EDGE CURRENTS FOR THE TIME-FRACTIONAL, HALF-PLANE, SCHRÖDINGER EQUATION WITH CONSTANT MAGNETIC FIELD

PETER D. HISLOP AND ÉRIC SOCCORSI

ABSTRACT. We study the large-time asymptotics of the edge current for a family of time-fractional Schrödinger equations with a constant, transverse magnetic field on a half-plane $(x, y) \in \mathbb{R}_x^+ \times \mathbb{R}_y$. The TFSE is parameterized by two constants (α, β) in $(0, 1]$, where α is the fractional order of the time derivative, and β is the power of i in the Schrödinger equation. We prove that for fixed α, there is a transition in the transport properties as β varies in $(0, 1]$: For $0 < \beta < \alpha$, the edge current grows exponentially in time, for $\alpha = \beta$, the edge current is asymptotically constant, and for $\beta > \alpha$, the edge current decays in time. We prove that the mean square displacement in the $y \in \mathbb{R}$-direction undergoes a similar transport transition. These results provide quantitative support for the comments of Laskin [13] that the latter two cases, $\alpha = \beta$ and $\alpha < \beta$, are the physically relevant ones.

Dedicated to the memory of our colleague and friend
Georgi Raikov

CONTENTS

1. Introduction and statement of the problem

There has been a lot of analysis of systems described by equations with space or time fractional derivatives, especially associated with parabolic and hyperbolic partial differential equations. In this article, we study the time-fractional Schrödinger equation (TFSE) for a physically motivated model. The TFSE studied in this paper has the form

$$i^\beta \partial_t^\alpha u(t,x) = Hu(x,t), \tag{1.1}$$

on the Hilbert space $L^2(\Omega)$, for certain self-adjoint Schrödinger operators H. The fractional time derivative is the Caputo derivative defined in (2.5). The parameters $\alpha, \beta \in (0,1]$. We show how the time-fractional derivative affects the large time behavior of the transport of the system. In particular, the model is described by a magnetic Schrödinger operator in a half-plane $\Omega := \mathbb{R}_x^+ \times \mathbb{R}_y$, with a Dirichlet boundary condition at $x = 0$. Such operators and geometry play a role in the integer quantum Hall effect.

These models allow us to examine the affect of a fractional time derivative on the transport properties of the model. The dynamics of the Schrödinger equation ($\alpha = \beta = 1$ in (1.1)) exhibits nontrivial edge currents, whose conductance is quantized, and which flow along the impenetrable edge $x = 0$ in the sense that they are localized in a small neighborhood of the edge. We now replace the dynamical Schrödinger equation by the time-fractional Schrödinger equation (1.1) and examine the edge current associated with these half-space operators. Our main results are that the edge current exhibits two types of extreme behavior depending on the relation between the two parameters (α, β): either the current decreases to zero as $t \to \infty$, or the edge current explodes as $t \to \infty$. These two distinctly different behaviors depend on how the time-fractional Schrödinger equation is defined. At the critical values, the edge current is asymptotically constant, reminiscent of the Schrödinger case ($\alpha = \beta = 1$) for which the edge current is constant.

In this paper, we study a two-parameter family of TFSE and compute the asymptotics of the edge current and the mean square displacement. We show, for example, that the edge current of the Naber model (1.2) is asymptotically a nonzero constant, whereas in the AYH model (1.3), it decays to zero. Furthermore, we study the mean-square displacement of the solutions in the y-direction and prove that for the Naber model, this is asymptotically ballistic, whereas for the AYH model, it decays to zero.

1.1. **Fractional quantum mechanics.** Fractional quantum mechanics (FQM) was introduced by Laskin in 1999, and developed by him in a series of papers [13, 14, 15]. The crucial component of FQM is the fractional Schrödinger equation (FSE). In the FSE, the normal Laplacian in the space variables, $-\Delta$, is replaced with a fractional Laplacian, defined through the functional calculus, $(-\Delta)^\alpha$, with $\alpha \in (0, 1]$:

$$-i\partial_t u + (-\Delta)^\alpha u + qu = 0, \ \alpha \in (0, 1].$$

The fractional Laplacian is a spatially non-local operator. However, due to the presence of the first-order time derivative in the FSE, FQM preserves all quantum mechanics fundamentals such as the probability conservation law and the energy conservation law, as follows from the unitarity of the time-evolution operator.

The regular Schrödinger equation has the mathematical appearance of a diffusion equation and can be derived by considering probability distributions. Feynman and Hibbs [4] used a Gaussian probability distribution in the space of all possible paths for a quantum mechanical particle, in order to derive the Schrödinger equation. Laskin derived the FSE by using Feynman's path integral approach with Lévy distributions, instead of Gaussian distributions, for the set of possible paths.

Naber [16] introduced the TFSE in 2004, by replacing the first-order time derivative in the Schrödinger equation with a fractional power $\alpha \in (0, 1]$, and raising i to the same power α:

$$-i^\alpha \partial_t^\alpha u - \Delta u + qu = 0. \tag{1.2}$$

We will refer to this TFSE as the **Naber model.** The fractional time derivative is defined to be the Caputo derivative, see (2.5).

One of the main reasons invoked by Naber for raising i to the power of the time derivative is the asymptotic stability of the time-evolution of the solutions to the TFSE with respect to α. By asymptotic stability, we mean that the infinite t behavior is independent of the order α of the derivative, see, for example, (1.16).

The Naber model was not derived from the classical Schrödinger equation by considering a non-Markovian evolution but the time-evolution of the solution to Naber's TFSE is reminiscent of the time-evolution of solutions to the regular Schrödinger equation. Nevertheless, the time-evolution of solutions in the Naber model for $0 < \alpha < 1$ is not unitary, so that it does not support the the basic quantum mechanical laws of probability conservation and energy conservation.

In 2013, Narahari Achar, Yale, and Hanneken [17] provided another model of a TFSE that differs from the Naber model. These authors presented a derivation that relies on a suitable representation of the action in the Feynman path integral motivated by the fractional dynamics of a free classical particle. In what we will call the **AYH model** of the TFSE, the operator $i\partial_t$ is replaced by $i\partial_t^\alpha$, with $\alpha \in (0, 1)$:

$$-i\partial_t^\alpha u - \Delta u + qu = 0. \tag{1.3}$$

Like the Naber model, the time-evolution is not unitary, and, consequently, the AYH model has the same shortcomings in that it violates the basic quantum mechanical laws.

However, Narahari Achar, Yale, and Hanneken [17] argued that this model is more physically natural than the one proposed by Naber. For example, the total probability associated with a solution $\psi(x,t)$ of the Schrödinger equation or TFSE is given by $\int_{\mathbb{R}^d} |\psi(x,t)|^2 dx$. The unitary time evolution of the solutions to the usual Schrödinger equation guarantees that this quantity is equal to one for all time (with a normalized initial condition). But, for the models of both Naber and AYH, this quantity depends on time. As $t \to \infty$, the total probability is greater than one in the Naber model, whereas for the AYH model, the total probability vanishes.

There have been several other papers on the TFSE since Naber's article [16] and we briefly mention the following works related to our results. Dong and Xu [3] (see also [19]) discussed a general space-time fractional Schrödinger equation combining the Caputo fractional time derivative with $\beta \in (0,2)$, with a fractional space derivative of order $\alpha \in (0,2)$ (the notation of [3]). The resulting fractional space-time Schrödinger equation has the form

$$i^\beta \partial_t^\beta u(x,t) = ((-\Delta)^{\frac{\alpha}{2}} u)(x,t) + V(x)u(x,t) = (H_\beta u)(x,t),$$

on the Hilbert space $L^2(\mathbb{R}^d)$ with an initial condition $u(0) = u_0$. The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is defined through the Fourier transform as the multiplication operator by $|k|^\alpha$.

Dong and Xu [3] studied the impact of the potential V on the time evolution of the solutions with initial conditions chosen to be eigenfunctions of the Hamiltonian $H_\alpha := (-\Delta)^{\frac{\alpha}{2}} + V$. With this choice, they find product solutions for which the time dependence is expressed through a Mittag-Leffler function. They determined that the long time behavior depends on the order β of fractional time derivative and the sign of the eigenvalue of H_α corresponding the the chosen eigenfunction for the initial condition. In particular, if the eigenvalue is positive, and $0 < \beta \leqslant 1$, the total probability of the solution never tends to zero, whereas if $1 < \alpha < 2$, the total probability of a solution tends to zero in certain cases.

Bayin [1] also explored the relation between the eigenvalues of H_α and the large-time behavior of the solutions. He studies solutions of the form $u(x,t) = X(x)T(t)$. Assuming the spectrum of H_α is discrete, as in [3], the function X is an eigenfunction of H_α. The temporal part T is given by a Mittag-Leffler function determined by β. Bayin proved the nonconservation of probability. He also gave the long-time behavior of product solutions to the TFSE following from the asymptotics of the Mittag-Leffler functions. There are similarities between these papers and our treatment in section 2.

P. Górka, H. Prado, and J. Trujillo [5] studied the abstract Naber model $i^\alpha \partial_t^\alpha u = Hu$, $\alpha \in (0,1)$ for a general nonnegative, self-adjoint operator H on a Hilbert space $\mathcal{H}$. They prove existence and uniqueness of strong solutions in $\mathcal{H}$ when ∂_t^α is the Caputo derivative, and the initial condition $u_0 \in D(H) \subset \mathcal{H}$. Among their results, they prove that the solution operator $U_\alpha(t)$ approaches the unitary time evolution e^{-itH} as $\alpha \to 1^-$. Their approach, based properties of

the Mittag-Leffler functions on the spectral theorem for H, has some similarities with ours, see section 2.

P. Górka, H. Prado, and D. J. Pons [6] continued this study and examined the time-asymtotic behavior of solutions to the abstract Naber model. They also proved that the global probability is not preserved under the evolution operator $U_\alpha(t)$: If $u_0 \in \mathcal{H}$, with $\|u_0\|_{\mathcal{H}} = 1$, then $\|U_\alpha(t)u_0\|_{\mathcal{H}}$ is $\mathcal{O}(\frac{1}{\alpha})$. This had already been noted by [3].

A. Iomin [9, 10] also studied the Naber model (1.2) in relation to the quantum comb model. He used an eigenfunction expansion of the spatial Hamiltonian and the Mittag-Leffler functions to express the Green's function as

$$G(x, x', t) = \sum_\lambda \phi_\lambda^*(x')\phi_\lambda(x)E_\alpha(\lambda(-it)^\alpha).$$

For $\alpha = \frac{1}{2}$, Iomin discussed similarities between the Green's function for the TFSE and the quantum comb model. In [11], Iomin studied the AYH model (**??**) and explored quantum mechanical properties of the model, such as the Heisenberg equations of motion. A review of TFSE is given in [12].

1.2. **The quantum Hall model.** We study the magnetic Schrödinger operator in a half-plane with a constant, transverse magnetic field of strength $b > 0$ and Dirichlet boundary conditions along the edge $x = 0$. The half-plane is $\Omega := \{(x, y) \mid x > 0, y \in \mathbb{R}\}$. The constant magnetic field is orthogonal to the half-plane and described by a vector potential $\mathbf{a}(x, y) := (a_1(x, y), a_2(x, y)) := b(0, x)$.

The magnetic Schrödinger operator is defined as follows. Let $p_x := -i\partial_x$ and $p_y := -i\partial_y$ be the two momentum operators. The two-dimensional magnetic Schrödinger operator $H(\mathbf{a})$ is defined on the dense domain $\mathrm{C}_0^\infty(\Omega) \subset \mathrm{L}^2(\Omega)$ by

$$H = H(\mathbf{a}) := (-\mathrm{i}\nabla - \mathbf{a})^2 = p_x^2 + (p_y + bx)^2. \tag{1.4}$$

The spectral properties of $H(\mathbf{a})$ are well-known. These are derived from the representation of $H(\mathbf{a})$ as a direct integral. Because of the translational invariance in the y-direction, the partial Fourier transform with respect to y provides this decomposition. Let $\mathcal{F}_y$ be the partial Fourier transform with respect to y, i.e.

$$\hat{\varphi}(x, k) = (\mathcal{F}_y\varphi)(x, k) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iky}\varphi(x, y)\mathrm{d}y, \ \varphi \in L^2(\Omega).$$

The direct integral decomposition of $H(\mathbf{a})$ is given by

$$\mathcal{F}_y H(\mathbf{a})\mathcal{F}_y^* = \int_{\mathbb{R}}^{\oplus} h_b(k) \ dk, \tag{1.5}$$

where the reduced Schrödinger operator $h_b(k)$ is given by

$$h_b(k) := -\partial_x^2 + (k + bx)^2, \ x > 0, \ k \in \mathbb{R}. \tag{1.6}$$

These operators act on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition at $x = 0$.

For all $k \in \mathbb{R}$ fixed, the effective potential $V_k(x) := (k+bx)^2$ is unbounded as x goes to infinity, so $h_b(k)$ has a compact resolvent. Let $\{\lambda_n(k), \ n \in \mathbb{N}\}$ denote the eigenvalues, arranged in non-decreasing order, of $h_b(k)$. Since all these

eigenvalues $\lambda_n(k)$ with $n \in \mathbb{N}$ are simple, each $k \mapsto \lambda_n(k)$ is a real analytic function in $\mathbb{R}$. The *dispersion curves* $\lambda_n(k)$, $n \in \mathbb{N}$, have been extensively studied in several articles (see e.g. [2, 7]). They are monotone decreasing functions of $k \in \mathbb{R}$, obeying

$$\lim_{k \to -\infty} \lambda_n(k) = +\infty \quad \text{and} \quad \lim_{k \to +\infty} \lambda_n(k) = E_n, \qquad (1.7)$$

for all $n \in \mathbb{N}$, where $E_n := (2n-1)b$ is the n-th Landau level. As a consequence, the spectrum of $H(\mathbf{a})$ is $\sigma(H(\mathbf{a})) = \overline{\cup_{n \geqslant 1} \lambda_n(\mathbb{R})} = [b, +\infty)$. The Landau levels E_n, $n \in \mathbb{N}$, are thresholds in the spectrum of $H(\mathbf{a})$.

For $n \in \mathbb{N}$ and $k \in \mathbb{R}$, we consider a normalized eigenfunction $\phi_n(\cdot, k)$ of $h_b(k)$ associated with $\lambda_n(k)$. It is well known that $\phi_n(\cdot, k)$ depends analytically on k. We define the n-th generalized Fourier coefficient of $u \in L^2(\Omega)$, by

$$u_n(k) := \langle \mathcal{F}_y u(\cdot, k), \phi_n(\cdot, k) \rangle_{L^2(\mathbb{R}^+)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} \hat{u}(x, k) \overline{\phi_n(x, k)} \, dx. \qquad (1.8)$$

Note that the inner product in (1.8) is linear in the first entry. In this setting, Parseval's Theorem yields

$$\|u\|_{L^2(\Omega)}^2 = \|\widehat{u}\|_{L^2(\Omega)}^2 = \sum_{n \geqslant 1} \|u_n\|_{L^2(\mathbb{R})}^2. \qquad (1.9)$$

1.3. **The edge current.** We define the *edge current* carried by a state as the time derivative of the expectation of the observable y in the time-evolved state. Here, the observable y denotes the multiplier by the coordinate y in $L^2(\Omega)$, with $\Omega = \mathbb{R}_x^+ \times \mathbb{R}_y$. Given an evolution equation, we denote by $u(t)$ the dynamical solution to the initial value problem with $u(t = 0) = u_0$. The edge current $J_y[u_0](t)$ is given by

$$J_y[u_0](t) := \frac{d}{dt} \langle u(t), y u(t) \rangle_{L^2(\Omega)}. \qquad (1.10)$$

For the ordinary Schrödinger equation, we have $u(t) = e^{-itH} u_0$. In this case, the time evolution can be represented by the Heisenberg variable $y(t) := e^{-itH} y e^{itH}$, all $t \in \mathbb{R}$. A current operator may be expressed in terms of the time derivative of $y(t)$. This is the velocity in the y-direction. As is well known, this operator is given by $\frac{dy(t)}{dt} = -i[H, y(t)] = -ie^{-itH}[H, y]e^{itH}$. In this case, the current operator has the form of the self-adjoint operator $J_y := -i[H, y] = -2(p_y + bx)$. The current carried by a time-evolved state $u(t)$, with initial condition u_0, can the be written in terms of the current operator:

$$\begin{aligned} J_y[u_0](t) &:= \frac{d}{dt} \langle u(t), y u(t) \rangle_{L^2(\Omega)} = -i \langle u(t), [H, y] u(t) \rangle_{L^2(\Omega)} \\ &= \langle u(t), J_y u(t) \rangle_{L^2(\Omega)} = \langle u_0, J_y(t) u_0 \rangle_{L^2(\Omega)}, \end{aligned} \qquad (1.11)$$

where the Heisenberg current observable is defined by $J_y(t) := e^{-itH} J_y e^{itH}$.

For the TFSE (1.1), the time evolution is not given by a unitary group. As discussed below, the time evolution is expressed, via the functional calculus, in terms of a Mittag-Leffler function. These functions, whose properties are

summarized in Appendix A, are defined by

$$E_{\alpha,\sigma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\sigma + \alpha k)}, \tag{1.12}$$

where $\Gamma(z)$ denotes the gamma function. It follows from (1.12) that $E_{1,1}(z) = e^z$. In terms of these functions, the solution to the TFSE may be written as

$$u(t) = E_{\alpha,1}((-i)^\beta t^\alpha H)u_0. \tag{1.13}$$

The current then has the form

$$J_y[u_0](t) := \frac{d}{dt}\langle u(t), yu(t)\rangle_{L^2(\Omega)} = \frac{d}{dt}\int_\Omega y|(E_{\alpha,1}((-i)^\beta t^\alpha H)u_0)(x,y)|^2 \ dx \ dy. \tag{1.14}$$

1.4. **Summary of results on the edge current.** The dynamics of the TFSE is governed by the initial-value problem:

$$i^\beta \partial_t^\alpha u(x,y;t) = H(\mathbf{a})u(x,y;t) = (p_x^2 + (p_y + bx)^2)u(x,y,t), \tag{1.15}$$

with the initial condition $u(t = 0) = u_0(x,y)$. The solution to the initial-value problem is

$$u(t) = E_{\alpha,1}((-i)^\beta t^\alpha H(\mathbf{a}))u_0,$$

see (1.13) and (1.12). The exponents $(\alpha, \beta) \in (0,1] \times [0,1]$. There are different dynamical asymptotic regimes depending on the relationship between these two exponents. Our general results are:

(1) **Case 1:** $0 < \beta \leqslant \alpha < 1$. For $\beta < \alpha$, the edge current grows exponentially in time. In the case $0 < \alpha = \beta < 1$, the edge current is asymptotically constant in time.

(2) **Case 2:** $0 < \alpha < \beta \leqslant 1$. The current exhibits decays to zero as $t \to \infty$ like $t^{-1-3\alpha}$.

In particular, we mention three special cases discussed in the literature:

(1) **Schrödinger equation** [7]: $\alpha = \beta = 1$. The solution to the initial-value problem is

$$u(t) = E_{1,1}(-itH(\mathbf{a}))u_0 = e^{-itH(\mathbf{a})}u_0.$$

The edge current is bounded for all time, see Remark 3.1. Lower bounds on the edge current were derived in [7].

(2) **Naber model** [16]: $\alpha = \beta \in (0,1)$. The solution to the initial-value problem is

$$u(t) = E_{\alpha,1}((-i)^\alpha t^\alpha H(\mathbf{a}))u_0.$$

The edge current is bounded for all time. This is similar to the Schrödinger case.

(3) **AYH model** [17]: $\alpha \in (0,1)$ and $\beta = 1$. The solution to the initial-value problem is

$$u(t) = E_{\alpha,1}(-it^\alpha H(\mathbf{a}))u_0.$$

The edge current decays to zero like $t^{-1-3\alpha}$.

These results illustrate that, for fixed $0 < \alpha < 1$, tuning the parameter $\beta \in (0,1]$ leads to qualitatively different behaviors in the long-time asymptotics of the current. The critical value $\beta = \alpha$ is a transition between exponential growth and inverse power decay. For $0 < \alpha < 1$ **fixed**, as $\beta \in (0,1)$ varies, we have

$$0 < \beta < \alpha \qquad J_y[u_0](t) \sim e^{ct}$$

$$\beta = \alpha \qquad J_y[u_0](t) \sim C_0$$

$$\alpha < \beta \leqslant 1 \quad J_y[u_0](t) \sim t^{-1-3\alpha}$$

$$(1.16)$$

The Schrödinger case $\alpha = \beta = 1$ is conservative and the edge current is constant in time. For other values of $\alpha = \beta$, the current is asymptotically constant.

1.5. **Outline of the paper.** In section 2, we prove the existence and uniqueness of the solution to the initial-value problem with u_0 chosen from an appropriate function space depending on the exponents (α, β). Section 3 presents the calculation of the edge current $J_y[u_0](t)$ in terms of the Mittag-Leffler functions. The main results on the asymptotic behavior of the edge currents, for various values of the exponents (α, β), are presented in section 4, including a the special cases of the models of Naber and of AYH. In section 6, we calculate the mean-square displacement (MSD) of the y-observable for the time-fractional Hall model for the ranges of exponents (α, β). Details concerning the Mittag-Leffler functions are summarized in the Appendix A.

1.6. **Acknowledgement.**

2. The well-posedness of constant-order time-fractional Schrödinger equations

We prove the existence and uniqueness of the solution to the initial-value problem for the time-fractional Schrödinger equation (TFSE) on general general subdomains $\Omega_0 \subset \mathbb{R}^d$, $d \geqslant 1$, with smooth boundary $\Gamma_0 := \partial\Omega_0$. For $\alpha, \beta \in (0,1)$, the initial-value problem for the time-fractional Schrödinger equation is:

$$-i^\beta \partial_t^\alpha u(\cdot, t) + Hu(\cdot, t) = 0, \text{ and } u(\cdot, t = 0) = u_0(\cdot). \qquad (2.1)$$

We assume that Ω_0 is open and not necessarily bounded. Let H be a lower semi-bounded (nonnegative) self-adjoint operator with domain $D(H) \subset L^2(\Omega_0)$ and having discrete spectrum. For the sake of simplicity, we assume that

$D(H^{\frac{1}{2}}) = H_0^1(\Omega_0)$, and that H has a compact resolvent. We denote by $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ the eigenvalues of H arranged in nondecreasing order and repeated with the multiplicity. We pick an orthonormal basis $\{\phi_n, \ n \in \mathbb{N}\}$ in $L^2(\Omega_0)$ of eigenvectors of H, satisfying

$$H\phi_n = \lambda_n\phi_n.$$

In order to use the results of this section, we recall that the quantum Hall model is described on $\Omega = \mathbb{R}_x^+ \times \mathbb{R}_y$, with the self-adjoint Schrödinger operator $H(\mathbf{a})$ on $L^2(\mathbb{R}_x^+ \times \mathbb{R}_y)$. By means of the partial Fourier transform with respect to y, see (1.5), the problem is reduced to the study of the family of fiber operators $h_b(k)$, $k \in \mathbb{R}$, acting on $L^2(\mathbb{R}_x^+)$. These nonnegative operators $h_b(k)$ have compact resolvent and, consequently, discrete, simple spectrum. So, the results of this section apply to the operators $h_b(k)$ with $\Omega_0 = \mathbb{R}_x^+$.

2.1. **Main results on the TFSE.** We recall the time-fractional Schrödinger equation for exponents $(\alpha, \beta) \in (0,1) \times (0,1)$. We will also consider the endpoints $\alpha, \beta \in \{0, 1\}$ separately. The TFSE is the initial-value problem

$$-i^\beta \partial_t^\alpha u(x,t) + Hu(x,t) = 0, \ (x,t) \in \Omega_0 \times (0,T) \tag{2.2}$$

with homogeneous Dirichlet boundary condition

$$u(x,t) = 0, \ (x,t) \in \Gamma_0 \times (0,T) \tag{2.3}$$

and initial state u_0

$$u(x,0) = u_0(x), \ x \in \Omega_0, \tag{2.4}$$

where ∂_t^α denotes the Caputo fractional derivative of order α. The Caputo fractional derivative ∂_t^α is defined by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds. \tag{2.5}$$

We note that $\partial_t^\alpha u \to \partial_t u$, as $\alpha \to 1$, if, for example, $u \in C^2(\mathbb{R})$. This follows from the formula (2.5) by an integration by parts.

We call $u(x,t)$, with $x \in \Omega_0$ and $t > 0$, a **solution** to (2.2)–(2.4) if the 3 following conditions hold simultaneously:

(1) $u(\cdot, t) \in D(H)$, for a.e. $t \in \mathbb{R}^+$;
(2) the fractional Schrödinger equation (2.2) holds in $L^2(\Omega_0)$ for $t \in \mathbb{R}^+$:

$$i^\beta \partial_t^\alpha u(\cdot, t) = Hu(\cdot, t);$$

(3) $u \in \mathcal{C}(\overline{\mathbb{R}^+}, L^2(\Omega_0))$, so $\lim_{t \downarrow 0} \|u(\cdot, t) - u_0\|_{L^2(\Omega_0)} = 0$.

Using an eigenfunction expansion, we prove the existence and uniqueness of solutions to the TFSE. We need to take the initial conditions from various function spaces in $L^2(\Omega_0)$ depending on the relationship between α and β. Recall that $D(H)$ denotes the domain of the self-adjoint operator H in $L^2(\Omega_0)$. For an orthonormal basis $\{\phi_n\}_{n \in \mathbb{N}}$ and $u_0 \in L^2(\Omega_0)$, we define the associated Fourier coefficients by $u_{0,n} := \langle u_0, \phi_n \rangle_{L^2(\Omega_0)}$. We denote by $C_F(\Omega_0)$ the dense set of initial conditions $u_0 \in L^2(\Omega_0)$ with finitely-many nonzero Fourier coefficients $u_{0,n}$.

Theorem 2.1. *Let $(\alpha, \beta) \in (0,1)^2$ be the parameters of the TFSE described in (2.2)-(2.4). We distinguish three cases:*

(1) *Case 1: $0 < \alpha < \beta \leqslant 1$. Let $u_0 \in L^2(\Omega_0)$.*

(2) *Case 2: $0 < \alpha = \beta \leqslant 1$. Let $u_0 \in D(H)$.*

(3) *Case 3: $0 \leqslant \beta < \alpha \leqslant 1$. Let $u_0 \in C_F(\Omega_0)$.*

In each case, there exists a unique solution $u \in C(\overline{\mathbb{R}^+}, L^2(\Omega_0)) \cap C(\mathbb{R}^+, D(H))$ to (2.2)–(2.4) of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_{0,n} E_{\alpha,1}((-i)^\beta \lambda_n t^\alpha) \phi_n(x), \; u_{0,n} := \langle u_0, \phi_n \rangle_{L^2(\Omega_0)}, \qquad (2.6)$$

such that

(1) *Case 1: For each $u_0 \in L^2(\Omega_0)$, there exists a finite constant $C_1 \geqslant 0$, so that*

$$\|u\|_{\mathcal{C}(\mathbb{R}^+, L^2(\Omega_0))} \leqslant C_1 \|u_0\|_{L^2(\Omega_0)}.$$

Moreover, we have $\partial_t^\alpha u \in C(\mathbb{R}^+, L^2(\Omega_0))$, and there exists a finite constant $\tilde{C}_1 > 0$ so that

$$\|u(\cdot, t)\|_{D(H)} + \|\partial_t^\alpha u(\cdot, t)\|_{L^2(\Omega_0)} \leqslant \tilde{C}_1 \|u_0\|_{L^2(\Omega_0)} t^{-\alpha},$$

for all $t > 1$.

(2) *Case 2: For each $u_0 \in D(H)$, there exists a finite constant $C_2 \geqslant 0$, so that*

$$\|u\|_{\mathcal{C}(\overline{\mathbb{R}^+}, L^2(\Omega_0))} \leqslant C_2 \|u_0\|_{D(H)}.$$

Moreover, we have $\partial_t^\alpha u \in C(\mathbb{R}^+, L^2(\Omega_0))$, and there exists a finite constant $\tilde{C}_2 \geqslant 0$, so that

$$\|u(\cdot, t)\|_{D(H)} + \|\partial_t^\alpha u(\cdot, t)\|_{L^2(\Omega_0)} \leqslant \tilde{C}_2 \|u_0\|_{D(H)}, \forall t \in \mathbb{R}^+.$$

(3) *Case 3: Let $\theta(\alpha, \beta) := \frac{\pi\beta}{2\alpha}$. For each $u_0 \in C_F(\Omega_0)$, there exists a finite constant $C_3 \geqslant 0$, and an index $n^* \in \mathbb{N}$, so that*

$$\|u\|_{\mathcal{C}(\overline{\mathbb{R}^+}, L^2(\Omega_0))} \leqslant C_3 (1 + t^\alpha \lambda_{n^*})^{2\left(\frac{1-\beta}{\alpha}\right)} e^{2t\lambda_{n^*}^{\frac{1}{\alpha}} \cos \theta(\alpha, \beta)} \|u_0\|_{L^2(\Omega_0)}^2$$

Moreover, we have $\partial_t^\alpha u \in C(\mathbb{R}^+, L^2(\Omega_0))$ and there exists a finite constant $\tilde{C}_3 \geqslant 0$, so that

$$\|u(\cdot, t)\|_{D(H)} + \|\partial_t^\alpha u(\cdot, t)\|_{L^2(\Omega_0)}$$
$$\leqslant \tilde{C}_3 (1 + t^\alpha \lambda_{n^*})^{2\left(\frac{1-\beta}{\alpha}\right)} e^{2t\lambda_{n^*}^{\frac{1}{\alpha}} \cos \theta(\alpha, \beta)} \|u_0\|_{L^2(\Omega_0)}^2. \qquad (2.7)$$

The constants C_j, $j = 1, 2, 3$ depend on α, β, Ω, and H.

2.2. **Proof of Theorem 2.1.** We divide the proof of Theorem 2.1 into two parts: existence and uniqueness.

Proof. Part 1: Existence.
1. We show that the solution exists as the L^2-norm limit of the sequence of partial sums given by the expansion of the solution with respect to the orthonormal basis ϕ_n of eigenfunctions of H. The relationship between the exponents α and β determine the choice of the initial conditions u_0 for which these expansion converge. This is controlled by the bounds on the Mittag-Leffler functions as described in section A.2. Given $u_0 \in L^2(\Omega_0)$, we let $u_{0,n} := \langle u_0, \phi_n \rangle_{L^2(\Omega_0)}$ be the nth-Fourier coefficient of u_0 relative to this orthonormal basis. We first consider the finite sum

$$S_N(x,t) := \sum_{n=1}^{N} u_{0,n} E_{\alpha,1}((-i)^{\beta} \lambda_n t^{\alpha}) \phi_n(x), \ N \in \mathbb{N}. \tag{2.8}$$

We define the function $F_{\alpha,\beta,n}(t)$, for $t \geqslant 0$, based on the bounds (A.4) and (A.5):

$$0 < \alpha < \beta \qquad\qquad F_{\alpha,\beta,n}(t) = \frac{C}{1+t^{\alpha}\lambda_n} \leqslant C_1 \tag{2.9}$$

$$\alpha = \beta \qquad\qquad F_{\alpha,\beta,n}(t) = C_2 \tag{2.10}$$

$$\beta < \alpha \leqslant 1 \quad F_{\alpha,\beta,n}(t) = C_3 \left[(1+t^{\alpha}\lambda_n)^{\frac{1-\beta}{\alpha}} e^{t\lambda_n^{\frac{1}{\alpha}} \cos\theta(\alpha,\beta)} + \frac{1}{1+t^{\alpha}\lambda_n} \right],$$
$$\tag{2.11}$$

where $\theta(\alpha, \beta) := \left(\frac{\pi\beta}{2\alpha} \right)$. The finite constants $C_j > 0$, for $j = 1, 2, 3$, depend on (α, β), but are independent of (n, t).
2. For any $t \geqslant 0$, we have

$$\|S_N(t)\|_{L^2(\Omega_0)}^2 = \sum_{n=1}^{N} |u_{0,n}|^2 \left| E_{\alpha,1}((-i)^{\beta} \lambda_n t^{\alpha}) \right|^2$$

$$\leqslant \sum_{n=1}^{N} |u_{0,n}|^2 F_{\alpha,\beta,n}^2(t). \tag{2.12}$$

For cases 1 and 2, the bounds (2.9) and (2.10) show that the function $F_{\alpha,\beta,n}(t)$ is uniformly bounded in (n, t). As a result, the right side of (2.12) is finite for any $u_0 \in L^2(\Omega_0)$. Since the bound on the partial sum is

$$\|S_N(t)\|_{L^2(\Omega_0)}^2 \leqslant C_j^2 \|u_0\|_{L^2(\Omega_0)}^2, \ j = 1, 2, \tag{2.13}$$

the L^2 limit exists. For case 3, the cosine factor in the exponential of (2.11) is positive since $0 \leqslant \frac{\beta}{\alpha} < 1$. Since the eigenvalues λ_n are increasing functions of n, the initial state u_0 has to be chosen so that

$$\sum_{n} |u_{0,n}|^2 (1+t^{\alpha}\lambda_n)^{2\left(\frac{1-\beta}{\alpha}\right)} e^{2t\lambda_n^{\frac{1}{\alpha}} \cos\theta(\alpha,\beta)},$$

is finite. A sufficient condition for this is that $\{n \in \mathbb{N} \mid u_{0,n} \neq 0\}$ is finite, that is, we require $u_0 \in C_F(\Omega_0)$, the set of initial conditions $u_0 \in L^2(\Omega_0)$ with finitely-many nonzero Fourier coefficients $u_{0,n}$. Let n^* denote the largest index for which $u_{0,n} \neq 0$. In this case, we obtain

$$\|S_N(t)\|_{L^2(\Omega_0)}^2 \leqslant C_3^2 (1 + t^\alpha \lambda_{n^*})^{2\left(\frac{1-\beta}{\alpha}\right)} e^{2t\lambda_{n^*}^{\frac{1}{\alpha}} \cos\theta(\alpha,\beta)} \|u_0\|_{L^2(\Omega_0)}^2. \qquad (2.14)$$

The L^2 limit exists but the norm increases exponentially in time. It follows from this, that for cases 1 and 2, with $u_0 \in L^2(\Omega_0)$, or case 3, with $u_0 \in C_F(\Omega_0)$, we have $u(t)$ in (2.6) exists for all $t > 0$. In addition, the analyticity properties of the Mittag-Leffler functions ensure that $u \in \mathcal{C}(\overline{\mathbb{R}^+}, L^2(\Omega_0))$.
3. We next show that $u(t) \in D(H)$ provided u_0 satisfies the conditions in Theorem 2.1. For case 1, if $t > 0$, the first bound of Lemma A.1 shows that

$$\begin{aligned}
\|Hu(\cdot,t)\|_{L^2(\Omega_0)}^2 &= \sum_{n=1}^\infty \lambda_n^2 |u_{0,n}|^2 |E_{\alpha,1}(-i\lambda_n t^\alpha)|^2 \\
&\leqslant C^2 \sum_{n=1}^\infty |u_{0,n}|^2 \frac{\lambda_n^2}{(1 + \lambda_n t^\alpha)^2} \leqslant C^2 t^{-2\alpha} \|u_0\|_{L^2(\Omega_0)}^2. (2.15)
\end{aligned}$$

For case 2, we have $u_0 \in D(H)$, so that for any $t > 0$,

$$\begin{aligned}
\|Hu(\cdot,t)\|_{L^2(\Omega_0)}^2 &= \sum_{n=1}^\infty \lambda_n^2 |u_{0,n}|^2 |E_{\alpha,1}(-i\lambda_n t^\alpha)|^2 \\
&\leqslant C_2^2 \|u_0\|_{D(H)}^2. \qquad (2.16)
\end{aligned}$$

For case 3, we have $u_0 \in C_F(\Omega_0)$, and since $C_F(\Omega_0) \subset D(H)$, we have for any $t > 0$,

$$\|Hu(\cdot,t)\|_{L^2(\Omega_0)}^2 \leqslant C_3^2 (1 + t^\alpha \lambda_{n^*})^{2\left(\frac{1-\beta}{\alpha}\right)} e^{2t\lambda_{n^*}^{\frac{1}{\alpha}} \cos\theta(\alpha,\beta)} \|u_0\|_{D(H)}^2. \qquad (2.17)$$

As a consequence we have $Hu \in \mathcal{C}(\mathbb{R}^+, L^2(\Omega_0))$, and hence $u \in \mathcal{C}(\mathbb{R}^+, D(H))$.
4. We next prove that $u(t)$ solves the TFSE. For all $\lambda > 0$, we have

$$\partial_t^\alpha E_{\alpha,1}((-i)^\beta \lambda t^\alpha) = (-i)^\beta \lambda E_{\alpha,1}((-i)^\beta \lambda t^\alpha), \ t > 0.$$

Therefore, in light of (2.15), we obtain for all $t \in \mathbb{R}^+$, that

$$\begin{aligned}
\partial_t^\alpha u(\cdot,t) &= (-i)^\beta \sum_{n=1}^\infty \lambda_n u_{0,n} E_{\alpha,1}((-i)^\beta \lambda_n t^\alpha) \phi_n \\
&= (-i)^\beta Hu(\cdot,t).
\end{aligned}$$

Therefore, for $t \in \mathbb{R}^+$, the function $u(x,y,t)$ is a solution to (2.2) in $L^2(\Omega_0) \cap D(H)$ and $\partial_t^\alpha u \in \mathcal{C}(\mathbb{R}^+, L^2(\Omega_0))$.

5. We establish the property that $u(t)$ converges to u_0 as $t \to 0$. For all $t \in \mathbb{R}^+$, we have

$$
\begin{aligned}
\|u(\cdot, t) - u_0\|_{L^2(\Omega_0)}^2 &= \sum_{n=1}^{\infty} |u_{0,n}|^2 |E_{\alpha,1}(-i\lambda_n t^\alpha) - 1|^2 \\
&\leqslant \sum_{n=1}^{\infty} |u_{0,n}|^2 \left(|E_{\alpha,1}(-i\lambda_n t^\alpha)| + 1 \right)^2 \\
&\leqslant C(t) \|u_0\|_{L^2(\Omega_0)}^2,
\end{aligned}
\tag{2.18}
$$

where $C(t) > 0$ is a constant in cases 1 and 2, and given by the expression on the right side of (2.17) in case 3. Since $\lim_{t\downarrow 0} (E_{\alpha,1}(-i\lambda_n t^\alpha) - 1) = 0$, for all $n \in \mathbb{N}$, inequality (2.18) and the dominated convergence theorem yield that $\lim_{t\downarrow 0} \|u(\cdot, t) - u_0\|_{L^2(\Omega_0)} = 0$. As a consequence, u given by (2.6) is a solution to (2.2)–(2.4).

Part 2: Uniqueness.

6. In order to show that this weak solution is unique, we assume that $u_0 = 0$ and prove that (2.2)–(2.4) admits only the trivial solution. To do so, we take the scalar product in $L^2(\Omega_0)$ of both sides of (2.2), with ϕ_n, for some $n \in \mathbb{N}$. Putting $u_n(t) := \langle u(\cdot, t), \phi_n \rangle_{L^2(\Omega_0)}$, we get that

$$
-i^\beta \partial_t^\alpha u_n(t) + \lambda_n u_n(t) = 0.
$$

Moreover, we have $\lim_{t\downarrow 0} u_n(t) = \langle \lim_{t\downarrow 0} u(\cdot, t), \phi_n \rangle_{L^2(\Omega_0)} = 0$ from the expansion (2.6). Thus, $u_n(t) = 0$ for a.e. $t \in \mathbb{R}^+$. This shows that $u(\cdot, t) = \sum_{n \geqslant 1} u_n(t)\phi_n = 0$ in $L^2(\Omega_0)$ for all $t \in \mathbb{R}^+$. $\qquad\square$

3. Calculation of the edge current

We consider the initial-value problem for the general TFSE with exponents $(\alpha, \beta) \in (0, 1]$:

$$
(-i^\beta \partial_t^\alpha + H(\mathbf{a}))u(x, y, t) = 0,
\tag{3.1}
$$

with initial condition $u(x, y, t = 0) = u_0(x, y)$. The solution is given in terms of the Mittag-Leffler functions:

$$
u(\cdot, t) = (E_{\alpha,1}((-i)^\beta t^\alpha H(\mathbf{a}))u_0)(\cdot),
\tag{3.2}
$$

where $(\cdot)$ stands for the spatial coordinates $(x, y) \in \Omega := \mathbb{R}_x^+ \times \mathbb{R}_y$. To compute the edge current $J_y[u_0]$, we need

$$
\partial_t u(\cdot, t) = (-i)^\beta t^{\alpha-1}(E_{\alpha,\alpha}((-i)^\beta t^\alpha H(\mathbf{a}))H(\mathbf{a})u_0)(\cdot),
\tag{3.3}
$$

where we used the identity (A.2) for the derivative of the Mittag-Leffler function. This provides a compact expression for the edge current:

$$
\begin{aligned}
J_y[u_0](t) &:= \partial_t \langle yu(\cdot, t), u(\cdot, t) \rangle_{L^2(\Omega)} \\
&= 2t^{\alpha-1} \operatorname{Re}\{(-i)^\beta \langle yE_{\alpha,\alpha}((-i)^\beta t^\alpha H(\mathbf{a}))H(\mathbf{a})u_0, E_{\alpha,1}((-i)^\beta t^\alpha H(\mathbf{a}))u_0 \rangle_{L^2(\Omega)}\}.
\end{aligned}
\tag{3.4}
$$

3.1. **Choice of the initial state** u_0. We select an initial state that is a truncation in the k-variable of the first eigenfunction $\phi_1(x, k)$ of the fiber operator $h_b(k)$ with eigenvalue $\lambda_1(k)$. We choose a compactly supported function $\chi(k)$ with support in a closed interval $I \subset \mathbb{R}$ so that $\lambda_1(I) \subset (b, 3b)$. We select $u_0(x, y)$ by taking

$$\widehat{u_0}(x, k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iky} u_0(x, y) \, dy := \chi(k)\phi_1(x, k). \tag{3.5}$$

We recall that we choose $\phi_1(x, k)$ to be real.

3.2. **Formula for the edge current.** To facilitate the calculation of $J_y[u_0]$, we define two functions

$$u_\alpha := E_{\alpha,1}((-i)^\beta t^\alpha H(\mathbf{a}))u_0 \tag{3.6}$$

$$v_\alpha := E_{\alpha,\alpha}((-i)^\beta t^\alpha H(\mathbf{a}))u_0. \tag{3.7}$$

Using these two functions, we may express the current in (3.4) as

$$J_y[u_0](t) = 2t^{\alpha-1} \operatorname{Re}\{(-i)^\beta \langle H(\mathbf{a})v_\alpha, yu_\alpha \rangle\}. \tag{3.8}$$

Let us reduce the matrix element

$$K_{u_0}(t) := \langle H(\mathbf{a})v_\alpha, yu_\alpha \rangle. \tag{3.9}$$

Using the choice of u_0 in (3.5), we obtain for the partial Fourier transforms with respect to y:

$$\widehat{u_\alpha}(x, k, t) = E_{\alpha,1}((-i)^\beta t^\alpha \lambda_1(k))\chi(k)\phi_1(x, k), \tag{3.10}$$

and

$$\widehat{v_\alpha}(x, k, t) = E_{\alpha,\alpha}((-i)^\beta t^\alpha \lambda_1(k))\chi(k)\phi_1(x, k). \tag{3.11}$$

Substituting these into the expression (3.9) for $K_{u_0}(t)$ yields

$$K_{u_0}(t) = \int_{\mathbb{R}} \langle h_b(k)\widehat{v_\alpha}(\cdot, k, t), \widehat{yu_\alpha}(\cdot, k, t) \rangle_{L^2(\mathbb{R}^+)} \, dk. \tag{3.12}$$

We note that

$$\widehat{yu_\alpha}(x, k, t) = i\partial_k(E_{\alpha,1}((-i)^\beta t^\alpha \lambda_1(k))\chi(k)\phi_1(x, k)). \tag{3.13}$$

We obtain from (3.12)-(3.13):

$$\begin{aligned} K_{u_0}(t) &= \int_{\mathbb{R}} \lambda_1(k)\chi(k)E_{\alpha,\alpha}((-i)^\beta t^\alpha \lambda_1(k))\langle \phi_1(\cdot, k), \\ &\quad i\partial_k(\chi(k)\phi_1(\cdot, k)E_{\alpha,1}((-i)^\beta t^\alpha \lambda_1(k)))\rangle_{L^2(\mathbb{R}^+)} \, dk. \end{aligned} \tag{3.14}$$

The inner product in (3.14) may be expanded to obtain

$$\begin{aligned} &\langle \phi_1(\cdot, k), i\partial_k(E_{\alpha,1}((-i)^\beta t^\alpha \lambda_1(k))\chi(k)\phi_1(\cdot, k))\rangle_{L^2(\mathbb{R}^+)} \\ &= -i\partial_k(\overline{E_{\alpha,1}((-i)^\beta t^\alpha \lambda_1(k))}\chi(k)), \end{aligned} \tag{3.15}$$

since $\langle \phi_1(\cdot, k), \phi_1(\cdot, k))\rangle_{L^2(\mathbb{R}^+)} = 1$, and $\langle \phi_1(\cdot, k), \partial_k \phi_1(\cdot, k))\rangle_{L^2(\mathbb{R}^+)} = 0$, since $\phi_1(x, k)$ may be chosen to be real. Consequently, the expression for $K_{u_0}(t)$ is

$$K_{u_0}(t) = -i \int_{\mathbb{R}} \lambda_1(k)E_{\alpha,\alpha}((-i)^\beta t^\alpha \lambda_1(k))\chi(k)\partial_k(\overline{E_{\alpha,1}((-i)^\beta t^\alpha \lambda_1(k))}\chi(k)) \, dk. \tag{3.16}$$

We write $E_{\alpha,j}(k,t) := E_{\alpha,j}((-i)^\beta t^\alpha \lambda_1(k))$, for $j = 1, \alpha$, for short. We decompose $K_{u_0}(t) := -i[K_{u_0}^{(1)}(t) + K_{u_0}^{(2)}(t)]$, where

$$K_{u_0}^{(1)}(t) := \int_{\mathbb{R}} \lambda_1(k)\chi(k)\chi'(k)E_{\alpha,\alpha}(k,t)\overline{E_{\alpha,1}(k,t)} \, dk, \qquad (3.17)$$

and

$$K_{u_0}^{(2)}(t) := \int_{\mathbb{R}} \lambda_1(k)\chi^2(k)E_{\alpha,\alpha}(k,t)\partial_k\overline{E_{\alpha,1}(k,t)} \, dk. \qquad (3.18)$$

We may reduce the form of $K_{u_0}^{(2)}(t)$ using the fact that

$$\partial_k E_{\alpha,1}(k,t) = \frac{1}{\alpha}(-i)^\beta t^\alpha \lambda_1'(k)E_{\alpha,\alpha}(k,t). \qquad (3.19)$$

From this, we may write

$$K_{u_0}^{(2)}(t) := \frac{i^\beta t^\alpha}{\alpha} \int_{\mathbb{R}} \lambda_1(k)\lambda_1'(k)\chi^2(k)|E_{\alpha,\alpha}(k,t)|^2 \, dk. \qquad (3.20)$$

The final formula for the current is:

$$\begin{aligned}
J_y[u_0](t) &= 2t^{\alpha-1}\operatorname{Re}\left\{(-i)^{1+\beta}\int_{\mathbb{R}}\left[\lambda_1(k)\chi(k)\chi'(k)E_{\alpha,\alpha}(k,t)\overline{E_{\alpha,1}(k,t)}\right.\right. \\
&\qquad \left.\left. +i^\beta\frac{t^\alpha}{\alpha}\lambda_1(k)\lambda_1'(k)\chi^2(k)|E_{\alpha,\alpha}(k,t)|^2\right] \, dk\right\} \\
&= 2t^{\alpha-1}\int_{\mathbb{R}}\lambda_1(k)\chi(k)\chi'(k)\operatorname{Re}\left\{(-i)^{1+\beta}E_{\alpha,\alpha}(k,t)\overline{E_{\alpha,1}(k,t)}\right\} \, dk.
\end{aligned}$$
$$(3.21)$$

Remark 3.1. *The formula (3.21) reduces to the classical result when $\alpha = \beta = 1$:*

$$J_y[u_0](t) = -2\left\{\int_{\mathbb{R}}\lambda_1(k)\chi(k)\chi'(k)|E_{1,1}(k,t)|^2 \, dk\right\}. \qquad (3.22)$$

We recall that $E_{1,1}(z) = e^z$ so that $E_{1,1}(k,t) = e^{-i\lambda_1(k)t}$, and hence

$$\begin{aligned}
J_y[u_0](t) &= -2\left\{\int_{\mathbb{R}}\lambda_1(k)\chi(k)\chi'(k) \, dk\right\} \\
&= \int_{\mathbb{R}}\lambda_1'(k)\chi^2(k) \, dk.
\end{aligned} \qquad (3.23)$$

This result agrees with the calculation of the edge current for the Schrödinger equation using the unitary group since $u(t) = e^{-iH(\mathbf{a})t}u_0$ so that one may use (1.11), with initial condition (3.5). Explicit positive lower bounds on $\lambda_1'(k)$ are given in [7, Theorem 2.1].

Remark 3.2. *It is interesting to compute the edge current for $\alpha = 1$ and $\beta \in (0,1)$. In this case, the solution to the initial-value problem is similar to the Schrödinger case: $u(t) = e^{(-i)^\beta H(\mathbf{a})t}u_0$. Using (3.21), with the fact that $E_{1,1}(k,t) = e^{(-i)^\beta \lambda_1(k)t}$, we find that the edge current has the form:*

$$J_y[u_0](t) = 2\cos\left(\frac{\pi}{2}(1+\beta)\right)\int_{\mathbb{R}}\lambda_1(k)\chi'(k)\chi(k)e^{2t\lambda_1(k)\cos\left(\frac{\pi\beta}{2}\right)} \, dk. \qquad (3.24)$$

We note that when $\beta = 1$, we obtain (3.23).

4. Asymptotic behavior of the edge current

In this section, we use formula (3.21) of section 3 in order to calculate the asymptotic behavior in time of the edge currents. We first summarize the asymptotic expansions of the Mittag-Leffler functions.

4.1. Asymptotic expansions of Mittag-Leffler functions.
In order to analyze the long-time behavior of $J_y[u_0](t)$, we need the large t behavior of the functions $E_{\alpha,\alpha}(k,t)$ and $E_{\alpha,1}(k,t)$. We use formulas (A.4)-(A.5) of Appendix A.3 to write the expansions of $E_{\alpha,\sigma}(k,t)$ to fourth order in κ. For this, we recall the notation:

$$E_{\alpha,\sigma}(k,t) := E_{\alpha,\sigma}((-i)^\beta t^\alpha \lambda_1(k)),$$

for $\sigma = 1, \alpha$. The choice of the expansion depends on the relationship between α and β. To apply the asymptotics of the Mittag-Leffler functions, we note that $z = (-i)^\beta \kappa$, with $\kappa := t^\alpha \lambda_1(k) > 0$, so that $|\arg z| = \frac{\pi\beta}{2} < \pi$, for $\beta \in [0,1]$. The Mittag-Leffler functions have different asymptotic behaviors in z depending on the sector in which $|z| \to \infty$. These sectors are determined by a real parameter μ satisfying (A.3). We have two cases:

Case 1: $\beta \leqslant \alpha$. $|\arg z| < \mu$ for all μ satisfying (A.3). The asymptotics (A.4) hold.

Case 2: $\alpha < \beta$. All $\mu \in \left(\frac{\pi\alpha}{2}, \min(\pi\alpha, \frac{\pi\beta}{2})\right)$ satisfy (A.3), and we have $\mu < |\arg z| < \pi$. The asymptotics (A.5) hold.

From this analysis, the fact that $z = (-i)^\beta \kappa$, with $\kappa := t^\alpha \lambda_1(k) > 0$, and the asymptotics (A.4)-(A.5), we obtain:

Case 1. $\beta \leqslant \alpha$.

$$E_{\alpha,1}(z) = \frac{1}{\alpha} e^{(-i)^{\frac{\beta}{\alpha}} \kappa^{\frac{1}{\alpha}}} - \frac{i^\beta}{\kappa\Gamma(1-\alpha)} - \frac{i^{2\beta}}{\kappa^2\Gamma(1-2\alpha)} - \frac{i^{3\beta}}{\kappa^3\Gamma(1-3\alpha)} + \mathcal{O}(\kappa^{-4}) \tag{4.1}$$

$$E_{\alpha,\alpha}(z) = \frac{1}{\alpha}(-i)^{\frac{\beta(1-\alpha)}{\alpha}} \kappa^{\frac{1-\alpha}{\alpha}} e^{(-i)^{\frac{\beta}{\alpha}} \kappa^{\frac{1}{\alpha}}} - \frac{i^{2\beta}}{\kappa^2\Gamma(-\alpha)} - \frac{i^{3\beta}}{\kappa^3\Gamma(-2\alpha)} + \mathcal{O}(\kappa^{-4}) \tag{4.2}$$

Case 2. $\beta > \alpha$.

$$E_{\alpha,1}(z) = -\frac{i^\beta}{\kappa\Gamma(1-\alpha)} - \frac{i^{2\beta}}{\kappa^2\Gamma(1-2\alpha)} - \frac{i^{3\beta}}{\kappa^3\Gamma(1-3\alpha)} + \mathcal{O}(\kappa^{-4}) \tag{4.3}$$

$$E_{\alpha,\alpha}(z) = -\frac{i^{2\beta}}{\kappa^2\Gamma(-\alpha)} - \frac{i^{3\beta}}{\kappa^3\Gamma(-2\alpha)} + \mathcal{O}(\kappa^{-4}) \tag{4.4}$$

We note that $\Gamma(-n)^{-1} = 0$, for $n \in \mathbb{N} \cup \{0\}$.

We recall that the integrand of the edge current contains the factor:

$$\text{Re}\left\{(-i)^{1+\beta} E_{\alpha,\alpha}(k,t)\overline{E_{\alpha,1}(k,t)}\right\}, \tag{4.5}$$

whose asymptotic expansion we now compute for each case. We note that as long as $\alpha, \sigma \geqslant 0$, the Mittag-Leffler function satisfies

$$\overline{E_{\alpha,\sigma}(z)} = E_{\alpha,\sigma}(\bar{z}).$$

4.2. Asymptotic expansion for the current in Case 1: $0 < \beta \leqslant \alpha < 1$.
We first compute the leading term of (4.5) using the asymptotics (4.1)-(4.2). The result is

$$\frac{1}{\alpha^2}\cos\left[\left(\frac{\pi\beta}{2\alpha}\right)(1-\alpha) + \frac{\pi}{2}(1+\beta)\right]\lambda_1(k)^{\frac{1-\alpha}{\alpha}}t^{1-\alpha}e^{2t\lambda_1(k)^{\frac{1}{\alpha}}\cos\left(\frac{\pi\beta}{2\alpha}\right)}. \tag{4.6}$$

This term is positive since $0 < \frac{\beta}{\alpha} \leqslant 1$. The leading term grows exponentially in time. The other terms in the expansion are of lower order. To simplify the notation, we introduce two angles:

$$\theta := \frac{\pi\beta}{2\alpha}, \ \gamma(k) := \lambda_1(k)^{\frac{1}{\alpha}}\sin\theta. \tag{4.7}$$

We express the next lower-order term in (4.5) as follows:

$$-t^{1-2\alpha}e^{t\lambda_1(k)^{\frac{1}{\alpha}}\cos\theta}\left[\frac{1}{\alpha\Gamma(1-\alpha)}\cos\left(t\gamma(k) + \theta + \frac{\pi}{2}(1+\beta)\right)\lambda_1(k)^{\frac{1-2\alpha}{\alpha}}\right]. \tag{4.8}$$

The remainder is seen to be $\mathcal{O}(t^{-2\alpha})$.

As a consequence of (4.6)-(4.8), the edge current (3.21) may be expressed as

$$\begin{aligned}
J_y[u_0](t) &= 2t^{\alpha-1}\,\text{Re}\left\{(-i)^{1+\beta}\int_{\mathbb{R}}\left[\lambda_1(k)\chi(k)\chi'(k)E_{\alpha,\alpha}(k,t)\overline{E_{\alpha,1}(k,t)}\right]\,dk\right\}\\
&= \frac{2}{\alpha^2}\cos\left(\theta(1-\alpha) + \frac{\pi}{2}(1+\beta)\right)\int_{\mathbb{R}}\lambda_1^{\frac{1}{\alpha}}(k)\chi(k)\chi'(k)e^{2t\lambda_1^{\frac{1}{\alpha}}(k)\cos\theta}\,dk\\
&\quad -\frac{2t^{-\alpha}}{\alpha\Gamma(1-\alpha)}\int_{\mathbb{R}}\cos\left(t\gamma(k) + \theta + \frac{\pi}{2}(1+\beta)\right)\lambda_1^{\frac{1-\alpha}{\alpha}}(k)\chi(k)\chi'(k)e^{t\lambda_1^{\frac{1}{\alpha}}(k)\cos\theta}\,dk\\
&\quad +\mathcal{O}(t^{-1-\alpha}).
\end{aligned} \tag{4.9}$$

The main characteristic of the edge current in Case 1 with $0 < \beta < \alpha < 1$ is the exponential growth of the current with time. However, for the case $\alpha = \beta$, the factor in the exponential $\cos\theta = 0$ and the edge current is asymptotically constant in time, see section 5.1.

4.3. Asymptotic expansion for the current in Case 2: $0 < \alpha < \beta \leqslant 1$.
The edge current in Case 2 decays with time. Using the asymptotics for Case2, (4.3)-(4.4), we find:

$$\begin{aligned}
J_y[u_0](t) &= \frac{2}{t^{1+3\alpha}}\left(\int_{\mathbb{R}}\frac{1}{\lambda_1^3(k)}\chi(k)\chi'(k)\,dk\right)\\
&\quad \times\cos\left(\frac{(\beta+1)\pi}{2}\right)\left[\frac{1}{\Gamma(1-2\alpha)\Gamma(-\alpha)} - \frac{1}{\Gamma(1-\alpha)\Gamma(-2\alpha)}\right]\\
&\quad +\mathcal{O}(t^{-1-4\alpha}).
\end{aligned} \tag{4.10}$$

In the regime of Case 2, $0 < \alpha < \beta \leqslant 1$, the edge current decays like $t^{-1-3\alpha}$.

5. Asymptotics of the edge current in special cases

In this section, we discuss the results of section 4 for two well known models.

5.1. **Asymptotics of the current for the Naber model.** The original TFSE model proposed by Naber [16] consists of taking $0 < \alpha = \beta < 1$. This model is an example of Case 1. We note that for $\alpha = \beta > 0$, the term in the exponent $(-i)^{\frac{\beta}{\alpha}} = -i$ and the exponential factors in (4.1) and (4.2) have modulus one. Consequently, the edge current (4.9) in this case has the expansion:

$$
\begin{aligned}
J_y[u_0](t) &= 2t^{\alpha-1}\,\mathrm{Re}\left\{(-i)^{1+\alpha}\int_{\mathbb{R}}\left[\lambda_1(k)\chi(k)\chi'(k)E_{\alpha,\alpha}(k,t)\overline{E_{\alpha,1}(k,t)}\right]\,dk\right\}\\
&= \frac{1}{\alpha^2}\int_{\mathbb{R}}(\lambda_1(k)^{\frac{1}{\alpha}})'\chi^2(k)\,dk\\
&\quad +\frac{2t^{-\alpha}}{\alpha\Gamma(1-\alpha)}\int_{\mathbb{R}}\lambda_1^{\frac{1-\alpha}{\alpha}}(k)\chi(k)\chi'(k)\cos\left(\frac{\pi\alpha}{2}+t\lambda_1^{\frac{1}{\alpha}}(k)\right)\,dk\\
&\quad +\mathcal{O}(t^{-1-\alpha}).
\end{aligned}
\tag{5.1}
$$

Thus, the edge current in the Naber model is a asymptotically constant in time. Taking $\alpha = \beta = 1$, the leading term of (5.1) is the same as (3.23) for the Schrödinger model, and the next term in the expansion vanishes.

5.2. **Asympotics of the current for the AYH model.** The TFSE model proposed by Narahari Achar, et. al., [17] consists of taking $0 < \alpha < 1$ and $\beta = 1$. This is an example of Case 2. Evaluating the asymptotic formula in (4.10), we obtain

$$
\begin{aligned}
J_y[u_0](t) &= -\frac{2}{t^{1+3\alpha}}\left(\int_{\mathbb{R}}\frac{1}{\lambda_1^3(k)}\chi(k)\chi'(k)\,dk\right)\left[\frac{1}{\Gamma(1-2\alpha)\Gamma(-\alpha)}-\frac{1}{\Gamma(1-\alpha)\Gamma(-2\alpha)}\right]\\
&\quad +\mathcal{O}(t^{-1-4\alpha}).
\end{aligned}
\tag{5.2}
$$

The edge current decays to zero as $t \to \infty$. We note that, unlike the Naber model, when $\alpha \to 1^-$, the leading term in (5.2) vanishes.

6. Mean square displacement in the time-fractional Hall model

In a related paper [8], we studied the effect of the fractional time derivative on the free propagation properties of wave packets in $\mathbb{R}^d$, $d \geqslant 1$, through the evolution of mean square displacement. As another indicator of the anomalous transport induced by a fractional time derivative in the quantum Hall model, we compute the mean square displacement (MSD) of a wave packet in the y direction for the TFSE:

$$
i^{\beta}\partial_t^{\alpha}u = H(\mathbf{a})u,\ u(t=0) = u_0,
\tag{6.1}
$$

with an initial condition u_0 as defined in (3.5).

We recall that the solution to (6.1) $u(x, y, t)$, with $(x, y) \in \Omega := \mathbb{R}_x^+ \times \mathbb{R}_y$, is given by

$$
u(x, y, t) = E_{\alpha,1}((-i)^{\beta}t^{\alpha}H(\mathbf{a}))u_0(x, y),
\tag{6.2}
$$

so that after a partial Fourier transform with respect to the y variable, we have

$$\widehat{u}(x, k, t) = E_{\alpha,1}((-i)^\beta t^\alpha \lambda_1(k)) \chi(k) \phi_1(x, k). \tag{6.3}$$

We define the MSD for the solution $u(x, y, t)$ in (6.2) by

$$D_2(u_0, t) := \langle u(t), y^2 u(t) \rangle_{L^2(\Omega)}. \tag{6.4}$$

Similar to calculations in section 3, by means of (3.13), we easily find

$$D_2(u_0, t) = \int_{\mathbb{R}} |\partial_k \widehat{u}(x, k, t)|^2 \ dx \ dk, \tag{6.5}$$

where, using (6.3) and (A.2) for the derivative of the Mittag-Leffler function, we obtain,

$$\partial_k \widehat{u}(x, k, t) = I + II + III. \tag{6.6}$$

Using the notation $E_{\alpha,j}(k, t) := E_{\alpha,j}((-i)^\beta t^\alpha \lambda_1(k))$, for $j = 1, \alpha$, as above, we have for the three terms in (6.6):

$$\begin{aligned}
I &= E_{\alpha,\alpha}(k, t)(-i)^\beta t^\alpha \lambda_1'(k) \chi(k) \phi_1(x, k), \\
II &= E_{\alpha,1}(k, t) \chi'(k) \phi_1(x, k), \\
III &= E_{\alpha,1}(k, t) \chi(k) \partial_k \phi_1(x, k).
\end{aligned} \tag{6.7}$$

Using the properties of the eigenfunction $\phi_1(x, k)$, and its orthogonality to $\partial_k \phi_1(x, k)$, we find that the MSD is the sum of four terms

$$D_2(u_0, t) = A + B + C + F,$$

where the square terms are

$$\begin{aligned}
A &= \int_{\mathbb{R}} |E_{\alpha,\alpha}(k, t)|^2 t^{2\alpha} (\lambda_1'(k))^2 \chi^2(k) \ dk, \\
B &= \int_{\mathbb{R}} |E_{\alpha,1}(k, t)|^2 (\chi'(k))^2 \ dk, \\
C &= \int_{\mathbb{R}} |E_{\alpha,1}(k, t)|^2 \chi^2(k) \left(\int_{\mathbb{R}^+} |\partial_k \phi_1(x, k)|^2 dx \right) \ dk,
\end{aligned} \tag{6.8}$$

and the one nonvanishing cross term is

$$F = 2t^\alpha \int_{\mathbb{R}} \mathrm{Re} \left\{ (-i)^\beta E_{\alpha,\alpha}(k, t) \overline{E_{\alpha,1}(k, t)} \right\} \lambda_1'(k) \chi'(k) \chi(k) \ dk. \tag{6.9}$$

Case 1. For $0 < \beta < \alpha < 1$, the asymptotics (4.1)-(4.2) indicate that the MSD increases exponentially in time. However, for the case $0 < \alpha = \beta < 1$, the Naber model, the asymptotic behavior of the MSD is a result of the asymptotics given in (4.1)-(4.2), for $\alpha = \beta$:

$$D_2(u_0, t) = \frac{t^2}{\alpha^2} \int_{\mathbb{R}} \lambda_1(k)^{2\left(\frac{1-\alpha}{\alpha}\right)} \lambda_1'(k)^2 \chi_1^2(k) \ dk + \mathcal{O}(1). \tag{6.10}$$

Hence, the MSD $D_2(u_0, t)$ is ballistic in the y-direction for the model of Naber (1.2) (with $-\Delta$ replaced by $H(\mathbf{a})$, and $q = 0$).

Case 2: $0 < \alpha < \beta \leqslant 1$. The expansions (4.3)-(4.4) lead to the following expression for the asymptotics of the MSD:

$$
\begin{aligned}
D_2(u_0, t) \;=\; & \frac{1}{t^{2\alpha}} \left\{ \frac{1}{\Gamma(-\alpha)^2} \int_{\mathbb{R}} \frac{\lambda'(k)^2}{\lambda_1^4(k)} \chi^2(k) \, dk \right. \\
& \left. + \frac{1}{\Gamma(1-\alpha)^2} \int_{\mathbb{R}} \frac{(\chi'(k)^2 + \chi^2(k)\Phi(k))}{\lambda_1^2(k)} \chi^2(k) \, dk \right\} + \mathcal{O}(t^{-3\alpha}),
\end{aligned}
$$

(6.11)

where

$$
\Phi(k) := \int_{\mathbb{R}^+} |\partial_k \phi_1(x, k)|^2 dx.
$$

In this case, the MSD in the y-direction decays like $t^{-2\alpha}$.

In summary, the MSD in the y-direction behaves like t^2 for the Naber model and like $t^{-2\alpha}$ for the AYH model. For fixed α, the MSD, like the edge current, exhibits a transport transition between $0 < \beta < \alpha$ and $\alpha \leqslant \beta \leqslant 1$.

A. MITTAG LEFFLER FUNCTIONS

A.1. **Definitions.** The Mittag-Leffler functions are generalizations of the exponential function. They are indexed by two parameters (α, σ):

$$
E_{\alpha,\sigma}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \sigma)}.
$$

(A.1)

These are entire functions of $z \in \mathbb{C}$ for $\alpha, \sigma > 0$. Note that $E_{1,1}(z) = e^z$. The function $E_{\alpha,1}(-\lambda t^\alpha)$ satisfies the fractional differential equation:

$$
(\partial_t^\alpha + \lambda) E_{\alpha,1}(-\lambda t^\alpha) = 0.
$$

We need the derivative of the Mittag-Leffler function $E_{\alpha,1}(z)$ with respect to z.

$$
\frac{dE_{\alpha,1}}{dz}(z) = \sum_{m=0}^{\infty} \frac{(m+1)z^m}{\Gamma(\alpha(m+1)+1)} = \frac{1}{\alpha} \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \alpha)} = \frac{1}{\alpha} E_{\alpha,\alpha}(z), \quad \text{(A.2)}
$$

using the property that $\Gamma(z+1) = z\Gamma(z)$.

A.2. **Bounds on Mittag-Leffler functions.** The following technical result, needed by the proof of Theorem 2.1, is a byproduct of [18, (1.147), (1.148)]. The asymptotic expansions are presented in (A.4) and (A.5). We consider only the case $\alpha \in (0, 1]$.

Lemma A.1. [18, Theorems 1.5 and 1.6] *Let* $\sigma \in \mathbb{R}$, $\alpha < 2$, *and* $\mu \in \left(\frac{\pi\alpha}{2}, \min\{\pi\alpha, \pi\}\right)$.

(1) *There exists a finite constant* $C = C(\alpha, \sigma) > 0$ *so that*

$$
|E_{\alpha,\sigma}(z)| \leqslant C \left((1+|z|)^{\frac{1-\sigma}{\alpha}} e^{\mathrm{Re}(z^{\frac{1}{\alpha}})} + \frac{1}{1+|z|} \right),
$$

for all $|\arg z| \leqslant \mu$ *and* $|z| \geqslant 0$.

(2) *There exists a finite constant $C = C(\alpha, \sigma) > 0$ so that*

$$|E_{\alpha,\sigma}(z)| \leqslant \frac{C}{1 + |z|},$$

for all $\mu < |\arg z| \leqslant \pi$ and $|z| \geqslant 0$.

A.3. Asymptotic expansions of the Mittag-Leffler functions.

The asymptotics of the Mittag-Leffler functions are presented in section 1.2.7 of Podnubny [18], Theorems 1.3 and 1.4. The asymptotics as $|z| \to \infty$ depend on the location of $z \in \mathbb{C}$. For any $\alpha \in (0, 1)$ (actually, $\alpha \in (0, 2)$ is allowed), and $\sigma \in \mathbb{C}$, choose any $\mu > 0$ satisfying

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\} = \pi\alpha. \tag{A.3}$$

Theorem 1.3 of section 1.2.7 of [18] states that if $|\arg z| \leqslant \mu$, then, for any $p \geqslant 1$, we have:

$$E_{\alpha,\sigma}(z) = \frac{1}{\alpha} z^{\frac{1-\sigma}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\sigma - \alpha k)} + \mathcal{O}(|z|^{-p-1}). \tag{A.4}$$

Theorem 1.4 of section 1.2.7 of [18] states that if $\mu \leqslant |\arg z| \leqslant \pi$, then, for any $p \geqslant 1$, we have:

$$E_{\alpha,\sigma}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\sigma - \alpha k)} + \mathcal{O}(|z|^{-p-1}). \tag{A.5}$$

We note that $\Gamma(-n)^{-1} = 0$, for $n \in \mathbb{N} \cup \{0\}$.

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Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027, USA
  E-mail address: `peter.hislop@uky.edu`

Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France
  E-mail address: `eric.soccorsi@univ-amu.fr`