the eigenfunctions $\psi_m(.;k)$ of the harmonic oscillator $h_L(k) = p_x^2 + (Bx-k)^2)$ applying without change to the case of the strip geometry examined here, the following estimate,

$$|\langle\varphi_j(.;k), V_0P_n\varphi_j(.;k)\rangle| \geq \frac{1}{2(n+1)B}(\omega_j(k) - E_n(B))(E_{n+1}(B) - \omega_j(k)), \tag{2.23}$$

holds for all $k \in \omega_j^{-1}(\Delta_n)_-$. We recall that P_n denotes the projection on the eigenspace spanned by the first n eigenfunctions $\psi_m(.;k)$ of $h_L(k)$,

$$P_n\varphi_j(x;k) \equiv \sum_{m=0}^{n} \alpha_m^{(j)}(k)\psi_m(x;k), \tag{2.24}$$

with

$$\alpha_m^{(j)}(k) \equiv \langle\varphi_j(.;k), \psi_m(.;k)\rangle, \tag{2.25}$$

and that the explicit expression of $\psi_m(x;k)$ is

$$\psi_m(x;k) = \frac{1}{\sqrt{2^m m!}} \left(\frac{B}{\pi}\right)^{1/4} H_m(B^{1/2}(x-k/B))\mathrm{e}^{-B/2(x-k/B)^2}, \qquad (2.26)$$

where H_m denotes the m^{th} Hermite polynomial function as in [10].

The strategy consists in computing an upper bound on $|\langle\varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle|$, involving the trace $\mathcal{V}_0^2 \varphi_j(-\ell/2;k)^2$. To do that, we expand $P_n\varphi_j(.;k)$ as in (2.24), in $\langle\varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle$, getting:

$$|\langle\varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle| \leq \mathcal{V}_0 \sum_{m=0}^{n} |\alpha_m^{(j)}(k)| \int_{|x|\geq \ell/2} |\varphi_j(x;k)||\psi_m(x;k)|dx. \qquad (2.27)$$

The set $|x| > \ell/2$ is the classically forbidden region for electrons with energy less than $\mathcal{V}_0$, so

$$0 \leq \varphi_j(x;k) \leq \varphi_j(\pm\ell/2;k)\mathrm{e}^{\mp(\mathcal{V}_0-\omega_j(k))^{1/2}(x\mp\ell/2)}, \ \pm x \geq \ell/2, \qquad (2.28)$$

according to Proposition 8.3 in [10]. Henceforth by substituting the corresponding exponentially decreasing term for $\varphi_j(.;k)$ in (2.27), we have

$$\begin{aligned} & |\langle\varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle| \\ \leq \ & \mathcal{V}_0 \sum_{m=0}^{n} |\alpha_m^{(j)}(k)| \left((I_{m,-}^{(j)})\varphi_j(-\ell/2;k) + (I_{m,+}^{(j)})\varphi_j(L/2;k) \right), \end{aligned} \qquad (2.29)$$

where

$$I_{m,\pm}^{(j)} \equiv \int_{\pm x \geq \ell/2} |\psi_m(x;k)|\mathrm{e}^{\mp(\mathcal{V}_0-\omega_j(k))^{1/2}(x\mp\ell/2)}dx. \qquad (2.30)$$

Step 2 : Trace Function Estimate

In view of bounding the integrals $I_{m,\pm}^{(j)}$ we first define the constant

$$\mathcal{H}_m \equiv \sup_{u\in\mathbb{R}} H_m(u)\mathrm{e}^{-u^2/2}. \qquad (2.31)$$

Then we substitute the following obvious consequence of (2.26) and (2.31)

$$|\psi_m(x;k)| \leq \left(\frac{B}{\pi}\right)^{1/4} \frac{\mathcal{H}_m}{\sqrt{2^m m!}}, \qquad (2.32)$$

for $|\psi_m(x;k)|$ in (2.30), and get:

$$I_{m,\pm}^{(j)} \leq \left(\frac{B}{\pi}\right)^{1/4} \frac{\mathcal{H}_m}{\sqrt{2^m m!}} \frac{1}{(\mathcal{V}_0 - \omega_j(k))^{1/2}}. \tag{2.33}$$

Now combining (2.29) with (2.33), provides

$$\begin{aligned} & |\langle \varphi_j(.,k), V_0 P_n \varphi_j(.,k)\rangle_{L^2(\mathbb{R}^2)}| \qquad (2.34) \\ \leq & \frac{\mathcal{V}_0}{(\mathcal{V}_0 - \omega_j(k))^{1/2}} \left(\frac{B}{\pi}\right)^{1/4} \left(\sum_{m=0}^{n} \frac{\mathcal{H}_m}{\sqrt{2^m m!}} |\alpha_m^{(j)}(k)|\right) (\varphi_j(-\ell/2;k) + \varphi_j(\ell/2;k)) \,. \end{aligned}$$

Let us define the constant $\mathcal{H}^{(n)}$ by

$$\mathcal{H}^{(n)} \equiv \left(\sum_{m \leq n} \frac{\mathcal{H}_m^2}{2^m m!}\right)^{1/2}. \tag{2.35}$$

By applying the Cauchy-Schwarz inequality to the sum in (2.34), and using the normalization condition

$$\sum_{m=0}^{n} |\alpha_m^{(j)}(k)|^2 = \|P_n \varphi_j(\cdot;k)\|^2 \leq 1,$$

we end up getting:

$$\begin{aligned} & |\langle \varphi_j(.,k), V_0 P_n \varphi_j(.,k)\rangle_{L^2(\mathbb{R}^2)}| \\ \leq & \frac{\mathcal{V}_0}{(\mathcal{V}_0 - \omega_j(k))^{1/2}} \left(\frac{B}{\pi}\right)^{1/4} \mathcal{H}^{(n)} (\varphi_j(-\ell/2;k) + \varphi_j(\ell/2;k)) \,. \end{aligned}$$

This together with (2.22) and (2.23) yield

$$\mathcal{V}_0^{1/2} \varphi_j(-\ell/2;k) \geq f_n(B\ell^2) B^{3/4}, \tag{2.36}$$

where

$$f_n(B\ell^2) = \frac{\pi^{1/4}}{2(n+1)\mathcal{H}^{(n)}} \left(1 - \frac{\omega_j(k)}{\mathcal{V}_0}\right)^{1/2} (a-1)(3-c) - \nu_n^{1/2} (B\ell^2)^{-1/4},$$

provided $B\ell^2 \geq \zeta_n$. Further for all $\mathcal{V}_0 \geq E_{n+1}(B)$ we have

$$1 - \frac{\omega_j(k)}{\mathcal{V}_0} \geq \frac{3-c}{2n+3},$$

uniformly in $k \in \omega_j^{-1}(\Delta_n)_-$, so $f_n(B\ell^2)$ can be made greater than $(a-1)(3-c)^{3/2}/(2(n+1)(2n+3)^{1/2}\mathcal{H}^{(n)})$ by taking $B\ell^2$ sufficiently large, and (2.16) then follows from (2.36), (2.21) and (2.22).

2.4 Perturbation of Edge Currents

We now consider the perturbation of the edge currents by adding a bounded impurity Potential $V_1(x,y)$ to H_0. As in section 2.3 of [10] for unbounded geometries, we prove that the lower bound on the edge currents is stable with respect to these perturbations provided $\|V_1\| = \|V_1\|_\infty$ is sufficiently small.

Theorem 2.2 *Let $n \in \mathbb{N}$ and Δ_n be as in (2.9). Let β_n be defined as in Lemma 2.2, $B\ell^2 \geq \beta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$. Let Δ be a subinterval of Δ_n with $|\Delta| < \delta_n$, where $\delta_n = \delta_n(B, \ell, \mathcal{V}_0) > 0$ is as in Lemma 2.1. We consider a larger interval $\tilde{\Delta}$ containing Δ, with same midpoint E_m and size $|\tilde{\Delta}| < \delta_n$. Let $V_1(x,y)$ be a bounded potential and let $E(\Delta)$ be the spectral projection for $H_1 = H_0 + V_1$ and the interval Δ. Let $\psi \in L^2(\mathbb{R}^2)$ be a state satisfying $\psi = E(\Delta)\psi$. Let $\phi \equiv E_0(\tilde{\Delta})\psi$ and $\xi \equiv E_0(\tilde{\Delta}^c)\psi$, so that $\psi = \phi + \xi$. Let ϕ have an expansion as in (2.4) with coefficients $\beta_j(k)$ satisfying the condition (2.17) for the interval $\tilde{\Delta}$. Then we have*

$$-\langle \psi, V_y \psi \rangle \geq \left(\frac{\gamma^2}{2+\gamma^2} C_n (a-1)^2 (3-c)^3 - F_n \right) B^{1/2} \; \|\psi\|^2, \tag{2.37}$$

where $C_n > 0$ is the constant defined in Lemma 2.2, and

$$\begin{aligned} & F_n \equiv F_n(B, \|V_1\|, |\Delta|, |\tilde{\Delta}|) \\ = \; & \left(\frac{|\Delta| + 2\|V_1\|}{|\tilde{\Delta}|} \right)^{1/2} \times \left[2 \left(2n + c + \frac{\|V_1\|}{B} \right)^{1/2} \right. \\ & \left. + \frac{\gamma^2}{2+\gamma^2} C_n (a-1)^2 (3-c)^3 \left(\frac{2}{|\tilde{\Delta}|} \right)^{3/2} \left(\frac{|\Delta| + 2\|V_1\|}{|\tilde{\Delta}|} \right)^{3/2} \right]. \end{aligned} \tag{2.38}$$

Further for a fixed level n, if $|\Delta|$ and $\|V_1\|$ are sufficiently small compared with $|\tilde{\Delta}|$, there is a constant $\tilde{C}_n > 0$ independent of B such that

$$-\langle \psi, V_y \psi \rangle \geq \tilde{C}_n B^{1/2} \|\psi\|^2. \tag{2.39}$$

Proof.
We write the function ψ as

$$\psi = E_0(\tilde{\Delta})\psi + E_0(\tilde{\Delta}^c)\psi \equiv \phi + \xi, \tag{2.40}$$

use the self-adjointness of V_y in $L^2(\mathbb{R}^2)$,

$$\begin{aligned} \langle \psi, V_y \psi \rangle &= \langle \phi, V_y \phi \rangle \\ &+ \langle \psi, V_y \xi \rangle + \langle V_y \xi, \phi \rangle, \end{aligned} \tag{2.41}$$

and find out that

$$-\langle \psi, V_y \psi \rangle \geq -\langle \phi, V_y \phi \rangle - 2\|V_y \xi\|_{L^2(\mathbb{R}^2)} \|\psi\|, \tag{2.42}$$

by the Cauchy-Schwarz inequality. The result then follows from Theorem 2.1 provided we have a good bound on $\|\xi\|$ and on $\|V_y \xi\|$. To this end we argue as in section 2.3 in [10], write

$$\begin{aligned} \|\xi\|^2 &= \langle \psi, \xi \rangle \\ &= \langle (H_0 - E_m)\psi, (H_0 - E_m)^{-1} \xi \rangle \\ &\leq \|(H - E_m - V_1)\psi\| \|(H_0 - E_m)^{-1} \xi\|, \end{aligned}$$

then we combine

$$\|(H - E_m - V_1)\psi\| \leq \left(\frac{|\Delta|}{2} + \|V_1\| \right) \|\psi\|,$$

with

$$\|(H_0 - E_m)^{-1} \xi\| \leq \mathrm{dist}^{-1}(E_m, \tilde{\Delta}^c) \|\xi\| = \left(\frac{2}{|\tilde{\Delta}|} \right) \|\xi\|,$$

and find that

$$\|\xi\| \leq \left(\frac{|\Delta| + 2\|V_1\|}{|\tilde{\Delta}|} \right) \|\psi\|. \tag{2.43}$$

Further we notice that

$$\langle \xi, H_0 \xi \rangle = \langle \xi, H \xi \rangle - \langle \xi, V_1 \xi \rangle = \langle \psi, H \xi \rangle - \langle \xi, V_1 \xi \rangle$$

so we end up getting

$$\|V_y \xi\|^2 \leq \langle \xi, H_0 \xi \rangle \leq ((2n + c)B + \|V_1\|) \ \|\xi\| \ \|\psi\|. \tag{2.44}$$

The lower bound on the main term in (2.42) follows from the estimate (2.18):

$$\begin{aligned} -\langle \phi, V_y \phi \rangle &\geq \frac{\gamma^2}{2 + \gamma^2} C_n (a-1)^2 (3-c)^3 \ B^{1/2} \ \left(\sum_{j=0}^{n} \int_{\omega_j^{-1}(\tilde{\Delta}_n)} |\beta_j(k)|^2 \ dk \right) \\ &\geq \frac{\gamma^2}{2 + \gamma^2} C_n (a-1)^2 (3-c)^3 \ B^{1/2} (\|\psi\|^2 - \|\xi\|^2), \end{aligned} \tag{2.45}$$

since

$$\sum_{j=0}^{n} \int_{\omega_j^{-1}(\tilde{\Delta})} |\beta_j(k)|^2 \, dk = \|\phi\|^2 = \|\psi\|^2 - \|\xi\|^2.$$

Combining this lower bound (2.45), with the estimate on $\|\xi\|$ in (2.43), and $\|V_y \xi\|$ in (2.44), we find (2.37) with the constant (2.38). This completes the proof. ■

If the distance from the midpoint E_m of Δ to Δ_n^c is not smaller than δ_n we may choose an interval $\tilde{\Delta}$ with size of the order of δ_n. In this case any state $\psi = E(\Delta)\psi$ satisfying the assumptions of Theorem 2.2 carries a current of size $B^{1/2}$ provided $(\|V_1\| + |\Delta|)/\delta_n$ is small enough. If δ_n is of size $\mathcal{O}(B)$, this indicates that the edge current survives in presence of perturbations V_1 sufficiently small compared with B.

2.5 Soft Confining Potentials

The estimation of edge currents can be generalized to the case of various confining potentials like polynomial confining potentials

$$V_0(x) = B^{1+(p/2)}(|x| - (\ell/2))^p \chi_{\{|x|>\ell/2\}}(x), \; p > 1. \tag{2.46}$$

As a preamble to the investigation of these models, we shall examine the straight parabolic channel model studied by Exner, Joye and Kovarik in [3]. In this case the confining potential is defined by

$$V_0(x) = \mathcal{V}_0^2 x^2, \; \mathcal{V}_0 > 0,$$

and it turns out this model is completely solvable, making the estimation of the edge currents rather straightforward in this particular case.

In both cases V_0 is a function of x alone so the direct sum decomposition (2.1)-(2.2) remains valid, the fibered operators $h_0(k)$, $k \in \mathbb{R}$, having compact resolvent since $\lim_{x \to +\infty}(\hat{V}_k(x) + V_0(x)) = +\infty$. We use the same notations as in the previous sections and note $\omega_j(k)$, $j \in \mathbb{N}$ the eigenvalues of $h_0(k)$. In light of the proof of Theorem 2.1 we remark that it is enough to give an estimation of $\omega_j'(k)$ for $k \in \omega_j^{-1}(\Delta_n)$ where Δ_n is as in (2.9). In order to avoid the inadequate increase of the size of the article we shall state the corresponding results without proof. For more details we refer to the archived version [11].

As an introducing remark we first address the model studied by Exner, Joye and Kovarik. For this model, the electron is confined to a parabolic channel of infinite extent in the y-direction. For any $E > 0$, the plane $\mathbb{R}^2$ is divided into a classically allowed region given by $|x| < \sqrt{E/\mathcal{V}_0}$, and the complementary classically forbidden region. The reduced, unperturbed Hamiltonian is given by

$$\begin{aligned} h_0(k) &= p_x^2 + (k - Bx)^2 + \mathcal{V}_0^2 x^2 \\ &= p_x^2 + \left(B_0 x - \frac{B}{B_0} k\right)^2 + \left(\frac{\mathcal{V}_0}{B_0}\right)^2 k^2, \end{aligned}$$

for the modified field strength $B_0 = (B^2+\mathcal{V}_0^2)^{1/2}$. Since this is simply a shifted harmonic oscillator Hamiltonian, it is completely solvable. The dispersion curves are parabolas with equation $\omega_j(k) = (2j+1)B_0 + (\mathcal{V}_0/B_0)^2 k^2$ so the set $\omega_j^{-1}(\Delta_n)$ for the interval $\Delta_n = [(2n+a)B_0, (2n+c)B_0]$, $1 < a < c < 3$, is explicitly known:

$$\omega_j^{-1}(\Delta_n)_- = [-k_j^{(n)}(c), -k_j^{(n)}(a)],\ k_j^{(n)}(x) \equiv \frac{B_0^{3/2}}{\mathcal{V}_0}(2(n-j)+x-1)^{1/2},\ x = a, c.$$

From this and $(-\omega_j'(k)) = -2(\mathcal{V}_0/B_0)^2 k \geq 2(\mathcal{V}_0/B_0)^2 k_j^{(n)}(a)$ then follows that

$$-\omega_j'(k) \geq 2(2(n-j)+a-1)^{1/2}\left(\frac{\mathcal{V}_0}{B_0^{1/2}}\right),\ k \in \omega_j^{-1}(\Delta_n)_-.$$

For the polynomial model (2.46) the confining potential V_0 is an even function so this is the case for the dispersion curves too, by repeating the arguments of Lemma 5.1. The corresponding fibered operators $h_0(k)$ still depending analytically on k, these functions are differentiable (see [12] or [16]). Their derivative can be estimated with the following

Lemma 2.3 *Let $n \in \mathbb{N}$, Δ_n be the same as in (2.9), and (for the sake of simplicity) $\mathcal{V}_0 = (2n+c)B$. Then, there are two constants $\zeta_n = \tilde{\zeta}_n(a) \geq 0$, depending only on n and a, and $C_{n,p} = C_{n,p}(a,c) > 0$, depending on n, a, c and p, such that we have*

$$(-\omega_j'(k)) \geq C_{n,p}(a-1)^2(3-c)^2 B^{1/2},\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0, 1, \dots, n,$$

provided $B\ell^2 \geq \zeta_n$.

2.6 Dirichlet Boundary Conditions

We denote the Landau Hamiltonian $H_L(B)$ on the space $L^2((-\ell/2, \ell/2) \times \mathbb{R})$ with Dirichlet boundary conditions along $x = (\pm\ell/2)$ by H_0^D. This unperturbed operator admits a direct integral decomposition with respect to the y-variable. We denote by $h_0^D(k)$ the corresponding fibered operator with eigenvalues $\omega_j^D(k)$ and eigenfunctions $\varphi_j^D(x; k)$.

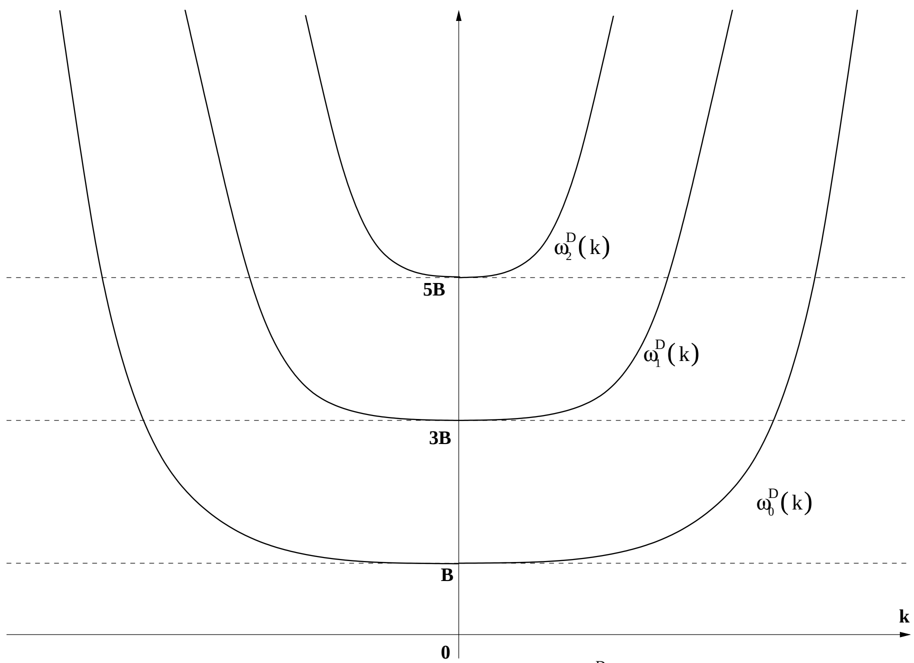

Figure 3: Dispersion Curves of H_0^D

These eigenfunctions provide an eigenfunction expansion of any state, as in (2.4), and we denote the coefficients of this expansion by $\beta_j^D(k)$. Many properties of the dispersion curves $\omega_j^D(k)$ can be derived from [9] and [13], such as

$$(\omega_{j+1}^D(k) - \omega_j^D(k)) > 0, \ k \in \mathbb{R}. \tag{2.47}$$

Since ω_j^D is a continuous function in $\mathbb{R}$, (2.47) entails

$$\delta_{j,l}^D(K) \equiv \inf_{k \in K} |\omega_l^D(k) - \omega_j^D(k)| > 0, \ j \neq l, \tag{2.48}$$

for all compact subset K of $\mathbb{R}$.

The perturbed operator is denoted by $H_D \equiv H_0^D + V_1$, on the same Hilbert space. We let $E_0^D(\cdot)$ and $E_D(\cdot)$ denote the corresponding spectral families.

Theorem 2.3 *Consider the operators H_0^D and $H_D = H_0^D + V_1$, on $\mathcal{H} \equiv L^2((-\ell/2, \ell/2) \times \mathbb{R})$, with Dirichlet boundary conditions along $x = \pm\ell/2$, where $V_1(x, y)$ is bounded. Let $n \in \mathbb{N}$ and Δ_n, defined by (2.9), be sufficiently small so that there is a larger interval $\tilde{\Delta}$ containing Δ_n, with same midpoint, such that*

$$(\omega_j^D)^{-1}(\tilde{\Delta}) \cap (\omega_l^D)^{-1}(\tilde{\Delta}) = \emptyset, \ 0 \leq j \neq l \leq n.$$

Then for $B\ell^2$ sufficiently large (depending on a, c and n), any state $\psi \in E_D(\Delta_n)\mathcal{H}$ with coefficients satisfying the condition (2.17) carries an edge current satisfying the lower bound (2.39) provided $|\Delta_n|$ and $\|V_1\|$ are sufficiently small compared with $|\tilde{\Delta}|$.

We prove this theorem through a perturbation argument comparing H_0^D with $H_0 = H_L(B) + V_0$ in the large $\mathcal{V}_0$ regime. We begin with an estimate of the traces of the eigenfunctions $\varphi_j(x; k)$ of $h_0(k)$ on the lines $x = (\pm\ell/2)$.

Lemma 2.4 *Let $n \in \mathbb{N}$, Δ_n be given by (2.9). Let $\mathcal{V}_0 \geq E_{n+1}(B)$ and $B\ell^2 \geq \theta_n$ where $\theta_n = \theta_n(a, c)$ is as in Lemma 5.8 (and depends only on n, a and c). Then there is a constant $r_n = r_n(a, c, B\ell^2) > 0$ depending only on n, a, c and $B\ell^2$ such that for all $j = 0, 1, \ldots, n$ and all $l \in \mathbb{N}$, we have*

$$0 \leq \varphi_l(\pm\ell/2; k) \leq (4\pi\ell^{-3/2})\frac{l + r_n}{\mathcal{V}_0^{1/2}}, \ k \in \omega_j^{-1}(\Delta_n) \cup (\omega_j^D)^{-1}(\Delta_n). \quad (2.49)$$

Proof.
1. For all $l \in \mathbb{N}$ we get that

$$\int_{\mathbb{R}} \varphi_l'(x; k)^2 + \int_{\mathbb{R}} ((Bx - k)^2 + V_0(x))\varphi_l(x; k)^2 dx = \omega_l(k), \ k \in \mathbb{R}, \quad (2.50)$$

by multiplying the eigenvalue equation (1.5) by $\varphi_l(.; k)$ and integrating over $\mathbb{R}$. From this and the Feynmann-Hellmann formula then follows that

$$\omega_l'(k) = 2\int_{\mathbb{R}} (k - Bx)\varphi_l(x; k)^2 dx \leq 2(\omega_l(k))^{1/2}, \quad (2.51)$$

and consequently

$$\omega_l(k)^{1/2} \leq \omega_l(0)^{1/2} + |k|,\ k \in \mathbb{R},\ l \in \mathbb{N}, \tag{2.52}$$

by integrating (2.51) over $[0, |k|]$. Since

$$\omega_l(k) \leq \omega_l^D(k),\ k \in \mathbb{R},\ l \in \mathbb{N}, \tag{2.53}$$

from the Max-Min principle, (2.52) then yields

$$\omega_l(k)^{1/2} \leq (\omega_l^D(0))^{1/2} + |k|,\ k \in \mathbb{R},\ l \in \mathbb{N}. \tag{2.54}$$

Further, the quadratic part B^2x^2 in $h_0^D(0)$ being bounded by $(B\ell/2)^2$, $\omega_l^D(0)$ is easily seen (see [16]) to be bounded as

$$\omega_l^D(0) \leq \left(\frac{2\pi l}{\ell}\right)^2 + \left(\frac{B\ell}{2}\right)^2,\ l \in \mathbb{N}. \tag{2.55}$$

Moreover taking into account (2.53) we deduce from Lemma 5.8 there are two constants τ_n and θ_n, depending only on n, a and c, such that

$$|k| \leq \frac{BL}{2} + \tau_n B^{1/2},\ k \in \omega_j^{-1}(\Delta_n) \cup (\omega_j^D)^{-1}(\Delta_n), \tag{2.56}$$

provided $B\ell^2 \geq \theta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$.
2. Let $\rho \in C^3(\mathbb{R})$ be a bounded real-valued function and A denote the self-adjoint operator $\rho(x)p_x + p_x\rho(x)$ in $L^2(\mathbb{R})$, with domain $H^1(\mathbb{R})$. Any function φ in the domain of $h_0(k)$ belonging to $H^1(\mathbb{R})$, $\langle [A, h_0(k)]\varphi, \varphi\rangle$ can be defined as $\langle h_0(k)\varphi, A\varphi\rangle - \langle A\varphi, h_0(k)\varphi\rangle$, and we find that

$$\begin{aligned}\langle [A, h_0(k)]\varphi, \varphi\rangle &= 4\langle \rho'\varphi', \varphi'\rangle - 4B\langle \rho(Bx-k)\varphi, \varphi\rangle - \langle \rho'''\varphi, \varphi\rangle \\ &\quad -2\mathcal{V}_0(\rho(\ell/2)\varphi(\ell/2)^2 - \rho(-\ell/2)\varphi(-\ell/2)^2),\end{aligned} \tag{2.57}$$

through standard computations. In the particular case where φ is an eigenfunction $\varphi_l(.;k)$ of $h_0(k)$, the scalar product $\langle -i[A, h_0(k)]\varphi, \varphi\rangle$ vanishes according to the Virial Theorem, so (2.57) yields

$$\begin{aligned}& 2\mathcal{V}_0(\rho(\ell/2)\varphi_l(\ell/2;k)^2 - \rho(-\ell/2)\varphi_l(-\ell/2;k)^2) \\ = \ & 4\langle \rho'\varphi_l'(.;k), \varphi_l'(.;k)\rangle - 4B\langle \rho(Bx-k)\varphi_l(.;k), \varphi_l(.;k)\rangle \\ & -\langle \rho'''\varphi_l(.;k), \varphi_l(.;k)\rangle.\end{aligned} \tag{2.58}$$

Let ρ be such that $\rho(x) = \pm 1$ for $\pm x \geq (\ell/2)$ and $0 \leq \rho^{(i)}(x) \leq (2/\ell)^i$ for $|x| \leq (\ell/2)$ and $i = 1, 2, 3$, in such a way that (2.58) entails :

$$\begin{aligned} & \mathcal{V}_0(\varphi_l(\ell/2;k)^2 + \varphi_l(-\ell/2;k)^2) \\ \leq \; & (4/\ell)\int_{-1}^{1}\varphi_j'(x;k)^2 dx + 2B\left(\int_{-1}^{1}(Bx-k)^2\varphi_l(x;k)^2 dx\right)^{1/2} \\ & +(4/\ell^3)\int_{-1}^{1}\varphi_l(x;k)^2 dx. \end{aligned}$$

The result now follows from this, (2.50) and (2.54)–(2.56). ■

The traces estimate (2.49) is a key ingredient in proving the local convergence of the dispersion curves to those for the Dirichlet problem.

Lemma 2.5 *The dispersion curves $\omega_j(k)$, $j \in \mathbb{N}$, are monotonic increasing functions of $\mathcal{V}_0$. Further, for $n \in \mathbb{N}$ fixed and Δ_n as in (2.9), for $\mathcal{V}_0 \geq E_{n+1}(B)$ and for $B\ell^2 \geq \theta_n$ as in Lemma 2.4, there is a constant $s_n(a, c, B\ell^2) > 0$ independent of $\mathcal{V}_0$, such that*

$$0 \leq \omega_j^D(k) - \omega_j(k) \leq s_n \frac{B^{3/4}}{\mathcal{V}_0^{1/2}}, \;\; k \in \omega_j^{-1}(\Delta_n), \;\; j = 0, \ldots, n. \tag{2.59}$$

Proof.
The Hamiltonians $h_0(k)$ being analytic operators in the parameter $\mathcal{V}_0$, we have

$$\frac{\partial \omega_j}{\partial \mathcal{V}_0}(k) = \int_{|x|\geq \ell/2} \varphi_j(x;k)^2 \; dx \geq 0, \tag{2.60}$$

by the Feynman-Hellmann Theorem, which shows that the dispersion curves are monotone increasing with respect to $\mathcal{V}_0$. Furthermore, the rate of increase in (2.60) slows as $\mathcal{V}_0 \to \infty$. This follows from the pointwise upper bound on $\varphi_j(x, k)$ restricted to $|x| \leq \ell/2$. In particular, from (2.28) and the trace estimate (2.49) of Lemma 2.4, we have

$$\begin{aligned} 0 \leq \frac{\partial \omega_j}{\partial \mathcal{V}_0}(k) \;\; & \leq \;\; \varphi_j(-\ell/2;k)^2 \int_{-\infty}^{-\ell/2} \mathrm{e}^{2(\mathcal{V}_0 - \omega_j(k))^{1/2}(x+\ell/2)} \; dx \\ & + \;\; \varphi_j(\ell/2;k)^2 \int_{\ell/2}^{+\infty} \mathrm{e}^{-2(\mathcal{V}_0 - \omega_j(k))^{1/2}(x-\ell/2)} \; dx \\ & \leq \;\; s_n \left(\frac{B}{\mathcal{V}_0}\right)^{3/2}, \end{aligned} \tag{2.61}$$

for $B\ell^2 \geq \theta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$, the constant $s_n > 0$ depending only on n, a, c and $B\ell^2$. To prove the rate of convergence (2.59), we use the eigenvalue equation (1.5) and take the inner product in $(-\ell/2, \ell/2)$ with the Dirichlet eigenfunction φ_l^D. After integration by parts, and an application of the eigenvalue equation for φ_l^D, one obtains,

$$\begin{aligned} & (\omega_l^D(k) - \omega_j(k))\langle \varphi_l^D(\cdot; k), \varphi_j(\cdot; k)\rangle \\ = \ & (\varphi_l^D)'(\ell/2; k)\varphi_j(\ell/2; k) - (\varphi_l^D)'(-\ell/2; k)\varphi_j(-\ell/2; k). \end{aligned} \tag{2.62}$$

The estimate (2.49) in Lemma 2.4 implies that the right side of (2.62) vanishes as $\mathcal{V}_0 \to \infty$, that is

$$\begin{aligned} & |\omega_l^D(k) - \omega_j(k)| \ |\langle \varphi_l^D(\cdot; k), \varphi_j(\cdot; k)\rangle| \\ \leq \ & \big(|(\varphi_l^D)'(-\ell/2; k)| + |(\varphi_l^D)'(\ell/2; k)|\big) \, (4\pi\ell^{-3/2}) \frac{l + r_n}{\mathcal{V}_0^{1/2}}. \end{aligned} \tag{2.63}$$

We next show that $|\langle \varphi_j^D(\cdot; k), \varphi_j(\cdot; k)\rangle|$ is uniformly bounded from below as $\mathcal{V}_0 \to \infty$, proving the convergence of the eigenvalues. To show this, let χ_i and χ_e denote the characteristic functions onto the interval lines $(-\ell/2, \ell/2)$ and $(-\infty, -\ell/2] \cup [\ell/2, +\infty)$, respectively. We first note that

$$\|\varphi_j(\cdot; k)\|^2 = 1 = \|\chi_i \varphi_j(\cdot; k)\|^2 + \|\chi_e \varphi_j(\cdot; k)\|^2,$$

and the upper bound on the eigenfunction φ_j outside the strip $(-\ell/2, \ell/2)$ (2.28), together with (2.49), imply that

$$\|\chi_e \varphi_j(\cdot; k)\| \leq \mathcal{O}(\mathcal{V}_0^{-1}),$$

so that

$$\|\chi_i \varphi_j(\cdot; k)\| \geq 1 - \mathcal{O}(\mathcal{V}_0^{-3/4}), \tag{2.64}$$

as $\mathcal{V}_0 \to \infty$ and $k \in \omega_j^{-1}(\Delta_n)$. Now, for $l \neq j$, it follows from (2.48) and the monotonicity of the dispersion curves in $\mathcal{V}_0$ that

$$|\omega_l^D(k) - \omega_j(k)| \geq |\omega_l^D(k) - \omega_j^D(k)| \geq \delta_{l,j}^D(\omega_j^{-1}(\Delta_n)) > 0.$$

So it follows from this and from (2.63) that for $l \neq j$

$$\langle \varphi_l^D(\cdot; k), \varphi_j(\cdot; k)\rangle \to 0, \quad \text{as} \ \ \mathcal{V}_0 \to \infty.$$

If, in addition, the matrix element $\langle \varphi_j^D(\cdot; k), \varphi_j(\cdot; k)\rangle$ also vanished as $\mathcal{V}_0 \to \infty$, this would contradict (2.64) as the family $\{\varphi_l^D(\cdot; k)\}$ is an orthonormal basis. It follows that this matrix element must be bounded from below uniformly in $\mathcal{V}_0$ as $\mathcal{V}_0 \to \infty$. Consequently, the dispersion curves must converge as $\mathcal{V}_0 \to \infty$ with the specified rate. ■

In light of the estimates (2.49) and (2.59) we argue now as in the proof of Lemma 3.3 in [10] and obtain the convergence of the projection $P_j(k)$, for the eigenvalue $\omega_j(k)$ of $h_0(k)$, to the projector $P_0^D(k)$, for the eigenvalue $\omega_j^D(k)$ of $h_0^D(k)$, when $\mathcal{V}_0$ tends to infinity, with $B\ell^2$ fixed and sufficiently large.

Lemma 2.6 *Let $n \in \mathbb{N}$ and Δ_n be given by (2.9). Let $P_j(k)$, respectively $P_j^D(k)$, for $j = 0, \ldots, n$, be the projection onto the one-dimensional subspace of $h_0(k)$, respectively $h_0^D(k)$, corresponding to the eigenvalue $\omega_j(k)$, respectively $\omega_j^D(k)$. Let $\mathcal{V}_0 \geq E_{n+1}(B)$ and $B\ell^2 \geq \theta_n$ as in Lemmas 2.4 and 2.5. Then, there exists a finite constant $t_n = t_n(a, c, B\ell^2) > 0$, such that for all $j = 0, 1, \ldots, n$ we have*

$$\|P_j(k) - P_j^D(k)\| \leq \frac{t_n}{\mathcal{V}_0^{1/2}}, \; k \in (\omega_j^D)^{-1}(\Delta_n) \cup \omega_j^{-1}(\Delta_n). \tag{2.65}$$

With reference to Theorem 2.2 and the two estimates (2.48) and (2.65), and recalling from the proof of Lemma 2.5 that $\langle \varphi_j^D(\cdot; k), \varphi_j(\cdot; k)\rangle \geq D_0 > 0$, as $\mathcal{V}_0 \to \infty$, Theorem 2.3 is now obtained by just mimicking the proof of Theorem 3.1 in [10].

3 Two-Edge Geometries: Spectral Properties and the Mourre Estimate

We now examine the spectral properties of the Hamiltonian $H_1 = H_0 + V_1$, for suitable perturbations V_1, for two-edge geometries, paralleling the study in sections 4 and 5 of [10] for one-edge geometries. We use the commutator method of Mourre [1, 15]. For two-edge geometries, an analysis of the dispersion curves for H_0 showed that $\omega_j'(k)$ does not have fixed sign. Consequently, the local commutator used for the one-edge geometries in section 2.5, does not immediately apply. We first construct an appropriate conjugate operator S_α for H_0 with a general confining potential $V_0(x)$. By standard arguments [1], this proves the existence of absolutely continuous spectrum of H_0 at energies away from the Landau levels for sufficiently large B. Of course, the spectral properties of H_0 can be obtained directly from the direct integral decomposition (2.1) and an analysis of the spectrum of $h_0(k)$ defined by (2.2). This proves that the spectrum of H_0 is everywhere purely absolutely continuous. The advantage of the Mourre method, however, is that we can obtain the stability of the absolutely continuous spectrum between Landau levels under two classes of perturbations V_1. We prove that the spectrum of H_1 is purely absolutely continuous if 1) $V_1(x, y)$ is periodic with respect to y with sufficiently small period or 2) $V_1(x, y)$ has some decay in y-direction. These results are similar to those of Exner, Joye, and Kovarik [3]. We point out that for the more general class of perturbations V_1 treated in sections 4 and 5 of [10], such as random potentials, we do not know the spectral type of the operator H_1. However, we still know that there are states carrying nontrivial edge currents. As follows from the work of Ferrari and Macris [5, 6], the existence of edge currents is not tied to the spectral properties of H_1. Indeed, the cylinder geometry model shows that the full Hamiltonian may have only pure point spectrum, yet there are nontrivial edge currents. Hence, the existence of edge currents is not directly tied to the existence of continuous spectrum. We will discuss this in more detail in section 4.

3.1 The Mourre Inequality for H_0

We construct a conjugate operator for $H_0 = H_L + V_0$, where the confining potential V_0 depends only on x, as above. Let $U_\alpha = \mathrm{e}^{i\alpha p_y}$, for $p_y = -i\partial_y$, and for any $\alpha \in \mathbb{R}$, be the translation group in the y-direction defined by

$$(U_\alpha g)(y) = g(y + \alpha). \tag{3.1}$$

Since the representation is unitary, the operator S_α defined by

$$S_\alpha = \frac{i}{2}(U_\alpha y - yU_{-\alpha}) \tag{3.2}$$

is easily seen to be selfadjoint on the domain D_y of the operator multiplication by y, since U_α preserves this domain.

We next compute the commutator $i[H_0, S_\alpha]$, $\alpha \in \mathbb{R}$. The operator S_α commutes with p_x and V_0. Since $V_y = p_y - Bx$, it is easy to check that

$$[V_y, S_\alpha] = \frac{1}{2}(U_\alpha - U_{-\alpha}) = i\sin(\alpha p_y), \tag{3.3}$$

so that

$$i[H_0, S_\alpha] = -2\sin(\alpha p_y) V_y, \tag{3.4}$$

as a quadratic form on $D(H_0) \cap D_y$, or as an operator identity on the core $C_0^\infty(\mathbb{R}^2)$. We also need to compute the double commutator $[[H_0, S_\alpha], S_\alpha]$. By formula (3.4), we find that

$$[[H_0, S_\alpha], S_\alpha] = 2i[\sin(\alpha p_y), V_y] = 0. \tag{3.5}$$

Consequently, a positive commutator will imply absolutely continuous spectrum (cf. [1]) in the range of the corresponding spectral projector.

Proposition 3.1 *Let n be in $\mathbb{N}$ and Δ_n be defined by (2.9). Then there are two constants $\varrho_n = \varrho_n(a, c) > 0$ and $\tau_n = \tau_n(a, c) > 0$ depending only on n, a and c, such that for all $B\ell^2 \geq \varrho_n$, all $\mathcal{V}_0 \geq E_{n+1}(B)$, any subinterval Δ of Δ_n such that $|\Delta| < \delta_n$, where $\delta_n = \delta_n(B, \ell, \mathcal{V}_0)$ is as in Lemma 2.1, and all $\alpha > 0$ satisfying*

$$(\alpha B^{1/2}) \in (0, \tau_n] \cap \left(\bigcup_{m \in \mathbb{N}} \left[\frac{((2/3) + 4m)\pi}{B^{1/2}\ell}, \frac{((4/3) + 4m)\pi}{B^{1/2}\ell} \right] \right), \tag{3.6}$$

we have

$$-iE_0(\Delta)[H_0, S_\alpha]E_0(\Delta) \geq (C_n/2)(a-1)^2(3-c)^3 B^{1/2} E_0(\Delta), \tag{3.7}$$

where $C_n > 0$ is defined in Lemma 2.2 and depends only on n.

Proof.
1. We first derive a general expression for $\langle \psi, [H_0, iS_\alpha]\psi \rangle$, for $\psi \in E_0(\Delta)L^2(\mathbb{R}^2) \cap D_y$ and $\alpha \in \mathbb{R}$. For any $\psi \in D(H_0) \cap D_y$, it follows from (3.4) that

$$-\langle \psi, [H_0, iS_\alpha]\psi \rangle = 2 \int_{\mathbb{R}} \sin(\alpha k) \langle \hat{\psi}(\cdot; k), \hat{V}_y \hat{\psi}(\cdot; k) \rangle \, dk, \tag{3.8}$$

where, as above, $\hat{u}$ denotes the partial Fourier transform of u with respect to y. Taking ψ in $E_0(\Delta)L^2(\mathbb{R}^2)$ and writing it as in (2.4), we find that

$$-\langle \psi, [H_0, iS_\alpha]\psi \rangle = \sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta)} \sin(\alpha k) \omega_j'(k) |\beta_j(k)|^2 dk, \tag{3.9}$$

according to the Feynman-Hellmann formula and the vanishing of the cross-terms established in Lemma 2.1. The ω_j's being even functions by Lemma 5.1, (3.9) can then be rewritten as

$$-\langle\psi,[H_0,iS_\alpha]\psi\rangle = \sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta)_-} \sin(\alpha k)\omega_j'(k)(|\beta_j(k)|^2 + |\beta_j(-k)|^2)dk. \tag{3.10}$$

In order to prove a Mourre estimate, it is necessary to bound the right side of (3.10) from below by a positive constant times $\|\psi\|^2$.
2. Let $\mathcal{V}_0 \geq E_{n+1}(B)$ and $B\ell^2 \geq \theta_n$ so Lemma 5.8 holds true: we have

$$\omega_j^{-1}(\Delta_n)_- \subset \left[-\frac{BL}{2} - \kappa_n B^{1/2}, -\frac{BL}{2} + \kappa_n B^{1/2}\right],\ j=0,1,\ldots,n, \tag{3.11}$$

where both $\theta_n > 0$ and $\kappa_n > 0$ depend only on n, a and c. Set

$$\tau_n = \frac{\pi}{6\kappa_n} \tag{3.12}$$

and assume that

$$B\ell^2 \geq \varrho_n \equiv \max\left(\theta_n, \left(\frac{2\pi}{3\tau_n}\right)^2\right). \tag{3.13}$$

It is clear from (3.12)-(3.13) that $\tau_n\varrho_n^{1/2} \geq (2\pi)/3$ so there is $\alpha \in (0, \tau_n B^{-1/2}]$ such that

$$\alpha B\ell = (\alpha B^{1/2})(B\ell^2)^{1/2} \in \bigcup_{m\in\mathbb{N}} \left[\left(\frac{2}{3}+4m\right)\pi, \left(\frac{4}{3}+4m\right)\pi\right]. \tag{3.14}$$

Further, α being taken in $(0, \tau_n B^{-1/2}]$, we have $0 < \alpha\kappa_n B^{1/2} \leq \pi/6$, whence

$$(-\alpha k) \in \bigcup_{m\in\mathbb{N}} \left[\frac{\pi}{6}+2m\pi, \frac{5\pi}{6}+2m\pi\right],\ k \in \omega_j^{-1}(\Delta_n)_-,\ j=0,1,\ldots,n,$$

from (3.11) and (3.14). As a consequence we have $\sin(-\alpha k) \geq 1/2$ for all $k \in \omega_j^{-1}(\Delta_n)_-$, and thus for all $k \in \omega_j^{-1}(\Delta)_-$, and the result follows from this, (3.10) and Lemma 2.2. ■

3.2 Perturbation Theory and Spectral Stability

The benefit of a local positive commutator is its stability under perturbations. We consider two types of perturbations of H_0: 1) Perturbations periodic in the y-direction, and 2) Perturbations decaying in the y-direction. As we mention below, these conditions on the perturbations are much weaker than what is required using scattering theoretic methods. In light of the positive commutator result (3.7) we will treat two classes of perturbations: perturbations 1) periodic in the y-direction and 2) decaying in the y-direction.

Perturbations Periodic in the y-Direction

We first consider perturbations $V_1(x, y)$ satisfying $V_1(x, y+T) = V_1(x, y)$, for certain $T > 0$. Due to the y-periodicity of V_1, the main property we will use in this section is the basic identity

$$[V_1, U_T] = 0. \tag{3.15}$$

Proposition 3.2 *Let $n \in \mathbb{N}$ and Δ_n be as in (2.9). Let $B\ell^2 \geq \varrho_n$, where ϱ_n is as in Proposition 3.1, and $\mathcal{V}_0 \geq E_{n+1}(B)$. Let Δ and $\tilde{\Delta}$ be as in Theorem 2.2. Let $V_1(x, y)$ be a periodic bounded potential with period T satisfying (3.6) and let $E(\Delta)$ be the spectral projection for $H = H_0 + V_1$ and the interval Δ. Then we have*

$$-iE(\Delta)[H, iS_T]E(\Delta) \geq \left((C_n/2)(a-1)^2(3-c)^3 - G_n\right) B^{1/2} E(\Delta),$$

where

$$\begin{aligned} G_n &\equiv G_n(B, \|V_1\|, |\Delta|, |\tilde{\Delta}|) \\ &= (C_n/2)(a-1)^2(3-c)^3 \left(\frac{|\Delta| + 2\|V_1\|}{|\tilde{\Delta}|}\right)^2 \\ &\quad +2\left(2n + c + \frac{\|V_1\|}{B}\right)^{1/2} \left(\frac{|\Delta| + 2\|V_1\|}{|\tilde{\Delta}|}\right)^{1/2}. \end{aligned}$$

Then upon taking $|\Delta|$ and $\|V_1\|$ sufficiently small compared with $|\tilde{\Delta}|$ we obtain

$$-iE(\Delta)[H_0 + V_1, S_T]E(\Delta) \geq (C_n/3)B^{1/2}E(\Delta_n),$$

where $C_n > 0$ is defined in Lemma 2.2.

Proof.
We decompose $\psi \in E(\Delta)L^2(\mathbb{R}^2)$ as in (2.40), use (3.15), and find that

$$\begin{aligned} \langle \psi, [H, iS_T]\psi\rangle &= \langle \psi, [H_0, iS_T]\psi\rangle \\ &= \langle \phi, [H_0, iS_T]\phi\rangle + G(\phi, \xi), \end{aligned} \tag{3.16}$$

where the perturbation term $G(\phi, \xi)$ has the expression

$$\begin{aligned} G(\phi, \xi) &= \int_{\mathbb{R}} \sin(Tk)\langle \hat{\xi}(., k), \hat{V}_y(k)\hat{\xi}(., k)\rangle dk \\ &\quad +2\mathrm{Re}\left(\int_{\mathbb{R}} \sin(Tk)\langle \hat{\phi}(., k), \hat{V}_y(k)\hat{\xi}(., k)\rangle dk\right), \end{aligned}$$

$\hat{\xi}(., k)$ and $\hat{\phi}(., k)$ denoting respectively the partial Fourier Transform in the y direction of $\xi(., y)$ and $\phi(., y)$. It follows from (3.16) that

$$-\langle \psi, [H, iS_T]\psi\rangle \geq -\langle \phi, [H_0, iS_T]\phi\rangle - 2\|V_y\xi\|\|\psi\|, \tag{3.17}$$

the main term $(-\langle \phi, [H_0, iS_T]\phi\rangle)$ being treated by Proposition 3.1. Namely, for T satisfying (3.6) we have

$$-\langle \phi, [H_0, iS_T]\phi\rangle \geq (C_n/2)(a-1)^2(3-c)^3 B^{1/2}\|\phi\|^2. \tag{3.18}$$

Further $\|\xi\|$ and $\|V_y\xi\|$ being bounded as in (2.43)-(2.44) we deduce from (3.17)-(3.18) that

$$\begin{aligned} & -\langle \psi, [H, iS_T]\psi\rangle \\ \geq & \left[(C_n/2)(a-1)^2(3-c)^3\left(1-\left(\frac{|\Delta| + 2\|V_1\|}{|\tilde{\Delta}|}\right)^2\right)\right. \\ & \left. -2(2n + c + (\|V_1\|/B))^{1/2}\left(\frac{|\Delta| + 2\|V_1\|}{|\tilde{\Delta}|}\right)^{1/2}\right] B^{1/2}\|\psi\|^2. \end{aligned} \tag{3.19}$$

It is clear now that the pre-factor of $B^{1/2}\|\psi\|^2$ in the r.h.s. of (3.19) can be made positive by taking $|\Delta|$ and $\|V_1\|$ sufficiently small relative to the difference $|\tilde{\Delta}|$. ■

Perturbations Decaying in the y-Direction

We now consider an impurity potential $V_1 = V_1(x,y) \in L^\infty(\mathbb{R}^2)$ decaying fast enough in the y-direction in the sense that $yV_1(x,y)$ remains bounded in $\mathbb{R}^2$:

$$\|yV_1\|_\infty < \infty. \tag{3.20}$$

The reason for this additional assumption is the identity,

$$\begin{aligned} & 2[V_1, iS_\alpha] \\ = \quad & (V_1(x, y+\alpha) - V_1(y))U_\alpha y - (V_1(x, y-\alpha) - V_1(x,y))yU_{-\alpha}, \end{aligned}$$

obtained by a straightforward computation. This yields

$$|\langle \psi, [V_1, iS_\alpha]\psi\rangle| \leq (2\|yV_1\|_\infty + |\alpha|\|V_1\|_\infty)\|\psi\|^2,\ \psi \in D(H_0) \cap D_y,\ \alpha \in \mathbb{R},$$

thus, arguing as in the proof of Proposition 3.2, we obtain the following

Proposition 3.3 *Let n, Δ_n, $B\ell^2$, $\mathcal{V}_0$, Δ and $\tilde{\Delta}$ be as in Proposition 3.2. Let V_1 be a bounded potential satisfying (3.20). Let $E(\Delta)$ be the spectral projection for $H = H_0 + V_1$ and the interval Δ. Then there is $\alpha \in (0, \tau_n B^{-1/2}]$, where $\tau_n >$ is as in Proposition 3.1, such that we have*

$$-iE(\Delta)[H_0 + V_1, S_\alpha]E(\Delta) \geq (C_n/3)B^{1/2}E(\Delta_n),$$

provided $|\Delta|$, $\|V_1\|$ and $\|yV_1\|$ are sufficiently small compared with $|\tilde{\Delta}|$.

Remark on the Stability of the Absolutely Continuous Spectrum for Strips

Following the idea developed by Macris, Martin and Pulé in [14] for the half-plane geometry, we can actually prove $H_0 + V_1$ has purely absolutely continuous spectrum for the two-edge geometry if the perturbation V_1 is bounded and integrable in $\mathbb{R}^2$. This class of perturbations is weaker than the classes considered above for which we proved the existence of absolutely continuous spectrum away from the Landau levels since, roughly speaking, the L^1-condition requires decay in all directions. The proof of this result relies on the diamagnetic inequality (see [1], [17]):

$$|\mathrm{e}^{-tH_L}u| \leq \mathrm{e}^{t\Delta}|u|,\ u \in L^2(\mathbb{R})^2,\ t \in \mathbb{R}_+. \tag{3.21}$$

Here $(-\Delta)$ denotes the nonnegative Laplacian in $\mathbb{R}^2$ and (3.21) holds true for all B. As the confining potential V_0 is nonnegative in $\mathbb{R}^2$, Kato's inequality (3.21) still holds by substituting H_0 for H_L, giving

$$|\mathrm{e}^{-tH_0}u| \leq \mathrm{e}^{t\Delta}|u| \text{ and } |\mathrm{e}^{-tH}u| \leq \mathrm{e}^{t\|V_1\|_\infty}\mathrm{e}^{t\Delta}|u|, \ u \in L^2(\mathbb{R})^2, \ t \in \mathbb{R}_+, \quad (3.22)$$

since V_1 is bounded. It follows by explicit calculation that $|V_1|^{1/2}\mathrm{e}^{t\Delta}$ belongs to the Schmidt class $\mathcal{B}_2(L^2(\mathbb{R}^2))$ so that the same is true for $|V_1|^{1/2}\mathrm{e}^{-tH_0}$ and $|V_1|^{1/2}\mathrm{e}^{-tH}$ by (3.22), with the following estimates:

$$\||V_1|^{1/2}\mathrm{e}^{-tH_0}\|_{\mathcal{B}_2(L^2(\mathbb{R}^2))} = \frac{\|V_1\|_1}{\sqrt{2\pi t}} \text{ and } \||V_1|^{1/2}\mathrm{e}^{-tH}\|_{\mathcal{B}_2(L^2(\mathbb{R}^2))} = \mathrm{e}^{t\|V_1\|_\infty}\frac{\|V_1\|_1}{\sqrt{2\pi t}}. \quad (3.23)$$

Let $\mathcal{B}_1(L^2(\mathbb{R}^2))$ denote the trace class. To estimate the trace norm of $\mathrm{e}^{-tH} - \mathrm{e}^{-tH_0}$, we use Duhamel's formula

$$\mathrm{e}^{-tH} = \mathrm{e}^{-tH_0} - \int_0^t \mathrm{e}^{sH}V_1\mathrm{e}^{-sH_0}ds. \quad (3.24)$$

Due to the estimates (3.23), the Hölder inequality for the trace norm, and (3.24), we obtain

$$\begin{aligned} \|\mathrm{e}^{-tH} - \mathrm{e}^{-tH_0}\|_{\mathcal{B}_1(L^2(\mathbb{R}^2))} &\leq \int_0^t \|\mathrm{e}^{(s-t)H}V_1\mathrm{e}^{-sH_0}\|_{\mathcal{B}_1(L^2(\mathbb{R}^2))}ds \\ &\leq \frac{\|V_1\|_1^2\mathrm{e}^{t\|V_1\|_\infty}}{2\pi}\int_0^t \frac{ds}{\sqrt{s(t-s)}} < \infty. \quad (3.25) \end{aligned}$$

Whence $\mathrm{e}^{-tH} - \mathrm{e}^{-tH_0}$ is a trace class operator for all $t > 0$ so H_1 has an absolutely continuous spectrum by the Kato-Rosenblum Theorem and the fact that H_0 has purely absolutely continuous spectrum.

4 Bounded, Two-Edge, Cylindrical Geometry

We address now the case of a quantum device with bounded cylindrical geometry. More precisely, the charged particle is assumed to be moving on the cylinder C_D of circumference $D > 0$ and confined along the cylinder axis by two boundaries separated by the distance $\ell > 0$. We define the infinite cylinder as $C_D = \mathbb{R} \times J = \{(x, y) \mid x \in \mathbb{R}, y \in J\}$, where $J = [-D/2, D/2]$ is an interval with length D, and identify $y = -D/2$ with $y = D/2$. The trajectories of the particle will be bounded in the x-direction by confining potentials.

Let us give now a precise statement of the model. The Landau Hamiltonian $H_L = p_x^2 + (p_y - Bx)^2$ is endowed with y-periodic boundary conditions

$$\varphi(x, -D/2) = \varphi(x, D/2) \text{ and } \partial_y \varphi(x, -D/2) = \partial_y \varphi(x, D/2), \tag{4.1}$$

making it self-adjoint in $L^2(C_D)$. As in the preceding sections, the quantum particle is confined in the x-direction to the strip $[-\ell/2, \ell/2]$ by adding to H_L the sharp confining potential V_0 (1.2). The spectrum of $H_0 = H_L + V_0$ consists of eigenvalues for energies below $\mathcal{V}_0$. We shall prove that suitable states $\varphi = E_0(\Delta_n)\varphi$, $\Delta_n \subset (E_n(B), E_{n+1}(B))$, carry a current of size $B^{1/2}$, and that this current survives in presence of a sufficiently small perturbation. Thus, the existence of the edge current is independent of the spectral type of the operator.

This result is in accordance with (and complements) the one obtained by Ferrari and Macris, who have extensively investigated this model ([5], [6], [7], [8]) in the particular case where $D = L$. They consider an Anderson-type random potential V_ω and prove with large probability (under a rather technical assumption on the spectra of the Hamiltonians $H_0^{(l)}$ and $H_0^{(r)}$ obtained respectively by removing the left or the right wall from H_0) that the spectrum of the random Hamiltonian $H_\omega = H_0 + V_\omega$ in an energy interval $(B + \|V_\omega\|_\infty, 3B - \|V_\omega\|_\infty)$ consists in the union of two sets σ_l and σ_r. The eigenvalues in σ_α, $\alpha = l, r$, are actually small perturbations of eigenvalues $E_j^{(l)}$ of the half-plane Hamiltonian $H_0^{(\alpha)} + V_\omega$ and they show the edge current carried by an associated eigenstate $\varphi_j^{(\alpha)}$ is of size D (with opposite signs depending on whether $\alpha = l$ or r). Their analysis extends to the case where ℓ is at least of size $\log D$.

The remaining of this section is organized as follows. After arguing $\sigma(H_0)$ is pure point, we estimate the current carried by an eigenstate of H_0, then we extend this estimate to the case of a convenient wave packet $\varphi = E_0(\Delta_n)\varphi$ for $\Delta_n \subset (E_n(B), E_{n+1}(B))$ and in presence of a perturbation V_1 sufficiently small relative to B. We point out that the estimates on the edge currents given in the remaining of this section are obtained unconditionally on the size of ℓ and B and they hold for general wave packets with energy in between two consecutive Landau levels.

4.1 The Spectra of H_L and H_0

Let us define the Fourier transform $\mathcal{F}$ as $\mathcal{F}\varphi(x) = (\hat{\varphi}_p(x))_{p\in\mathbb{Z}}$, where

$$\hat{\varphi}_p(x) = \int_J \varphi(x,y)\frac{\mathrm{e}^{-ik_p y}}{\sqrt{D}}dy \text{ and } k_p = \frac{2\pi}{D}p, \tag{4.2}$$

for any $p \in \mathbb{Z}$ and a.e. $x \in \mathbb{R}$. It is unitary from $L^2(C_D)$ endowed with the usual scalar product onto $l^2(\mathbb{Z}; L^2(\mathbb{R}))$. Due to the periodic boundary conditions (4.1) and the fact V_0 it is well known that $\mathcal{F}H_L\mathcal{F}^* = \sum_{p\in\mathbb{Z}}^{\oplus} h_L(k_p)$, where $h_L(k)$ still denotes the operator $p_x^2 + (k - Bx)^2$ in $L^2(\mathbb{R})$, and hence that the spectrum of H_L is thus pure point with $\sigma(H_L) = (2\mathbb{N}+1)B$, each eigenvalue having infinite multiplicity.

We turn now to describing the spectrum of $H_0 = H_L + V_0$. The confining potential V_0 being a function of x alone we have

$$\mathcal{F}H_0\mathcal{F}^* = \sum_{p\in\mathbb{Z}}^{\oplus} h_0(k_p), \tag{4.3}$$

where $h_0(k)$ is defined by (1.4) and has a compact resolvent. We recall the eigenvalues of $h_0(k)$ are denoted $\omega_m(k)$, $m \in \mathbb{N}$, the corresponding normalized eigenfunction being called $\varphi_m(x;k)$. Evidently $\{\Phi_m^{(p)},\ m \in \mathbb{N},\ p \in \mathbb{Z}\}$, where

$$\Phi_m^{(p)}(x,y) \equiv \varphi_m(x;k_p)\frac{\mathrm{e}^{ik_p y}}{\sqrt{D}},\ m \in \mathbb{N},\ p \in \mathbb{Z},$$

is an orthonormal basis of $L^2(C_D)$, and it follows from (4.3) that

$$H_0 = \sum_{m\geq 0}\sum_{p\in\mathbb{Z}} \omega_m(k_p)|\Phi_m^{(p)}\rangle\langle\Phi_m^{(p)}|.$$

As a consequence H_0 has pure point spectrum: $\sigma(H_0) = \{\omega_m(k_p),\ m \geq 0,\ p \in \mathbb{Z}\}$. Nevertheless and despite of the fact each eigenvalue $\omega_m(k_p)$, $(m,p) \in \mathbb{N}\times\mathbb{Z}$, has finite multiplicity, it is not guaranteed that the spectrum of H_0 is discrete. Indeed as $|p|$ goes to infinity, each $\omega_m(k_p)$ goes to $E_m(B)+\mathcal{V}_0$ by Lemma 5.2(i), so the eigenvalues lying in a neighborhood of $E_m(B)+\mathcal{V}_0$ may not be isolated.

4.2 Edge Currents: the Unperturbed Case

Let Δ_n for $n \geq 0$, be defined by (2.9) and $\varphi = E_0(\Delta_n)\varphi$.

We want to estimate the current carried by φ along the edges of the free sample C_D. It turns out (see below the estimate (4.11) of the current carried by a wave packet) this current is the weighted sum of the currents carried by all the eigenstates $\Phi_m^{(p)}$, $(m, p) \in \mathbb{N} \times \mathbb{Z}$, such that

$$\omega_m(k_p) \in \Delta_n. \tag{4.4}$$

We therefore start by estimating the current carried by such an eigenstate $\Phi_m^{(p)}$, for appropriate indices $m \in \mathbb{N}$ and $p \in \mathbb{Z}_-$. In a second step we extend this estimate to the case of the wave packet φ.

Current Carried by an Eigenstate

We consider an eigenfunction $\Phi_m^{(p)}$ of H_0 for some (m, p) in $\mathbb{N} \times \mathbb{Z}_-$ satisfying (4.4). The current carried by $\Phi_m^{(p)}$ along the left edge of the cylinder C_D is defined as the expectation $\langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle$ of the velocity operator $V_y = p_x - Bx$ in the y-direction. By arguing as in section 2.1 we find that

$$\langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle = \omega_m'(k_p),$$

so we may deduce from Lemma 2.2 the

Proposition 4.1 *Let Δ_n be defined by (2.9). Then, for any $(m, p) \in \mathbb{N} \times \mathbb{Z}_-$ satisfying (4.4), we have*

$$-\langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle \geq C_n (a-1)^2 (3-c)^3 B^{1/2},$$

provided $B\ell^2 \geq \beta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$ where $\beta_n > 0$ and $C_n > 0$ are as in Lemma 2.2.

Current Carried by a Wave Packet

We turn now to estimating the current carried along C_D by a the state $\varphi = E_0(\Delta)\varphi$, where Δ is a subinterval of Δ_n. We assume as in Proposition 4.1 that $B\ell^2 \geq \beta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$, and suppose that $|\Delta| < \delta_n(B, \ell, \mathcal{V}_0)$

so Lemma 2.1 holds true. The state φ decomposes in the orthonormal basis $\{\Phi_m^{(p)},\ m \in \mathbb{N},\ p \in \mathbb{Z}\}$ as

$$\varphi(x,y) = \sum_{p\in\mathbb{Z}} \sum_{\substack{0\,\leq\, m\,\leq\, n \\ \omega_m(k_p)\,\in\,\Delta_n}} \beta_m^{(p)} \Phi_m^{(p)}(x,y), \tag{4.5}$$

where

$$\beta_m^{(p)} = \langle \varphi, \Phi_m^{(p)} \rangle. \tag{4.6}$$

since $\mathcal{V}_0 \geq E_{n+1}(B)$ there are only a finite number of indices p involved in the sum (4.5). Indeed, we know from Lemma 5.2(i) that $\lim_{|k|\to+\infty} \omega_0(k) = E_0(B)+\mathcal{V}_0$ with $E_0(B)+\mathcal{V}_0 > (2n+c)B$, thus there is $p_n^* = p_n^*(B,\ell,\mathcal{V}_0,\Delta) \in \mathbb{N}$ such that

$$\omega_0(k_{p_n^*}) \in \Delta \text{ and } \omega_0(k_p) \notin \Delta \text{ for all } |p| > p_n^*. \tag{4.7}$$

Since $\omega_n(k) > \omega_0(k)$ for all $n \geq 1$ and $k \in \mathbb{R}$, we have $\omega_n(k_p) \notin \Delta$ for $|p| > p_n^*$, so (4.5) finally reduces to

$$\varphi(x,y) = \sum_{|p|\leq p_n^*} \sum_{\substack{0\,\leq\, m\,\leq\, n \\ \omega_m(k_p)\,\in\,\Delta}} \beta_m^{(p)} \Phi_m^{(p)}(x,y). \tag{4.8}$$

Therefore the current carried by φ along the left edge of the cylinder has the following expression:

$$\langle \varphi, V_y \varphi \rangle = \sum_{|p|,|p'|\leq p_n^*} \sum_{\substack{0\,\leq\, m,\, m'\,\leq\, n \\ \omega_m(k_p)\,\in\,\Delta \\ \omega_{m'}(k_{p'})\,\in\,\Delta}} \beta_m^{(p)} \overline{\beta_{m'}^{(p')}} \langle \Phi_m^{(p)}, v_y \Phi_{m'}^{(p')} \rangle. \tag{4.9}$$

Actually the crossed terms $\langle \Phi_m^{(p)}, V_y \Phi_{m'}^{(p')} \rangle$ in (4.9) vanish for $p \neq p'$. This can be seen from the two following basic identities

$$\mathcal{F}\Phi_m^{(p)}(x) = \left(\delta(s-p)\varphi_m(x;k_p)\right)_{s\in\mathbb{Z}},$$
$$\mathcal{F}\left(V_y \Phi_{m'}^{(p')}\right)(x) = \left(\delta(s-p')(k_{p'}-Bx)\varphi_{m'}(x;k_{p'})\right)_{s\in\mathbb{Z}},$$

and from the unitarity of $\mathcal{F}$:

$$\begin{aligned}\langle \Phi_m^{(p)}, V_y \Phi_{m'}^{(p')} \rangle &= \delta(p'-p)\langle \varphi_m(.;k_p), (k_p - Bx)\varphi_{m'}(.;k_p)\rangle \\ &= \delta(p'-p)\langle \Phi_m^{(p)}, V_y \Phi_{m'}^{(p)} \rangle.\end{aligned}$$

As a consequence, (4.9) can be rewritten as

$$\langle\varphi, V_y\varphi\rangle = \sum_{|p|\le p_n^*} \sum_{\substack{0\le m,m'\le n\\ \omega_m(k_p)\in\Delta\\ \omega_{m'}(k_p)\in\Delta}} \beta_m^{(p)}\overline{\beta_{m'}^{(p)}}\langle\Phi_m^{(p)}, V_y\Phi_{m'}^{(p)}\rangle, \tag{4.10}$$

with $\omega_m^{-1}(\Delta)\cap\omega_{m'}^{-1}(\Delta)=\emptyset$ for all $m\neq m'$ by Lemma 2.1 since $|\Delta|<\delta_n$. Hence we end up getting

$$\langle\varphi, V_y\varphi\rangle = \sum_{|p|\le p_n^*} \sum_{\substack{0\le m\le n\\ \omega_m(k_p)\in\Delta}} |\beta_m^{(p)}|^2\langle\Phi_m^{(p)}, V_y\Phi_m^{(p)}\rangle, \tag{4.11}$$

which shows that the current carried by φ is the $|\beta_m^{(p)}|^2$-weighted sum of the current carried by the eigenstates $\Phi_m^{(p)}$ with energy $\omega_m(k_p)$ in Δ. Now by combining (4.11) with Proposition 4.1 and mimicking the proof of Theorem 2.1 we obtain the

Theorem 4.1 *Let $n\in\mathbb{N}$, Δ_n, $B\ell^2$, $\mathcal{V}_0$, and Δ be as in Theorem 2.1. Let $\varphi = E_0(\Delta)\varphi$ and p_n^* be the smallest integer satisfying (4.7), so φ has expansion as in (4.8). Assume there is a constant $\gamma>0$ such that the coefficients $\beta_m^{(p)}$ defined by (4.6) satisfy*

$$|\beta_m^{(-p)}|^2 \ge (1+\gamma^2)|\beta_m^{(p)}|^2, \tag{4.12}$$

for all $m=0,1,\ldots,n$ and $p=0,1,\ldots,p_n^*$, such that $\omega_m(k_p)\in\Delta$. Then we have

$$-\langle\varphi, V_y\varphi\rangle \ge \frac{\gamma^2}{2+\gamma^2}C_n\,(a-1)^2\,(3-c)^3\,B^{1/2}\|\varphi\|^2,$$

the constant $C_n>0$ being as in Lemma 2.2.

4.3 Perturbation Theory

As in section 2.4 for the strip geometries we now consider the perturbation of the edge currents by adding a bounded impurity potential $V_1(x,y)$ to H_0, and show (using Theorem 4.1 and arguing in the same way as in the proof of Theorem 2.2) that the lower bound on the edge currents is stable with respect to these perturbations provided $\|V_1\|$ is not too large compared with the constant δ_n defined in Lemma 2.1.

Theorem 4.2 *Let n, Δ_n, $B\ell^2$, $\mathcal{V}_0$, Δ, $\tilde{\Delta}$ be as in Theorem 2.2. Let $V_1(x,y)$ be a bounded potential and let $E(\Delta)$ denote the spectral projection for $H_1 = H_0 + V_1$ and the interval Δ. Let $\psi = E(\Delta_n)\psi$. Let $\phi \equiv E_0(\tilde{\Delta})\psi$ and $\xi \equiv E_0(\tilde{\Delta}^c)\psi$, so that $\psi = \phi + \xi$. Let ϕ have an expansion as in (4.8) with coefficients $\beta_m^{(p)}$ satisfying (4.12). Then if $|\Delta|$ and $\|V_1\|$ are sufficiently small compared with $|\tilde{\Delta}|$ the conclusion of Theorem 2.2 holds true: there is a constant $\tilde{C}_n > 0$ independent of B such that*

$$-\langle \psi, V_y \psi \rangle \geq \tilde{C}_n B^{1/2} \|\psi\|^2.$$

5 Appendix : Basic Properties of the Eigenvalues and Eigenfunctions

The resolvent of the operator $h_0(k) = h(k) + V_0$, $k \in \mathbb{R}$, being compact since the effective potential $(Bx - k)^2 + V_0(x)$ is unbounded as $|x| \to \infty$, the spectrum of $h_0(k)$ is discrete with only ∞ as an accumulation point. We write the eigenvalues of $h_0(k)$ in increasing order and denote them by $\omega_j(k)$, $j \geq 0$. The normalized eigenfunction associated to $\omega_j(k)$ is $\varphi_j(x;k)$. We recall from Proposition 7.2 in [10] that the eigenvalues $\omega_j(k)$, $j \geq 0$, are simple for all $k \in \mathbb{R}$.

In this Appendix we collect the main properties of the eigenvalues and eigenfunctions of the operator $h_0(k)$ for an even confining potential V_0.

5.1 Symmetry Properties

Lemma 5.1 *For all $j \in \mathbb{N}$, ω_j is an even function and there is $\theta_j \in \{-1, 1\}$ such that*

$$\varphi_j(-x;-k) = \theta_j \varphi_j(x;k), \ x \in \mathbb{R}, \ k \in \mathbb{R}.$$

Proof.
Let $j \in \mathbb{N}$. The operation P that implements $x \to (-x)$ satisfies $P\mathrm{dom}\, h_0(k) = \mathrm{dom}\, h_0(-k)$ and $Ph_0(k) = h_0(-k)P$. This entails

$$h_0(-k)P\varphi_j(x;k) = \omega_j(k)P\varphi_j(x;k). \tag{5.1}$$

Whence $\omega_j(k)$ is an eigenvalue of $h_0(-k)$ so there is necessarily some $m_k \geq 0$ such that $\omega_j(k) = \omega_{m_k}(-k)$. Since this is true for any $q \neq k$, we can find $m_q \geq 0$ such that $\omega_j(q) = \omega_{m_q}(-q)$. Moreover ω_j being a continuous function, $\omega_{m_q}(-q)$ goes to $\omega_{m_k}(-k)$ as q goes to k, so $m_q = m_k$ by the simplicity of the eigenvalues. Therefore m_k does not depend on k. By writing now m instead of m_k we have shown that

$$\omega_j(k) = \omega_m(-k),\ \ k \in \mathbb{R}.$$

It follows in particular from this that $\omega_n(0) = \omega_m(0)$ so we get $m = n$ from the simplicity of the eigenvalues once more.
To prove the second part of this Lemma we substitute $(-k)$ for k in (5.1) and use the evenness of ω_j, getting

$$h_0(k)\varphi_j(-x;-k) = \omega_j(k)P\varphi_j(-x;-k).$$

Due to the simplicity of the real valued eigenfunction $\varphi_j(.;k)$ together with the normalization condition $\|\varphi_j(.;\pm k)\| = 1$, there is $\theta_j(k) \in \{-1, 1\}$ such that

$$\varphi_j(-x;-k) = \theta_j(k)\varphi_j(x;k),\ \ x \in \mathbb{R},\ \ k \in \mathbb{R}. \tag{5.2}$$

For all $k_0 \in \mathbb{R}$, there is $x_0 \in \mathbb{R}$ such that $\varphi_j(x_0;k_0) \neq 0$. Furthermore, $k \mapsto \varphi_j(x_0;k)$ being continuous about k_0, there is $\delta > 0$ such that

$$\varphi_j(x_0;k) \neq 0,\ \ k \in (k_0 - \delta, k_0 + \delta).$$

This combined with (5.2) shows that θ_j is continuous (and even analytic) about k_0. As a consequence θ_j is continuous in $\mathbb{R}$ whence it is constant. ■

5.2 Asymptotic Behavior and Separation of the Dispersion Curves

We now describe the asymptotic behavior of the dispersion curves and show that the dispersion curves remain separated.

Lemma 5.2 *For any $j \in \mathbb{N}$, we have :*

(i) $\lim_{|k| \to +\infty} \omega_j(k) = E_j(B) + \mathcal{V}_0$

(ii) $\inf_{k \in \mathbb{R}} (\omega_{j+1}(k) - \omega_j(k)) > 0.$

Proof.
(i) In light of Lemma 5.1 it is enough to show the result for $k > 0$. First we deduce from the obvious operator inequality $h_0(k) \leq h_L(k) + \mathcal{V}_0$, following from the definition $h_0(k) = h_L(k) + V_0(x)$, that

$$\omega_j(k) \leq E_j(B) + \mathcal{V}_0. \tag{5.3}$$

Let $\varepsilon \in (0, 1)$ and φ be a normalized function in the domain of $h_0(k)$. We have

$$\langle h_L(k)\varphi, \varphi\rangle \geq (1-\varepsilon)\langle h_L(k)\varphi, \varphi\rangle + \varepsilon \int_{|x|\leq \ell/2} (Bx-k)^2 |\varphi(x)|^2 dx,$$

whence

$$\langle h_0(k)\varphi, \varphi\rangle \geq (1-\varepsilon)\langle h_L(k)\varphi, \varphi\rangle + \mathcal{V}_0 - \varepsilon + R_\varepsilon, \tag{5.4}$$

where $R_\varepsilon \equiv \int_{|x|\leq \ell/2} (\varepsilon(Bx-k)^2 - \mathcal{V}_0) |\varphi(x)|^2 dx$. Since $\varepsilon(Bx-k)^2 - \mathcal{V}_0 \geq 0$ on $[-\ell/2, \ell/2]$ for all $k \geq k_\varepsilon \equiv (B\ell)/2 + (\mathcal{V}_0/\varepsilon)^{1/2}$, (5.4) entails

$$\langle h_0(k)\varphi, \varphi\rangle \geq (1-\varepsilon)\langle h_L(k)\varphi, \varphi\rangle + \mathcal{V}_0 - \varepsilon,\ k \geq k_\varepsilon.$$

Let $\mathcal{M}_j$ denote a j-dimensional submanifold of dom $h_0(k)$, $j = 0, 1, 2, \cdots, n$. It follows from the above inequality and the Max-Min Principle that

$$\begin{aligned} \omega_j(k) &\geq \min_{\varphi\in\mathcal{M}_j^\perp,\ \|\varphi\|=1} \langle h_0(k)\varphi, \varphi\rangle \\ &\geq \min_{\varphi\in\mathcal{M}_j^\perp,\ \|\varphi\|=1} (1-\varepsilon)\langle h(k)\varphi, \varphi\rangle + \mathcal{V}_0 - \varepsilon, \end{aligned}$$

so we obtain

$$\omega_j(k) \geq (1-\varepsilon)E_j(B) + \mathcal{V}_0 - \varepsilon,\ \forall k \geq k_\varepsilon,$$

by taking the max over the $\mathcal{M}_j$'s. Now the result follows from this and (5.3).
(ii) Let us suppose that $\inf_{k\in\mathbb{R}} (\omega_{j+1}(k) - \omega_j(k)) = 0$ for some $j \in \mathbb{N}$. There would also be a sequence $(k_m)_{m\geq 1}$ of real numbers, such that

$$0 \leq \omega_{j+1}(k_m) - \omega_j(k_m) < \frac{1}{m},\ m \geq 1. \tag{5.5}$$

Due to the evenness of ω_j and ω_{j+1}, the k_m could actually be chosen non-negative, and we know from Lemma 5.2(i) the sequence $(k_m)_{m\geq 1}$ would be necessarily bounded. Therefore we could build a subsequence $(k_{m'})_{m'}$ of $(k_m)_m$ that converges to $k^* \in \mathbb{R}_+$. Hence, by substituting m' for m in (5.5) and taking the limit as m' goes to infinity, we would have

$$\omega_j(k^*) = \omega_{j+1}(k^*),$$

since ω_j and ω_{j+1} are continuous functions. This would mean $\omega_j(k^*)$ is a doubly-degenerated eigenvalue of $h_0(k^*)$, a contradiction to the simplicity of the eigenvalues of $h_0(k)$, $k \in \mathbb{R}$. ■

5.3 Estimation of $\omega_j(k) - E_j(B)$ for $|k| < B\ell/2$

Let $k_0 \in (0, \ell/2)$ and $j \in \mathbb{N}$ be fixed. We show that upon choosing $B\ell^2$ sufficiently large, $\omega_j(k)$ can be made arbitrarily close to $E_j(B)$, uniformly in $k \in [-k_0, k_0]$.

Lemma 5.3 *Let $\epsilon \in (0,1)$. Then for all $j \in \mathbb{N}$ there is a constant $\eta_j > 0$ independent of k, B, $\mathcal{V}_0$ and ϵ such that we have*

$$0 \le \frac{\omega_j(k) - E_j(B)}{B} \le \eta_j \left((B\ell^2\epsilon^2)^{-3/4} + 2(B\ell^2\epsilon^2)^{-1/4} \right) \mathrm{e}^{-(B\ell^2\epsilon^2)/64}, \qquad (5.6)$$

for all $k \in [-B\ell/2(1-\epsilon), B\ell/2(1-\epsilon)]$ provided we have $B\ell^2\epsilon^2 \ge 1$.

Proof.
The left inequality being obvious it is enough to prove the right one. Let θ_ϵ be a real valued, even and twice continuously differentiable function in $\mathbb{R}$, such that

$$\theta_\epsilon(x) = \begin{cases} 1 & \text{if } x \in [-\ell/2(1-\epsilon/2), 0] \\ 0 & \text{if } x \in (-\infty, -\ell/2]. \end{cases}$$

It is evident that $\theta_\epsilon \psi_j(.;k)$ (where $\psi_j(x;k)$ still denotes the j^{th} normalized eigenfunction of $h_L(k)$) obviously belongs to the domain of $h_0(k)$. Moreover, the supports of V_0 and θ_ϵ being disjoint, we have

$$\begin{aligned} (h_0(k) - E_j(B))\theta_\epsilon(x)\psi_j(x;k) &= [h_0(k), \theta_\epsilon]\psi_j(x;k) \\ &= -\theta_\epsilon''\psi_j(x;k) + 2i\theta_\epsilon'\psi_j'(x;k), \end{aligned}$$

which leads to

$$\|(h_0(k) - E_j(B))\theta_\epsilon\psi_j(.;k)\| \le \|\theta_\epsilon''\psi_j(.;k)\| + 2\|\theta_\epsilon'\psi_j'(x;k)\|. \qquad (5.7)$$

Using that $k/B \in [-\ell/2(1-\epsilon), 0]$ together with the explicit expression (2.26) of $\psi_j(.;k)$ and the vanishing of θ'_ϵ outside $[-\ell/2, -\ell/2(1-\epsilon/2)] \cup [\ell/2(1-\epsilon/2), \ell/2]$, we see there are two constants α_j and β_j independent of k, B, $\mathcal{V}_0$ and ϵ, such that

$$\begin{cases} \|\theta''_\epsilon \psi_j(.;k)\| \leq \alpha_j B(B\ell^2\epsilon^2)^{-3/4} \mathrm{e}^{-(B\ell^2\epsilon^2)/64} \\ \|\theta'_\epsilon p_x \psi_j(x;k)\| \leq \beta_j B(B\ell^2\epsilon^2)^{-1/4} \mathrm{e}^{-(B\ell^2\epsilon^2)/64}. \end{cases}$$

This, combined with (5.7), entails

$$\|(h_0(k) - E_j(B))\theta_\epsilon \psi_j(.;k)\| \leq \gamma_j B \left((B\ell^2\epsilon^2)^{-3/4} + 2(B\ell^2\epsilon^2)^{-1/4} \right) \mathrm{e}^{-(B\ell^2\epsilon^2)/64}, \tag{5.8}$$

where $\gamma_j = \max(\alpha_j, \beta_j)$. Moreover bearing in mind that $\|\psi_j(.;k)\| = 1$ it follows from (2.26) that

$$\begin{aligned} \|\theta_\epsilon \psi_j(.;k)\|^2 &\geq \frac{1}{2^j j! \sqrt{\pi}} \int_{B^{1/2}(-\ell/2(1-\epsilon/2)-k/B)}^{B^{1/2}(\ell/2(1-\epsilon/2)-k/B)} H_j(y)^2 \mathrm{e}^{-y^2} dy \\ &\geq \frac{1}{2^j j! \sqrt{\pi}} \int_{-(B^{1/2}\ell\epsilon)/4}^{(B^{1/2}\ell\epsilon)/4} H_j(y)^2 \mathrm{e}^{-y^2} dy = \zeta_j > 0, \end{aligned}$$

by taking, say, $B\ell^2\epsilon^2 \geq 1$. This combined with (5.8) proves the result. ∎

The main consequence of Lemma 5.3 is the following

Lemma 5.4 *Let $n \in \mathbb{N}$, Δ_n be given by (2.9), and $\epsilon \in (0,1)$. Then there is a constant $\gamma_n = \gamma_n(a) > 0$ depending only on n and a such that*

$$\sup \omega_j^{-1}(\Delta_n)_- < -\frac{BL}{2}(1-\epsilon),\ j = 0, 1, \ldots, n, \tag{5.9}$$

uniformly in $\mathcal{V}_0 \geq 0$, provided we have $B\ell^2\epsilon^2 \geq \gamma_n$.

Proof.
It suffices to apply Lemma 5.3 and take $B\ell^2\epsilon^2$ sufficiently large so the r.h.s. of (5.6) is smaller than $B^{-1}(\inf \Delta_n - E_n(B)) = a-1$. This entails $\omega_j(k) < \inf \Delta_n$ for all $k \in [-(B\ell/2)(1-\epsilon), (B\ell/2)(1-\epsilon)]$ and the result follows. ∎

5.4 Estimation of the Eigenfunctions in the Classically Forbidden Zone

We prove in Lemma 5.5 for $j = 0, 1, \ldots, n$, and $k \in \omega_j^{-1}(\Delta_n)_-$, where Δ_n is as in Lemma 5.4, that the $\varphi_j(x; k)$, are exponentially decreasing functions in the domain $x > -(\ell/2)$ provided $B\ell^2$ is taken sufficiently large. This is the main tool for the proof of 1) Lemma 5.6, a technical result used in the estimation of $\omega_j'(k)$ given in Lemma 2.2, and 2) Lemma 5.7, which states the localization properties of the eigenfunctions in view of Proposition 2.1. Finally Lemma 5.8, who is particularly useful in Section 3, can be derived from Lemmas 5.4 and 5.7.

Lemma 5.5 *Let $n \in \mathbb{N}$, Δ_n be given by (2.9), $\varepsilon \in (0,1)$ and $x_\varepsilon = -(\ell/2)(1-\varepsilon)$. Then there is a constant $\mu_n = \mu_n(a) > 0$ depending only on n and a such that*

$$\varphi_j(x; k)^2 \leq (B\ell\varepsilon)\mathrm{e}^{-\frac{B\ell\varepsilon}{8}(x - x_\varepsilon)},\ x \geq x_\varepsilon,\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0, 1, \ldots, n,$$

uniformly in $\mathcal{V}_0 \geq 0$, provided $B\ell^2\epsilon^2 \geq \mu_n$.

Proof.
Set $\tilde{x}_\varepsilon = -(\ell/2)(1 - \varepsilon/2)$. We notice that it suffices to take $B\ell^2$ large enough so the effective potential $W_j(x; k)$ defined by (2.20) is positive for $k \in \omega_j^{-1}(\Delta_n)_-$ in the region $x \geq \tilde{x}_\varepsilon$. This follows from Lemma 5.4 with $\epsilon = \varepsilon/4$. Indeed in this case (5.9) involves

$$Bx - k \geq \frac{B\ell\varepsilon}{8},\ x \geq \tilde{x}_\varepsilon,\ k \in \omega_j^{-1}(\Delta_n)_-,$$

so by using the fact that $W_j(x; k) \geq (Bx - k)^2 - E_{n+1}(B)$ we get that

$$W_j(x; k) \geq \left(\frac{B\ell\varepsilon}{16}\right)^2,\ x \geq \tilde{x}_\varepsilon,\ k \in \omega_j^{-1}(\Delta_n)_-, \tag{5.10}$$

provided $B\ell^2\varepsilon^2 \geq \max(\gamma_n(a), 16^2(2n+3))$, where $\gamma_n(a)$ is defined in Lemma 5.4. As a consequence, the $H^1(\mathbb{R})$-solution $\varphi_j(.; k)$ to the differential equation $\varphi''(x) = W_j(x; k)\varphi(x)$ is exponentially decaying in the region $x \geq \tilde{x}_\varepsilon$. Namely for all $k \in \omega_j^{-1}(\Delta_n)_-$, we have

$$0 \leq \varphi_j(t; k) \leq \varphi_j(s; k)\mathrm{e}^{-\frac{B\ell\varepsilon}{8}(t-s)},\ \tilde{x}_\varepsilon \leq s \leq t, \tag{5.11}$$

from Proposition 8.3 in [10]. By combining (5.11) for $t = x_\varepsilon$ with (5.10), we find that

$$\varphi_j(x_\varepsilon;k)^2 \mathrm{e}^{-\frac{B\ell\varepsilon}{8}s} \leq \varphi_j(s;k)^2 \mathrm{e}^{-\frac{BL\varepsilon}{8}x_\varepsilon},\ \tilde{x}_\varepsilon \leq s \leq x_\varepsilon,\ k \in \omega_j^{-1}(\Delta_n)_-, \quad (5.12)$$

whence

$$\varphi_j(x_\varepsilon;k)^2 \leq \frac{(B\ell\varepsilon)/8}{\mathrm{e}^{\frac{B\ell\varepsilon}{8}(x_\varepsilon-\tilde{x}_\varepsilon)}-1},\ k \in \omega_j^{-1}(\Delta_n)_-, \quad (5.13)$$

by integrating (5.12) w.r.t. s over $(\tilde{x}_\varepsilon, x_\varepsilon)$ and using the normalization condition $\|\varphi_j(.;k)\| = 1$. Bearing in mind that $x_\varepsilon - \tilde{x}_\varepsilon = (\ell\varepsilon)/4$, we may take $B\ell^2\varepsilon^2$ sufficiently large so (5.13) involves

$$\varphi_j(x_\varepsilon;k)^2 \leq (B\ell\varepsilon)\mathrm{e}^{-\frac{B\ell\varepsilon}{8}(x_\varepsilon-\tilde{x}_\varepsilon)},\ k \in \omega_j^{-1}(\Delta_n)_-.$$

Now the result follows from this, (5.10) and (5.11). ■

We give now two corollaries of Lemma 5.5.

Lemma 5.6 *Let $n \in \mathbb{N}$ and Δ_n be given by (2.9). For $s \in \{+,-\}$ and $t > 0$, set*

$$g_s(t) = \begin{cases} t^{-1/2} & \text{if } s = + \\ 1 & \text{if } s = -. \end{cases}$$

Then there are two constants $\zeta_n = \zeta_n(a) > 0$ and $\nu_n = \nu_n(a) > 0$, depending on n and a, such that for all $B\ell^2 \geq \zeta_n$ we have

$$\mathcal{V}_0\varphi_j(\pm\ell/2;k)^2 \leq \nu_n g_\pm(B\ell^2)B^{3/2},\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0,1,\ldots,n, \quad (5.14)$$

uniformly in $\mathcal{V}_0 \geq 0$.

Proof.
1. Let j be $0,1,\ldots,n$. We start by proving there are two constants $\alpha_j > 0$ and $\beta_j > 0$ independent of B, ℓ, $\mathcal{V}_0$ and k such that we have

$$0 \leq \int_0^{+\infty}(Bx-k)\varphi_j(x;k)^2dx \leq \alpha_j B\ell\mathrm{e}^{-\beta_j B\ell^2},\ k \in \omega_j^{-1}(\Delta_n)_-, \quad (5.15)$$

provided $B\ell^2$ is sufficiently large. The left inequality in (5.15) being evident, it is enough to show the right one. By using the fact that $\omega_j(k) \leq E_{n+1}(B)$ and arguing as in the proof of Lemma 5.5, we deduce from Lemma 5.4 there is a constant $v_n(a) > 0$ depending only on n and a such that the potential $W_j(t;k)$ defined by (2.20) satisfies

$$W_j(t;k) \geq 0 \text{ and } 0 \leq Bt - k \leq 2W_j(t;k)^{1/2},\ t \geq 0,\ k \in \omega_j^{-1}(\Delta_n)_-,$$

upon taking $B\ell^2 \geq v_n(a)$. Now (5.15) follows immediately from this, the exponentially decaying behavior of $\varphi_j(.;k)$ in $\mathbb{R}_+$,

$$0 \leq \varphi_j(x;k) \leq \varphi_j(0;k)\mathrm{e}^{-\int_0^x W_j(t;k)^{1/2}dt},\ x \geq 0,\ k \in \omega_j^{-1}(\Delta_n)_-,$$

as stated in Proposition 8.2 in [10], and Lemma 5.5.
2. We write (2.58) for a bounded real-valued function $\rho \in C^3(\mathbb{R})$ such that $\rho(x) = 0$ if $x \leq 0$, and $\rho(x) = 1$ if $x \geq \ell/2$, and find

$$\begin{aligned} & 2\mathcal{V}_0\varphi_j(\ell/2;k)^2 \\ \leq\ & \|\rho'''\|_\infty \int_0^{\ell/2} \varphi_j(x;k)^2 dx + 4\|\rho'\|_\infty \int_0^{\ell/2} \varphi_j'(x;k)^2 dx \\ & +4B\|\rho\|_\infty \int_0^{+\infty} (Bx-k)\varphi_j(x;k)^2 dx. \end{aligned} \tag{5.16}$$

The first term in the r.h.s. of (5.16) is bounded by a constant times ℓ^{-3}. Due to the energy equation (2.50), the second one is bounded by a constant times $B\ell^{-1}$. Summing up (5.15) and (5.16) we then get that

$$\mathcal{V}_0\varphi_j(\ell/2;k)^2 \leq cB\ell^{-1}(1 + 4(2n+3) + \alpha_j B\ell^2 \mathrm{e}^{-\beta_j B\ell^2}),$$

for some $c > 0$ independent of B, ℓ, $\mathcal{V}_0$ and j, whence

$$\mathcal{V}_0\varphi_j(\ell/2;k)^2 \leq \nu_n (B\ell^2)^{-1/2} B^{3/2}, \tag{5.17}$$

upon taking $B\ell^2$ sufficiently large.
3. The end of the proof now follows from (2.21), (2.51) and (5.17). ■

Lemma 5.7 *Let $\alpha_n = (\mu_n + 1)^{1/2}$ and $x_n = -\ell/2 + \alpha_n B^{-1/2}$, where μ_n is defined in Lemma 5.5. Then for all $B > 0$, $\ell > 0$, $\mathcal{V}_0 > 0$ and $\Delta x \geq 0$, we have*

$$\int_{x_n+\Delta x}^{+\infty} \varphi_j(x;k)^2 dx \leq 8\mathrm{e}^{-(\alpha_n/4)B^{1/2}\Delta x},\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0, 1, \ldots, n.$$

Furthermore there are two constants $\theta_n = \theta_n(a,c) > 0$ and $\vartheta_n = \vartheta_n(a,c) > 0$ depending only on n, a, and c such that for all $\Delta x \geq 0$ we have

$$\int_{-\infty}^{-\ell/2-\Delta x} \varphi_j(x;k)^2 dx \leq \vartheta_n \mathrm{e}^{-2(3-c)^{1/2}B^{1/2}\Delta x},\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0,1,\ldots,n,$$

provided $B\ell^2 \geq \theta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$.

Proof.
Set $\varepsilon = 2\alpha_n/(B^{1/2}\ell)$ so $B\ell^2\varepsilon^2 > \mu_n$ and

$$\varphi_j(x;k)^2 \leq 2\alpha_n B^{1/2}\mathrm{e}^{-(\alpha_n/4)B^{1/2}x},\ x \geq x_n,\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0,1,\ldots,n, \tag{5.18}$$

according to Lemma 5.5. The first part of the result obviously follows by integrating (5.18) over $[x_n + \Delta x, +\infty)$.

To prove the second part we make $\mathcal{V}_0 \geq E_{n+1}(B)$ in such a way that the effective potential (2.20) is lower bounded as $W_j(x;k) \geq (3-c)B$ for all $x \leq -\ell/2$, and subsequently

$$\varphi_j(x;k)^2 \leq \varphi_j(-\ell/2;k)^2 \mathrm{e}^{2(3-c)^{1/2}B^{1/2}(x+\ell/2)},\ x \leq -\ell/2,\ k \in \omega_j^{-1}(\Delta_n)_-,$$

by Proposition 8.3 in [10]. Now the result immediately follows from this and from Lemma 5.6. ■

An immediate consequence of Lemmas 5.4 and 5.7 is the following

Lemma 5.8 *Let $n \in \mathbb{N}$ and Δ_n be defined by (2.9). Then there is a constant $\kappa_n = \kappa_n(a,c) > 0$ depending only on n, a and c such that*

$$\left| k + \frac{BL}{2} \right| \leq \kappa_n B^{1/2},\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0,1,\ldots,n, \tag{5.19}$$

provided $\mathcal{V}_0 \geq E_{n+1}(B)$ and $B\ell^2 \geq \theta_n$, where θ_n is as in Lemma 5.7.

Proof.
1. Let $k \in \omega_j^{-1}(\Delta_n)_-$, $j = 0,1,\ldots,n$, be of the form $k = -(B\ell/2)(1+\epsilon)$ for some $\epsilon > 0$. Next taking $\Delta x = (\ell\epsilon)/4$ in Lemma 5.7 and assuming that $B\ell^2 \geq \theta_n$ and $\mathcal{V}_0 \geq E_{n+1}(B)$, we get that

$$\int_{-\infty}^{-(\ell/2)(1+(\epsilon/2))} \varphi_j(x;k)^2 dx \leq \frac{\mathrm{e}^{-((3-c)^{1/2}/2)B^{1/2}\ell\epsilon}}{(3-c)^{1/2}}. \tag{5.20}$$

Further, the normalization condition $\|\varphi_j(.;k)\| = 1$ entails

$$\int_{-(\ell/2)(1+(\epsilon/2))}^{+\infty} (Bx-k)^2 \varphi_j(x;k)^2 dx$$
$$\geq \left(\frac{B\ell\epsilon}{4}\right)^2 \left(1 - \int_{-\infty}^{-(\ell/2)(1+(\epsilon/2))} \varphi_j(x;k)^2 dx\right), \tag{5.21}$$

since $x - k/B \geq (\ell\epsilon)/4$ for all $x \geq -(\ell/2)(1+(\epsilon/2))$. As a consequence we have

$$\omega_j(k) \geq \frac{B^2\ell^2\epsilon^2}{16}\left(1 - \frac{\mathrm{e}^{-((3-c)^{1/2}/2)B^{1/2}\ell\epsilon}}{(3-c)^{1/2}}\right), \tag{5.22}$$

by combining (5.20)-(5.21) with the basic estimate

$$\omega_j(k) \geq \int_{\mathbb{R}} (Bx-k)^2 \varphi_j(x;k)^2 dx.$$

Moreover k being in $\omega_j^{-1}(\Delta_n)_-$ we have $\omega_j(k) \leq (2n+c)B$, whence

$$2n + c \geq \frac{B\ell^2\epsilon^2}{16}\left(1 - \frac{\mathrm{e}^{-((3-c)^{1/2}/2)B^{1/2}\ell\epsilon}}{(3-c)^{1/2}}\right), \tag{5.23}$$

according to (5.22). The r.h.s. of (5.23) being an unbounded increasing function of $B\ell^2\epsilon^2$ depending only on c, while the l.h.s. depends only on n and c, we thus have

$$B\ell^2\epsilon^2 \leq \xi_n(c),$$

for some constant $\xi_n(c) > 0$ depending only on n and c. Therefore

$$\inf \omega_j^{-1}(\Delta_n)_- \geq -\frac{BL}{2} - \xi_n(c)^{1/2} B^{1/2}, \tag{5.24}$$

since $k = -((B\ell)/2)(1+\epsilon) \in \omega_j^{-1}(\Delta_n)_-$.
2. For the rest of the proof we recall from Lemma 5.4 that

$$\sup \omega_j^{-1}(\Delta_n)_- \leq -\frac{BL}{2} + \gamma_n(a)^{1/2} B^{1/2}, \tag{5.25}$$

where $\gamma_n(a) > 0$ depends only on n and a. The result then follows from (5.24)-(5.25) by setting $\kappa_n(a,c) = \max(\xi_n(c)^{1/2}, \gamma_n(a)^{1/2})$. ∎

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