Carleman Estimate for the Schrödinger Equation and Application to Magnetic Inverse Problems

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1 Introduction

Let $T \in (0, +\infty)$ and let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N} := \{1, 2, \ldots\}$, with sufficiently smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem (IBVP) for the magnetic Schrödinger equation

$$\begin{cases} -i\partial_t u - \Delta_{A_0} u + \rho_0 u = 0 & \text{in } Q := \Omega \times (0,T) \\ u = g & \text{on } \Sigma := \partial\Omega \times (0,T) \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

with initial state u_0 and non-homogeneous Dirichlet boundary condition g. Here, $\rho_0 : \Omega \to \mathbb{C}$ is a complex-valued electric potential and

$$\Delta_{A_0} := (\nabla + iA_0)\cdot(\nabla + iA_0) = \Delta + 2iA_0\cdot\nabla + i(\nabla\cdot A_0) - |A_0|^2 \tag{2}$$

denotes the magnetic Laplace operator associated with the magnetic vector potential $A_0 : \Omega \to \mathbb{R}^n$. In the particular case where $n = 3$, the magnetic field induced by the magnetic potential vector A_0 reads $\operatorname{curl} A_0 := \nabla \times A_0$.

1.1 What we are aiming for

In the present paper we examine the stability issue in the inverse problem of determining the electromagnetic potential (A_0, ρ_0) from finitely many partial Neumann boundary measurements over the entire time-span of the solution to (1), obtained by $n + 1$ times suitably changing the initial state u_0. More precisely, we shall actually establish the following three stability results:

i) *Case 1*: Assuming that A_0 is known, we stably determine the complex-valued electric potential ρ_0 from a single partial boundary measurement over the entire time-span of the normal derivative of the solution u to (1), measured on a sub-boundary $\Gamma_0 \subset \partial\Omega$. The result is valid for any two electrostatic potentials with difference ρ, whose imaginary part of the logarithmic gradient $\nabla \ln\left(\rho^{-1}\overline{\rho}\right)$ is uniformly bounded in Ω, see condition (14) below;

ii) *Case 2*: We prove simultaneous stable reconstruction of the magnetic vector potential A_0 (together with its divergence $\nabla \cdot A_0$) and the complex-valued electric potential ρ_0, through $n + 1$ partial Neumann observations of the solution, obtained by changing $n + 1$ times the initial condition u_0 suitably. This is proved provided that the logarithmic gradient of the difference of the electromagnetic potentials is uniformly bounded in Ω, see assumptions (15), (16) and (17);

iii) *Case 3*: Assuming that ρ_0 and the strength $|A_0|$ of the magnetic potential vector are known, we stably retrieve the direction of A_0 (together with the divergence), from $n+1$ partial Neumann data. In contrast with the two above results, there is no additional condition of the type of (14) or (15)-(17), imposed on the magnetic vector potential for this result to hold.

Our first claim (see Theorem 1.2 below) extends the stability results of [2] to the case of complex-valued electrostatic potentials. We refer to [24, Part 2, Section 14, Appendix B] for the physical relevance of complex-valued electric potentials appearing in the Schrödinger equation. Moreover, we point out that the second and third claims (see Theorems 1.3 and 1.4) are, to the best of our knowledge, the only stability results available in the mathematical literature for stationary magnetic potential vectors of the Schrödinger equation by finitely many local Neumann data.

1.2 Carleman estimate and time-symmetrization of the Schrödinger equation

In this article we aim for stable determination of the electromagnetic potential (A_0, ρ_0) in (1) through finitely many partial Neumann observations, by means of a Carleman estimate. We refer to [1, 2, 32, 36] for actual examples of Carleman inequalities for the Schrödinger equation. The idea of using Carleman estimates for solving inverse coefficient problems was first introduced by Bukhgeim and Klibanov in [9]. Since then, this technique has then been successfully applied by numerous authors to various types (parabolic, hyperbolic, elasticity, Maxwell, etc.) of inverse coefficient problems in bounded domains, see e.g. [4, 18, 21, 22, 35] and the references therein (recently, this method was adapted to the reconstruction of non compactly supported unknown coefficients in [5, 7, 19, 20]).

More specifically, in the framework of the Schrödinger equation, the authors of [2, 11, 36] use a Carleman inequality on the extended domain $\Omega \times (-T, T)$ in order to avoid observation data at $t = 0$ over Ω, appearing in Carleman estimates on Q. This imposes that the solution u to (1), extended to $\Omega \times (-T, T)$ by setting either $u(x,t) = \overline{u(x,-t)}$ or $u(x,t) = -\overline{u(x,-t)}$ for a.e. $(x,t) \in \Omega \times (-T, 0)$, depending on whether the initial state u_0 is taken real-valued or purely imaginary, be a solution to the Schrödinger equation in $\Omega \times (-T, T)$. It follows readily from the above time-symmetrization $u(\cdot, t) = \pm \overline{u(\cdot, -t)}$ that

$$(-i\partial_t - \Delta_{-A_0} + \overline{\rho_0})u(\cdot, t) = \pm(\overline{(-i\partial_t - \Delta_{A_0} + \rho_0)u})(\cdot, -t),\ t \in (-T, 0), \tag{3}$$

and hence that u is a solution to the Schrödinger equation in $\Omega \times (-T, T)$ if and only if $(-A_0, \overline{\rho_0}) = (A_0, \rho_0)$, i.e. $A_0 = 0$ and $\rho_0 \in \mathbb{R}$ (this is precisely the situation examined in [2], where Lipschitz stable reconstruction of the real-valued electrostatic potential ρ_0 is derived in absence of a magnetic potential), in which case the right hand side of (3) is zero.

As a conclusion, the time-symmetrization method implemented in [2] does not work in presence of a non-zero time-independent magnetic potential vector A_0 (notice that this is no longer true for odd time-dependent magnetic potentials: Indeed, when ρ_0 and A_0 depend on (x,t) then (3) reads $(-i\partial_t - \Delta_{-A_0(\cdot,t)} + \overline{\rho_0(\cdot,t)})u(\cdot,t) = \pm(\overline{(-i\partial_t - \Delta_{A_0(\cdot,-t)} + \rho_0(\cdot,-t))u})(\cdot,-t)$ for a.e. $t \in (-T, 0)$, so the extended solution u fulfills the magnetic Schrödinger equation in $\Omega \times (-T, T)$ if and only if we have $(-A_0(\cdot,t), \overline{\rho_0(\cdot,t)}) = (A_0(\cdot,-t), \rho_0(\cdot,-t))$, which corresponds to the framework of [11, 36]). Therefore, in contrast with [2, 11, 36], we cannot symmetrize the solution to (1) with respect to t in the framework of this paper. As a consequence we need a modified global Carleman estimate for the Schrödinger operator in Q, as compared to the ones of [2, 36] that are established in $\Omega \times (-T, T)$, in order to adapt the Bukhgeim-Klibanov (BK) method to the "stationary magnetic" Schrödinger equation investigated here.

1.3 State of the art: a short bibliography

There are numerous papers available in the mathematical literature, dealing with inverse coefficient problems from knowledge of the Dirichlet-to-Neumann (DN) map. However in the particular case of the stationary magnetic Schrödinger equation, the DN map Λ_{A_0} is invariant under gauge transformation of A_0, i.e. $\Lambda_{A_0 + \nabla\psi} = \Lambda_{A_0}$ for all $\psi \in C^1(\overline{\Omega})$ such that $\psi_{|\partial\Omega} = 0$, see e.g. [13]. Therefore the magnetic potential vector cannot be uniquely determined by the DN map and the best we can expect from knowledge of Λ_{A_0} is uniqueness modulo gauge transform of A_0. However it is well known that the magnetic field dA_0, i.e. the exterior derivative of A_0 defined as the 1-form $\sum_{j=1}^n (A_0)_j dx_j$ (if $n = 3$ then dA_0 is generated by the curl of A_0) is invariant under gauge transformation of A_0. As a matter of fact it was proved in [31] that the DN map uniquely determines the magnetic field provided the underlying magnetic vector potential is sufficiently small in a suitable class. The smallness assumption was removed in [28] for C^∞ magnetic vector potentials. Later on, this smoothness assumption was weakened to C^1 in [33] and to Dini continuous in [29]. In [14], the author proved that the electromagnetic potential of the Schrödinger equation in domains with several obstacles, is uniquely defined by the DN map. In [3] the magnetic field is stably retrieved by the dynamical DN map. The uniqueness and stability issues for time-dependent electromagnetic potentials of the Schrödinger equation are addressed in [15] and [17], respectively. All the above cited results were obtained with the full DN map, which is made of measurements of the solution taken on the whole boundary. The uniqueness problem by a local DN map was solved in [16] and it was shown in [34] that the magnetic field depends stably on the DN map measured on any sub-boundary that is slightly larger than half the boundary. This result was extended in [8] to arbitrary small sub-boundaries provided the magnetic potential is known in the vicinity of the boundary.

Notice that an infinite number of boundary observations of the solution to the magnetic Schrödinger equation were needed in all the above mentioned articles, in order to define the DN map. By contrast, the time independent and real-valued electric potential in a Schrödinger equation was stably retrieved from a single boundary measurement in [2, 26]. In these two papers, the observation zone fulfills a geometric condition related to geometric optics condition insuring observability. This geometric condition was relaxed in [27] upon assuming that the electrostatic potential is known near the boundary. In [11], the space varying part of the divergence free n dimensional magnetic potential was reconstructed by n partial Neumann data, by changing the initial state of the Schrödinger equation n times suitably. This result was extended in [6] to simultaneous recovery of the space varying part of the divergence free magnetic potential and the time-independent electric potential, by $n+1$ partial lateral observations of the solution. Recently, the BK method was adapted to a system of coupled Schrödinger equations in [12, 23].

1.4 Notations

Throughout this text $x := (x_1, \ldots, x_n)$ denotes a generic point of $\Omega \subset \mathbb{R}^n$ and we write $\partial_i := \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$. Next we set $\partial_{ij}^2 := \partial_i \partial_j$ for $i, j = 1, \ldots, n$, and as usual we use the notation ∂_i^2 instead of ∂_{ii}^2. For any multi-index $k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, we write $\partial_x^k := \partial_1^{k_1} \partial_2^{k_2} \ldots \partial_n^{k_n}$ and $|k| := \sum_{j=1}^n k_j$. Similarly, we put $\partial_t := \frac{\partial}{\partial t}$ and $\partial_\nu u = \frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$, where ν is the outward normal vector to the boundary $\partial\Omega$ and ∇ is the gradient operator with respect to the space variable x. The symbol $\cdot$ denotes the Euclidian scalar product in $\mathbb{R}^n$ and $\nabla \cdot$ stands for the divergence operator.

Let us now introduce the following functional spaces. For X, a manifold, we set

$$H^{r,s}(X \times (0,T)) := L^2(0,T; H^r(X)) \cap H^s(0,T; L^2(X)), \ r, s \in (0, +\infty),$$

where $H^r(X)$ stands for the usual Sobolev space of order r. For convenience, we sometimes use the notation $H^0(X) := L^2(X)$. When $X = \Omega$, we write $H^{r,s}(Q) = L^2(0,T; H^r(\Omega)) \cap H^s(0,T; L^2(\Omega))$ instead of $H^{r,s}(\Omega \times (0,T))$ and for $X = \partial\Omega$, we write $H^{r,s}(\Sigma) = L^2(0,T; H^r(\partial\Omega)) \cap H^s(0,T; L^2(\partial\Omega))$ instead of $H^{r,s}(\partial\Omega \times (0,T))$.

1.5 Existence and uniqueness results

Our first statement is a global existence and uniqueness result for the IBVP (1).

Proposition 1.1. *Assume that $\partial\Omega$ is C^2. For $M \in (0, +\infty)$, let $A_0 \in W^{2,\infty}(\Omega, \mathbb{R}^n)$ and $\rho_0 \in W^{1,\infty}(\Omega, \mathbb{C})$ satisfy*

$$\|A_0\|_{W^{2,\infty}(\Omega)} + \|\rho_0\|_{W^{1,\infty}(\Omega)} \le M. \tag{4}$$

Then, for all $g \in H^{\frac{7}{2},\frac{7}{4}}(\Sigma)$ *and all* $u_0 \in H^3(\Omega)$ *obeying*

$$g(\cdot,0) = u_0 \text{ on } \partial\Omega, \tag{5}$$

there exists a unique solution $u \in H^{2,1}(Q)$ *to the IBVP* (1)*. Moreover we have the estimate*

$$\|u\|_{H^{2,1}(Q)} \leq C\left(\|u_0\|_{H^3(\Omega)} + \|g\|_{H^{\frac{7}{2},\frac{7}{4}}(\Sigma)}\right), \tag{6}$$

where C *is a positive constant depending only on* T*,* Ω *and* M*.*

Under stronger regularity assumptions on Ω, A_0, ρ_0, u_0 and g than in Proposition 1.1, we have the following improved regularity result.

Theorem 1.1. *Fix* $m \in \mathbb{N}$ *and assume that* $\partial\Omega$ *is* $C^{2(m+1)}$*. For* $M \in (0,+\infty)$*, let* $A_0 \in W^{2(m+1),\infty}(\Omega,\mathbb{R}^n)$ *and* $\rho_0 \in W^{2m+1,\infty}(\Omega,\mathbb{C})$ *fulfill*

$$\|A_0\|_{W^{2(m+1),\infty}(\Omega)} + \|\rho_0\|_{W^{2m+1,\infty}(\Omega)} \leq M, \tag{7}$$

and pick $g \in H^{2\left(m+\frac{7}{4}\right),m+\frac{7}{4}}(\Sigma)$ *and* $u_0 \in H^{2m+3}(\Omega)$ *such that*

$$\partial_t^k g(\cdot,0) = \left(-i(-\Delta_{A_0} + \rho_0)\right)^k u_0 \text{ on } \partial\Omega \text{ for } k = 0,1,\ldots,m. \tag{8}$$

Then there exists a unique solution $u \in \bigcap_{k=0}^{m+1} H^{m+1-k}(0,T;H^{2k}(\Omega))$ *to* (1)*, satisfying*

$$\sum_{k=0}^{m+1} \|u\|_{H^{m+1-k}(0,T;H^{2k}(\Omega))} \leq C\left(\|u_0\|_{H^{2m+3}(\Omega)} + \|g\|_{H^{2\left(m+\frac{7}{4}\right),m+\frac{7}{4}}(\Sigma)}\right), \tag{9}$$

for some positive constant C *depending only on* T*,* Ω *and* M*.*

Remark 1.1. *Theorem 1.1 is quite similar to [25, Chapter 5, Theorem 12.1] but cannot be deduced from it (since* $\partial\Omega$ *is assumed to be smooth and, more importantly,* g *is identically zero in [25, Chapter 5, Theorem 12.1]). Moreover, as it will appear below, the regularity and the fixed boundary values imposed on the admissible unknown coefficients by the analysis of the inverse problem under consideration in this paper, are directly requested by the application of Theorem 1.1 and by the compatibility conditions* (8)*. Furthermore, we stress that the energy estimate* (9) *is the main tool for establishing Corollary 1.1 below asserting that the solution to* (1) *lies in* $W^{1,\infty}(0,T;L^\infty(\Omega))$*, which is essential for applying the BK method in Section 4. For all the above reasons and for convenience of the readers, we give a detailed proof of Theorem 1.1 in Section 2.*

Let N be the (unique) natural number fulfilling $\frac{n}{4}+1 < N \leq \frac{n}{4}+2$, i.e.

$$N \in \mathbb{N} \cap \left(\frac{n}{4}+1, \frac{n}{4}+2\right]. \tag{10}$$

Since the IBVP (1) admits a unique solution $u \in H^2(0,T;H^{2(N-1)}(\Omega))$, in virtue of Theorem 1.1 with $m = N$, and since $2(N-1) > \frac{n}{2}$ by (10), then the Sobolev imbedding theorem entails the following:

Corollary 1.1. *Under the conditions of Theorem 1.1 with* $m = N$*, the solution* u *to the IBVP* (1) *lies in* $W^{1,\infty}(0,T;L^\infty(\Omega))$ *and there exists a positive constant* C*, depending only on* T*,* Ω *and* M*, such that*

$$\|u\|_{W^{1,\infty}(0,T;L^\infty(\Omega))} \leq C\left(\|u_0\|_{H^{2N+3}(\Omega)} + \|g\|_{H^{2\left(N+\frac{7}{4}\right),N+\frac{7}{4}}(\Sigma)}\right). \tag{11}$$

1.6 Inverse problem: main results

Let $A_0 \in W^{2(N+1),\infty}(\Omega, \mathbb{R}^n)$ and $\rho_0 \in W^{2N+1,\infty}(\Omega, \mathbb{C})$ be fixed, where N is given by (10). For $M \in (0, +\infty)$ we define the set of admissible unknown magnetic vector potentials as

$$\mathcal{A}_M(A_0) := \{A \in W^{2(N+1),\infty}(\Omega, \mathbb{R}^n) : \ \|A\|_{W^{2(N+1),\infty}(\Omega)} \leq M \text{ and } \partial_x^k A = \partial_x^k A_0 \text{ on } \partial\Omega, \ |k| \leq 2N - 1\},$$

and the set of admissible unknown electric potentials as

$$\mathcal{Q}_M(\rho_0) := \{\rho \in W^{2N+1}(\Omega, \mathbb{C}) : \ \|\rho\|_{W^{2N+1,\infty}(\Omega)} \leq M \text{ and } \partial_x^k \rho = \partial_x^k \rho_0 \text{ on } \partial\Omega, \ |k| \leq 2(N-1)\}.$$

Remark 1.2. *The regularity and the fixed boundary values imposed by $\mathcal{A}_M(A_0)$ on the unknown magnetic potential vectors, and by $\mathcal{Q}_M(\rho_0)$ on the unknown electric potentials, are requested by the application of Theorem 1.1 in the mathematical analysis of the inverse problem under study in this article. This arises from the stability inequalities presented in Theorems 1.2, 1.3 and 1.4 below, where the size of the difference of two admissible unknown electromagnetic potentials (A_1, ρ_1) and (A_2, ρ_2), is estimated in terms of the boundary measurement of $u_1 - u_2$. Here u_j, for $j = 1, 2$, is the $W^{1,\infty}(0, T; L^\infty(\Omega))$-solution to the system (1) where (A_0, ρ_0) is replaced by (A_j, ρ_j). In view of Theorem 1.1, each u_j, $j = 1, 2$, is well-defined by Corollary 1.1, provided the Dirichlet data $g \in H^{2(N+\frac{7}{4}),N+\frac{7}{4}}(\Sigma)$, the initial state $u_0 \in H^{2N+3}(\Omega)$ and the electromagnetic coefficients $(A_j, \rho_j) \in W^{2(N+1),\infty}(\Omega, \mathbb{R}^n) \times W^{2N+1,\infty}(\Omega, \mathbb{C})$ fulfill the compatibility condition (8):*

$$\partial_t^k g(\cdot, 0) = (-i(-\Delta_{A_1} + \rho_1))^k u_0 = (-i(-\Delta_{A_2} + \rho_2))^k u_0 \ \text{on } \partial\Omega, \ k = 0, 1, \ldots, N. \tag{12}$$

One way to comply with the second equality of (12) is to impose fixed boundary values on (A_j, ρ_j), $j = 1, 2$, as specified in $\mathcal{A}_M(A_0) \times \mathcal{Q}_M(\rho_0)$. However, since (12) should be satisfied for any initial state u_0, we see that the aforementioned boundary conditions are necessary.

We first address the inverse problem of recovering the complex-valued electrostatic potential when the magnetic vector potential is known.

Theorem 1.2. *Assume that $\partial\Omega$ is $C^{2(N+1)}$ and let $u_0 \in H^{2N+3}(\Omega)$ and $g \in H^{2(N+\frac{7}{4}),N+\frac{7}{4}}(\Sigma)$ fulfill the compatibility condition (8) with $m = N$. Suppose moreover that*

$$\exists r_0 \in (0, +\infty), \ |u_0(x)| \geq r_0, \qquad x \in \Omega. \tag{13}$$

Fix $M \in (0, +\infty)$ and let $\rho_j \in \mathcal{Q}_M(\rho_0)$, $j = 1, 2$, satisfy

$$\left| \mathrm{Im}\left(\overline{(\rho_1 - \rho_2)} \nabla(\rho_1 - \rho_2) \right) \right| \leq M |\rho_1 - \rho_2|^2 \ \text{a.e. in } \Omega. \tag{14}$$

Then there exist a nonempty sub-boundary $\Gamma_0 \subset \partial\Omega$ and a positive constant C that depends only on Ω, T, M and (A_0, ρ_0), such that

$$\|\rho_1 - \rho_2\|_{L^2(\Omega)} \leq C \|\partial_\nu \partial_t (u_1 - u_2)\|_{L^2(\Gamma_0 \times (0,T))}.$$

Here, u_j for $j = 1, 2$, is the solution to the IBVP (1) associated with the electromagnetic potential (A_0, ρ_j), which is given by Theorem 1.1.

Let us now briefly comment on Theorem 1.2:

a) The assumption (13) allows for a far more flexible choice of initial input u_0 than in [2, 11], where it is required to be either real-valued or purely imaginary.

b) The condition (14) holds true provided either of the real or imaginary parts of the electrostatic potential, is known. Therefore Theorem 1.2 with $A_0 = 0$ extends the stability result of [2].

c) Arguing as in [2], we can prove at the expense of higher regularity on the coefficients and data of the magnetic Schrödinger equation that the following double-sided stability inequality

$$\|\rho_1-\rho_2\|_{H_0^1(\Omega)} \leq C_1\|\partial_\nu\partial_t(u_1-u_2)\|_{L^2(\Gamma_0\times(0,T))} \leq C_2\|\rho_1-\rho_2\|_{H_0^1(\Omega)},$$

holds for two positive constants C_1 and C_2.

d) There are actual classes of complex-valued electrostatic potentials fulfilling condition (14). For instance, this is the case of $\mathcal{E}_a := \{\rho(x) = a+\delta\langle x\rangle,\ \delta\in\mathbb{C}\}$, where $a\in\mathbb{C}$ is arbitrary and $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$ for all $x\in\mathbb{R}^n$. Indeed, for any $\rho_j(x) = a+\delta_j\langle x\rangle \in \mathcal{E}_a$, $j=1,2$, it holds true that

$$|\nabla(\rho_1-\rho_2)(x)| = |\delta_1-\delta_2|\frac{|x|}{\langle x\rangle} \leq |\delta_1-\delta_2| \leq \left(\min_{x\in\overline{\Omega}}\langle x\rangle\right)^{-1}|(\rho_1-\rho_2)(x)|,\ x\in\Omega.$$

e) The sub-boundary Γ_0 fulfills a specific geometrical condition expressed by (38). The same remark holds for the sub-boundary Γ_0 appearing in Theorems 1.3 and 1.4 below.

We next consider the inverse problem of determining the electromagnetic potential (A_0,ρ_0).

Theorem 1.3. *Assume that $\partial\Omega$ is $C^{2(N+1)}$. For $M\in(0,+\infty)$, let $\rho_j\in\mathcal{Q}_M(\rho_0)$ and $A_j\in\mathcal{A}_M(A_0)$, $j=1,2$, fulfill the three following conditions a.e. in Ω:*

$$|\nabla(\rho_1-\rho_2)| \leq M|\rho_1-\rho_2|, \tag{15}$$

$$\max_{i=1,\dots,n}\sum_{j=1}^n|\partial_i(A_1-A_2)_j| \leq M|A_1-A_2|, \tag{16}$$

$$|\nabla(\nabla\cdot(A_1-A_2))| \leq M|\nabla\cdot(A_1-A_2)|, \tag{17}$$

where $(A_1-A_2)_j$, for $j=1,\dots,n$, denotes the j-th component of A_1-A_2. Then, there exist a nonempty sub-boundary $\Gamma_0\subset\partial\Omega$ and a set

$$\{(u_0^k,g^k),\ k=0,1,\dots,n\} \in \left((H^{2N+3}(\Omega,\mathbb{C})\times H^{2\left(N+\frac{7}{4}\right),N+\frac{7}{4}}(\Sigma)\right)^{n+1},$$

where each pair (u_0^k,g^k) obeys the compatibility condition (8) with $m=N$, such that the stability inequality

$$\|\rho_1-\rho_2\|_{L^2(\Omega)} + \|A_1-A_2\|_{L^2(\Omega)} + \|\nabla\cdot A_1-\nabla\cdot A_2\|_{L^2(\Omega)} \leq C\sum_{k=0}^n\left\|\partial_\nu\partial_t(u_1^k-u_2^k)\right\|_{L^2(\Gamma_0\times(0,T))}$$

holds for some positive constant C, depending only on T, Ω, M and (A_0,ρ_0). Here, we denote by u_j^k, for $j=1,2$ and $k=0,\dots,n$, the solution to (1) associated with the initial state u_0^k, the boundary condition g^k and the electromagnetic potential (A_j,ρ_j).

We stress out that actual examples of classes of electromagnetic potentials fulfilling conditions (15)-(17) can be built in the same fashion as in Point d) of the remark following Theorem 1.2.

Remark 1.3. *In view of the third line of (44) and the estimates (54)-(55) and (59)-(60) established in the derivation of Theorem 1.3, presented in Section 4.3, the statement of the above result remains valid upon replacing the three conditions (15), (16) and (17) by the $2(n+1)$ following ones:*

$$\left|\sigma_i^k\right| \leq C\left(\|\rho\|^2_{L^2(\Omega)}+\|A\|^2_{L^2(\Omega)}+\|\nabla\cdot A\|^2_{L^2(\Omega)}\right),\ i=1,2,\ k=0,\dots,n, \tag{18}$$

with

$$\sigma_1^k := \mathrm{Im}\left(\left(2A\cdot\nabla u_0^k+u_0^k\nabla\cdot A\right)\left(2\mathbb{J}_A\nabla\overline{u_0^k}+\overline{u_0^k}\nabla(\nabla\cdot A)\right)+(\rho+S\cdot A)(\nabla\overline{\rho}+\mathbb{J}_AS)\left|u_0^k\right|^2\right),$$

$$\sigma_2^k := \text{ Re } \left((\rho + S \cdot A) u_0^k (2 \mathbb{J}_A \nabla \overline{u_0^k} + \overline{u_0^k} \nabla (\nabla \cdot A)) - (2A \cdot \nabla u_0^k + u_0^k \nabla \cdot A) \overline{u_0^k} (\nabla \overline{\rho} + \mathbb{J}_A S) \right).$$

Here we used the notations $A := A_1 - A_2$, $S := A_1 + A_2$, $\rho := \rho_1 - \rho_2$ and $\mathbb{J}_A$ stands for the Jacobian matrix of A. It is apparent that (18) *is fulfilled by any two electromagnetic potentials (A_1, ρ_1) and (A_2, ρ_2) obeying*(15), (16) *and* (17).

Finally, we consider the inverse problem of determining the direction of the magnetic vector potential when its strength, together with the electric potential, is known.

Theorem 1.4. *Assume that $\partial\Omega$ is $C^{2(N+1)}$. For $M \in (0, +\infty)$, let $A_j \in \mathcal{A}_M(A_0)$, $j = 1, 2$, be such that*

$$|A_1(x)| = |A_2(x)|, \ x \in \Omega. \tag{19}$$

Then there exist a nonempty sub-boundary $\Gamma_0 \subset \partial\Omega$, and a set

$$\{(u_0^k, g^k), \ k = 0, 1, \ldots, n\} \in \left(H^{2N+3}(\Omega, \mathbb{C}) \times H^{2\left(N+\frac{7}{4}\right), N+\frac{7}{4}}(\Sigma) \right)^{n+1}$$

of initial states u_0^k and boundary conditions g^k, fulfilling (8) *with $m = N$ for each $k = 0, 1, \ldots, n$, such that we have*

$$\|A_1 - A_2\|_{L^2(\Omega)} + \|\nabla \cdot A_1 - \nabla \cdot A_2\|_{L^2(\Omega)} \leq C \sum_{k=0}^{n} \left\| \partial_\nu \partial_t (u_1^k - u_2^k) \right\|_{L^2(\Gamma_0 \times (0,T))}$$

some positive constant C, depending only on T, Ω, M and (A_0, ρ_0). Here, u_j^k, for $j = 1, 2$ and $k = 0, \ldots, n$, is the solution to (1) *with initial state u_0^k, boundary condition g^k, magnetic potential vector $A_0 = A_j$ and electric potential ρ_0.*

We point out that if the divergence of the magnetic vector potentials is known, in which case we have $\nabla \cdot (A_1 - A_2) = 0$ everywhere in Ω, then it is easy to see from the derivation of Theorem 1.4 (given in Subsection 4.4) that the above stability inequality remains valid with only n local boundary measurements. Such a result is optimal in the sense that the n components of the vector-valued function representing the unknown magnetic vector potential are recovered with exactly n local boundary measurements of the solution.

1.7 Overview

The paper is organized as follows. In Section 2 we study the forward problem associated with (1) by proving Proposition 1.1 and Theorem 1.1. Section 3 is devoted to the derivation of a Carleman estimate for the Schrödinger equation in Q, needed by the analysis of the inverse problem under examination. Finally, Section 4 contains the proof of Theorems 1.2, 1.3 and 1.4.

2 Analysis of the direct problem

Let us first introduce the magnetic Dirichlet Laplacian in Ω. For $A_0 \in W^{2,\infty}(\Omega, \mathbb{R}^n)$ we denote by $-\Delta_{A_0}^D$ the self-adjoint operator generated in $L^2(\Omega)$ by the closed symmetric form

$$a(u, v) := \int_\Omega (\nabla + iA_0) u(x) \cdot \overline{(\nabla + iA_0) v(x)} dx, \ u, v \in H_0^1(\Omega).$$

It is well known that the Dirichlet Laplacian $\Delta_{A_0}^D$ acts on his domain $H_0^1(\Omega) \cap H^2(\Omega)$ as the operator Δ_{A_0} defined in (2).

Assume that $\rho_0 \in W^{1,\infty}(\Omega, \mathbb{C})$. Then, upon applying [10, Lemma 2.1] (with $X = L^2(\Omega)$, $U = i\Delta_{A_0}^D$ and $B(t) = -i\rho_0$ for all $t \in [0, T]$), we obtain:

Lemma 2.1. *For all $f \in H^{0,1}(Q)$ there exists a unique solution $v \in C([0,T], H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0,T], L^2(\Omega))$ to the Cauchy problem*

$$\begin{cases} (-i\partial_t - \Delta_{A_0}^D + \rho_0)v = f \\ v(\cdot, 0) = 0. \end{cases} \tag{20}$$

Moreover v satisfies the following energy estimate

$$\|v\|_{C^0([0,T],H^2(\Omega))} + \|v\|_{C^1([0,T],L^2(\Omega))} \leq C\|f\|_{H^{0,1}(Q)}. \tag{21}$$

Here and in the remaining part of this section, C denotes a positive constant depending only on Ω, T and M.

2.1 Proof of Proposition 1.1

Since $g \in H^{\frac{7}{2},\frac{7}{4}}(\Sigma)$ and $u_0 \in H^3(\Omega)$ fulfill (5) then [25, Section 4, Theorem 2.3] yields existence of $G \in H^{4,2}(Q)$ such that we have simultaneously $G(\cdot, 0) = u_0$ in Ω and $G = g$ on Σ, with the estimate

$$\|G\|_{H^{4,2}(Q)} \leq C\left(\|u_0\|_{H^3(\Omega)} + \|g\|_{H^{\frac{7}{2},\frac{7}{4}}(\Sigma)}\right). \tag{22}$$

Here and henceforth, C denotes a positive constant depending only on T, Ω and M. It is clear that u solves (1) if and only if the function $\tilde{u} := u - G$ is solution to the IBVP

$$\begin{cases} (-i\partial_t - \Delta_{A_0} + \rho_0)\tilde{u} = f_G & \text{in } Q \\ \tilde{u} = 0 & \text{on } \Sigma \\ \tilde{u}(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{23}$$

with $f_G := -(-i\partial_t - \Delta_{A_0} + \rho_0)G$. Next, as $G \in H^{4,2}(Q)$ yields $\partial_t G \in H^{2,1}(Q)$ with $\|\partial_t G\|_{H^{2,1}(Q)} \leq C\|G\|_{H^{4,2}(Q)}$ in virtue of [25][Section 4, Proposition 2.3], then we have $f_G \in H^{0,1}(Q)$ and

$$\|f_G\|_{H^{0,1}(Q)} \leq C\left(\|G\|_{H^{2,1}(Q)} + \|\partial_t G\|_{H^{2,1}(Q)}\right) \leq C\|G\|_{H^{4,2}(Q)}. \tag{24}$$

Therefore, applying Lemma 2.1 to (23), we get that there is a unique solution $\tilde{u} \in H^{2,1}(Q)$ to (23), such that

$$\|\tilde{u}\|_{H^{2,1}(Q)} \leq C\|f_G\|_{H^{0,1}(Q)},$$

according to (21). Finally, putting this together with the estimates $\|u\|_{H^{2,1}(Q)} \leq \left(\|\tilde{u}\|_{H^{2,1}(Q)} + \|G\|_{H^{2,1}(Q)}\right)$, (22) and (24), we obtain (6).

2.2 Proof of Theorem 1.1

We prove the statement of Theorem 1.1 for $m = 1$ only, as the rest of the proof is obtained in a similar fashion by induction on m.

Set $m = 1$ and put $z := \partial_t u$, where u is the $H^{2,1}(Q)$-solution to (1), given by Proposition 1.1. Then we have

$$\begin{cases} (-i\partial_t - \Delta_{A_0} + \rho_0)z = 0 & \text{in } Q \\ z = \partial_t g & \text{on } \Sigma \\ z(\cdot, 0) = z_0 & \text{in } \Omega, \end{cases} \tag{25}$$

where $z_0 := -i(-\Delta_{A_0} + \rho_0)u_0 \in H^3(\Omega)$. Moreover, as $g \in H^{\frac{11}{2},\frac{11}{4}}(\Sigma)$, we have $\partial_t g \in H^{\frac{7}{2},\frac{7}{4}}(\Sigma)$ by [25][Section 4, Proposition 2.3] and $\|\partial_t g\|_{H^{\frac{7}{2},\frac{7}{4}}(\Sigma)} \leq C\|g\|_{H^{\frac{11}{2},\frac{11}{4}}(\Sigma)}$. Therefore, in light of (25) and the compatibility condition (8) with $k = 1$, we infer from Proposition 1.1 that $z \in H^{2,1}(Q)$ satisfies

$$\|z\|_{H^{2,1}(Q)} \leq C\left(\|z_0\|_{H^3(\Omega)} + \|\partial_t g\|_{H^{\frac{7}{2},\frac{7}{4}}(\Sigma)}\right)$$

$$\leq \quad C\left(\|u_0\|_{H^5(\Omega)} + \|g\|_{H^{\frac{11}{2},\frac{11}{4}}(\Sigma)}\right). \tag{26}$$

This entails that $u \in \bigcap_{k=1}^{2} H^{2-k}(0,T;H^{2k}(\Omega))$ fulfills

$$\sum_{k=1}^{2} \|u\|_{H^{2-k}(0,T;H^{2k}(\Omega))} \leq C\left(\|u_0\|_{H^5(\Omega)} + \|g\|_{H^{\frac{11}{2},\frac{11}{4}}(\Sigma)}\right). \tag{27}$$

Next, with reference to (2), we infer for a.e. $t \in (0,T)$ from the first line of (1) that $u(\cdot,t)$ is solution to the following elliptic problem

$$\begin{cases} \Delta u(\cdot,t) = h(\cdot,t) & \text{in } \Omega \\ u(\cdot,t) = g(\cdot,t), \end{cases} \tag{28}$$

where $h(\cdot,t) := -iz(\cdot,t) - 2iA_0 \cdot \nabla u(\cdot,t) + \left(|A_0|^2 - i\nabla \cdot A_0 + \rho_0\right) u(\cdot,t) \in H^1(\Omega)$ and $g(\cdot,t) \in H^{\frac{11}{2}}(\partial\Omega) \subset H^{1+\frac{3}{2}}(\partial\Omega)$. Thus we have $u(\cdot,t) \in H^3(\Omega)$ by elliptic regularity, with

$$\begin{aligned} \|u(\cdot,t)\|_{H^3(\Omega)} &\leq C\left(\|h(\cdot,t)\|_{H^1(\Omega)} + \|g(\cdot,t)\|_{H^{\frac{5}{2}}(\partial\Omega)}\right) \\ &\leq C\left(\|z(\cdot,t)\|_{H^1(\Omega)} + \|u(\cdot,t)\|_{H^2(\Omega)} + \|g(\cdot,t)\|_{H^{\frac{11}{2}}(\partial\Omega)}\right). \end{aligned} \tag{29}$$

As a consequence we have $h(\cdot,t) \in H^2(\Omega)$ for a.e. $t \in (0,T)$, and since $\partial\Omega$ is C^4 and $g(\cdot,t) \in H^{\frac{11}{2}}(\partial\Omega) \subset H^{1+\frac{7}{2}}(\partial\Omega)$, we deduce from (28) and the elliptic regularity theorem that $u(\cdot,t) \in H^4(\Omega)$ fulfills

$$\begin{aligned} \|u(\cdot,t)\|_{H^4(\Omega)} &\leq C\left(\|h(\cdot,t)\|_{H^2(\Omega)} + \|g(\cdot,t)\|_{H^{\frac{9}{2}}(\partial\Omega)}\right) \\ &\leq C\left(\|z(\cdot,t)\|_{H^2(\Omega)} + \|u(\cdot,t)\|_{H^3(\Omega)} + \|g(\cdot,t)\|_{H^{\frac{11}{2}}(\partial\Omega)}\right). \end{aligned}$$

Putting this with (29) we get

$$\|u(\cdot,t)\|_{H^4(\Omega)} \leq C\left(\|z(\cdot,t)\|_{H^2(\Omega)} + \|u(\cdot,t)\|_{H^2(\Omega)} + \|g(\cdot,t)\|_{H^{\frac{11}{2}}(\partial\Omega)}\right), \ t \in (0,T).$$

Now, bearing in mind that u and z are both in $L^2(0,T;H^2(\Omega))$, and that $g \in L^2(0,T;H^{\frac{11}{2}}(\partial\Omega))$, we infer from the above estimate that $u \in L^2(0,T;H^4(\Omega))$ and that

$$\|u\|_{L^2(0,T;H^4(\Omega))} \leq C\left(\|z\|_{H^{2,1}(\Omega)} + \|u\|_{H^{2,1}(Q)} + \|g\|_{H^{\frac{11}{2},\frac{11}{4}}(\Sigma)}\right).$$

This, (6) and (26) lead to

$$\|u\|_{L^2(0,T;H^4(\Omega))} \leq C\left(\|u_0\|_{H^5(\Omega)} + \|g\|_{H^{\frac{11}{2},\frac{11}{4}}(\Sigma)}\right),$$

which, combined with (27), entails (9) with $m = 1$.

3 Global Carleman estimate

In this section, we establish a global Carleman estimate for the main part of the Schrödinger operator

$$L := -i\partial_t - \Delta \tag{30}$$

acting in $Q = \Omega \times (0,T)$. Carleman estimates for the Schrödinger operator in domains centered around $t = 0$ such as $\Omega \times (-T,T)$ were derived in [36] with a regular weight function and in [2] with a symmetric singular weight function. However, since the solution u to (1) cannot be time-symmetrized in the framework of this paper, we need to establish a Carleman estimate for the operator L in Q.

3.1 Settings and start of the calculation

To this end, we assume in the entire section that $u \in L^2(0,T;H_0^1(\Omega))$ and $Lu \in L^2(Q)$. Notice we further use that $\partial_\nu u \in L^2(\Sigma)$. Next we put $w := e^{s\alpha}u$, where $s \in (0,+\infty)$ and α is a real-valued smooth function we shall make precise in the next subsection, and set

$$Rw := e^{s\alpha}Lu = e^{s\alpha}L(e^{-s\alpha}w) = is(\partial_t\alpha)w + R_1w + R_2w = is(\partial_t\alpha)w + R_3w,$$

with

$$\begin{aligned} R_1w &:= -i\partial_t w - \Delta w - s^2|\nabla\alpha|^2 w, \\ R_2w &:= 2s\nabla\alpha\cdot\nabla w + s(\Delta\alpha)w, \\ R_3w &:= Rw - is(\partial_t\alpha)w = R_1w + R_2w. \end{aligned} \tag{31}$$

Since the function w is complex-valued, we denote by w_{re} its real part and by w_{im} its imaginary part, in such a way that $w = w_{\mathrm{re}} + iw_{\mathrm{im}}$. Similarly we decompose each R_jw, for $j = 1,2,3$, into the sum

$$R_jw = P_jw + iQ_jw,$$

where

$$\begin{aligned} P_1w &:= \partial_t w_{\mathrm{im}} - \Delta w_{\mathrm{re}} - s^2|\nabla\alpha|^2 w_{\mathrm{re}},\ Q_1w := -\partial_t w_{\mathrm{re}} - \Delta w_{\mathrm{im}} - s^2|\nabla\alpha|^2 w_{\mathrm{im}}, \\ P_2w &:= 2s\nabla\alpha\cdot\nabla w_{\mathrm{re}} + s(\Delta\alpha)w_{\mathrm{re}},\ Q_2w := 2s\nabla\alpha\cdot\nabla w_{\mathrm{im}} + s(\Delta\alpha)w_{\mathrm{im}}, \\ P_3w &:= \mathrm{Re}(Rw) + s(\partial_t\alpha)w_{\mathrm{im}},\ Q_3w := \mathrm{Im}(Rw) - s(\partial_t\alpha)w_{\mathrm{re}}. \end{aligned}$$

As we are aiming for computing $|R_3w|^2$ and since

$$|R_3w|^2 = \sum_{j=1}^{2}|R_jw|^2 + 2\,\mathrm{Re}\,\left((R_1w)\overline{R_2w}\right) = \sum_{j=1}^{2}|R_jw|^2 + 2(P_1w)P_2w + 2(Q_1w)Q_2w, \tag{32}$$

we start by expanding the two last terms in the right hand side of (32). We get that

$$\begin{aligned} 2(P_1w,P_2w)_{L^2(Q)} =& \int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{re}})\partial_t w_{\mathrm{im}}dxdt + \int_Q 2s(\Delta\alpha)w_{\mathrm{re}}\partial_t w_{\mathrm{im}}dxdt - \int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{re}})\Delta w_{\mathrm{re}}dxdt \\ &- \int_Q 2s(\Delta\alpha)w_{\mathrm{re}}\Delta w_{\mathrm{re}}dxdt - \int_Q 4s^3|\nabla\alpha|^2(\nabla\alpha\cdot\nabla w_{\mathrm{re}})w_{\mathrm{re}}dxdt - \int_Q 2s^3|\nabla\alpha|^2(\Delta\alpha)|w_{\mathrm{re}}|^2dxdt \\ &=: \sum_{k=1}^{6} I_k \end{aligned}$$

and

$$\begin{aligned} 2(Q_1w,Q_2w)_{L^2(Q)} =& -\int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{im}})\partial_t w_{\mathrm{re}}dxdt - \int_Q 2s(\Delta\alpha)w_{\mathrm{im}}\partial_t w_{\mathrm{re}}dxdt - \int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{im}})\Delta w_{\mathrm{im}}dxdt \\ &- \int_Q 2s(\Delta\alpha)w_{\mathrm{im}}\Delta w_{\mathrm{im}}dxdt - \int_Q 4s^3|\nabla\alpha|^2(\nabla\alpha\cdot\nabla w_{\mathrm{im}})w_{\mathrm{im}}dxdt - \int_Q 2s^3|\nabla\alpha|^2(\Delta\alpha)|w_{\mathrm{im}}|^2dxdt, \\ &=: \sum_{k=1}^{6} J_k, \end{aligned}$$

hence are left with the task of computing I_k and J_k for $j = 1,\dots,6$. We proceed by integration by parts. Bearing in mind that $w_{|\Sigma} = 0$, we find through direct calculations that

$$I_1 = \int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{re}})\partial_t w_{\mathrm{im}}dxdt$$

$$
\begin{aligned}
&= -\int_Q 4s(\Delta\alpha) w_{\mathrm{re}} \partial_t w_{\mathrm{im}} dxdt - \int_Q 4s w_{\mathrm{re}} (\nabla\alpha \cdot \partial_t \nabla w_{\mathrm{im}}) dxdt \\
&= -\int_\Omega 4s(\nabla\alpha\cdot\nabla w_{\mathrm{im}}) w_{\mathrm{re}} dx\Big|_{t=0}^{t=T} + \int_Q 4s(\partial_t\nabla\alpha \cdot \nabla w_{\mathrm{im}}) w_{\mathrm{re}} dxdt + \int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{im}}) \partial_t w_{\mathrm{re}} dxdt \\
&- \int_Q 4s(\Delta\alpha) w_{\mathrm{re}} \partial_t w_{\mathrm{im}} dxdt \\
&= -\int_\Omega 2s(\nabla\alpha\cdot\nabla w_{\mathrm{im}}) w_{\mathrm{re}} dx\Big|_{t=0}^{t=T} + \int_\Omega 2s(\nabla\alpha\cdot\nabla w_{\mathrm{re}}) w_{\mathrm{im}} dx\Big|_{t=0}^{t=T} + \int_\Omega 2s(\Delta\alpha) w_{\mathrm{im}} w_{\mathrm{re}} dx\Big|_{t=0}^{t=T} \\
&+ \int_Q 4s(\partial_t\nabla\alpha \cdot \nabla w_{\mathrm{im}}) w_{\mathrm{re}} dxdt + \int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{im}}) \partial_t w_{\mathrm{re}} dxdt - \int_Q 4s(\Delta\alpha) w_{\mathrm{re}} \partial_t w_{\mathrm{im}} dxdt,
\end{aligned}
$$

$$
I_2 = \int_Q 2s(\Delta\alpha) w_{\mathrm{re}} (\partial_t w_{\mathrm{im}}) dxdt,
$$

$$
\begin{aligned}
I_3 &= -\int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{re}}) \Delta w_{\mathrm{re}} dxdt \\
&= \int_\Sigma -4s(\nabla\alpha\cdot\nabla w_{\mathrm{re}}) \partial_\nu w_{\mathrm{re}} d\Sigma + \int_Q 4s \sum_{i,j=1}^n (\partial_{ij}^2\alpha)(\partial_i w_{\mathrm{re}})(\partial_j w_{\mathrm{re}}) dxdt + \int_Q 2s\nabla\alpha\cdot\nabla|\nabla w_{\mathrm{re}}|^2 dxdt \\
&= \int_\Sigma \left(-4s\nabla\alpha\cdot\nabla w_{\mathrm{re}}\partial_\nu w_{\mathrm{re}} + 2s(\partial_\nu\alpha)|\nabla w_{\mathrm{re}}|^2\right) d\Sigma + \int_Q 4s \sum_{i,j=1}^n (\partial_{ij}^2\alpha)(\partial_i w_{\mathrm{re}})(\partial_j w_{\mathrm{re}}) dxdt - \int_Q 2s\Delta\alpha|\nabla w_{\mathrm{re}}|^2 dxdt \\
&= -\int_\Sigma 2s(\partial_\nu\alpha)\left|\partial_\nu w_{\mathrm{re}}\right|^2 d\Sigma + \int_Q 4s \sum_{i,j=1}^n (\partial_{ij}^2\alpha)(\partial_i w_{\mathrm{re}})(\partial_j w_{\mathrm{re}}) dxdt - \int_Q 2s\Delta\alpha|\nabla w_{\mathrm{re}}|^2 dxdt,
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= -\int_Q 2s(\Delta\alpha) w_{\mathrm{re}} \Delta w_{\mathrm{re}} dxdt \\
&= \int_Q 2s(\Delta\alpha)|\nabla w_{\mathrm{re}}|^2 dxdt + \int_Q 2s(\nabla(\Delta\alpha)\cdot\nabla w_{\mathrm{re}}) w_{\mathrm{re}} dxdt \\
&= \int_Q 2s(\Delta\alpha)|\nabla w_{\mathrm{re}}|^2 dxdt - \int_Q s(\Delta^2\alpha)|w_{\mathrm{re}}|^2 dxdt,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= -\int_Q 4s^3|\nabla\alpha|^2(\nabla\alpha\cdot\nabla w_{\mathrm{re}}) w_{\mathrm{re}} dxdt \\
&= \int_Q 2s^3(\nabla\cdot(|\nabla\alpha|^2\nabla\alpha))|w_{\mathrm{re}}|^2 dxdt,
\end{aligned}
$$

$$
I_6 = -\int_Q 2s^3|\nabla\alpha|^2(\Delta\alpha)|w_{\mathrm{re}}|^2 dxdt,
$$

$$
J_1 = -\int_Q 4s(\nabla\alpha\cdot\nabla w_{\mathrm{im}}) \partial_t w_{\mathrm{re}} dxdt,
$$

$$
\begin{aligned}
J_2 &= -\int_Q 2s(\Delta\alpha)w_{\rm im}\partial_t w_{\rm re}dxdt\\
&= -\int_\Omega 2s(\Delta\alpha)w_{\rm im}w_{\rm re}dx\Big|_{t=0}^{t=T} + \int_Q 2s(\Delta\alpha)(\partial_t w_{\rm im})w_{\rm re}dxdt + \int_Q 2s(\partial_t\Delta\alpha)w_{\rm im}w_{\rm re}dxdt,
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= -\int_Q 4s(\nabla\alpha\cdot\nabla w_{\rm im})\Delta w_{\rm im}dxdt\\
&= \int_\Sigma -4s\nabla\alpha\cdot\nabla w_{\rm im}\partial_\nu w_{\rm im}d\Sigma + \int_Q 4s\sum_{i,j=1}^n(\partial_{ij}^2\alpha)(\partial_i w_{\rm im})(\partial_j w_{\rm im})dxdt + \int_Q 2s\nabla\alpha\cdot\nabla|\nabla w_{\rm im}|^2dxdt\\
&= -\int_\Sigma 2s(\partial_\nu\alpha)\left|\partial_\nu w_{\rm im}\right|^2 d\Sigma + \int_Q 4s\sum_{i,j=1}^n(\partial_{ij}^2\alpha)(\partial_i w_{\rm im})(\partial_j w_{\rm im})dxdt - \int_Q 2s\Delta\alpha|\nabla w_{\rm im}|^2dxdt,
\end{aligned}
$$

$$
\begin{aligned}
J_4 &= -\int_Q 2s(\Delta\alpha)w_{\rm im}\Delta w_{\rm im}dxdt\\
&= \int_Q 2s(\Delta\alpha)|\nabla w_{\rm im}|^2dxdt + \int_Q 2s\nabla(\Delta\alpha)\cdot\nabla w_{\rm im}w_{\rm im}dxdt\\
&= \int_Q 2s(\Delta\alpha)|\nabla w_{\rm im}|^2dxdt - \int_Q s(\Delta^2\alpha)|w_{\rm im}|^2dxdt,
\end{aligned}
$$

$$
\begin{aligned}
J_5 &= -\int_Q 4s^3|\nabla\alpha|^2(\nabla\alpha\cdot\nabla w_{\rm im})w_{\rm im}dxdt\\
&= \int_Q 2s^3(\nabla\cdot(|\nabla\alpha|^2\nabla\alpha))|w_{\rm im}|^2dxdt,
\end{aligned}
$$

$$
J_6 = -\int_Q 2s^3|\nabla\alpha|^2(\Delta\alpha)|w_{\rm im}|^2dxdt.
$$

Therefore we have

$$
2(P_1w,P_2w)_{L^2(Q)} + 2(Q_1w,Q_2w)_{L^2(Q)} = \sum_{k=1}^6(I_k+J_k) =: \mathrm{Main}_1 + \mathrm{Main}_2 + \mathrm{Lower} + \mathrm{Bndry},
$$

with

$$
\begin{aligned}
\mathrm{Main}_1 &:= \int_Q 4s\sum_{i,j=1}^n(\partial_{ij}^2\alpha)\big[(\partial_i w_{\rm re})\partial_j w_{\rm re} + (\partial_i w_{\rm im})\partial_j w_{\rm im}\big]dxdt,\\
\mathrm{Main}_2 &:= \int_Q 2s^3\big(\nabla|\nabla\alpha|^2\nabla\alpha\big)|w|^2dxdt\\
&\quad= \int_Q 4s^3\sum_{i,j=1}^n(\partial_{ij}^2\alpha)(\partial_i\alpha)(\partial_j\alpha)|w|^2dxdt,\\
\mathrm{Lower} &:= \int_Q 4s(\partial_t\nabla\alpha\cdot\nabla w_{\rm im})w_{\rm re}dxdt - \int_Q s(\Delta^2\alpha)|w|^2dxdt + \int_Q 2s(\partial_t\Delta\alpha)w_{\rm im}w_{\rm re}dxdt,\\
\mathrm{Bndry} &:= -\int_\Sigma 2s(\partial_\nu\alpha)\left(|\partial_\nu w_{\rm re}|^2 + |\partial_\nu w_{\rm im}|^2\right)d\Sigma + \int_\Omega 2s\left[(\nabla\alpha\cdot\nabla w_{\rm re})w_{\rm im} - (\nabla\alpha\cdot\nabla w_{\rm im})w_{\rm re}\right]dx\Big|_{t=0}^{t=T}.
\end{aligned}
$$

3.2 Weight functions

We introduce the weight functions

$$\alpha(x,t) = \frac{e^{\lambda\beta(x)} - e^{\lambda K}}{\ell^2(t)} \text{ and } \varphi(x,t) = \frac{e^{\lambda\beta(x)}}{\ell^2(t)},\ \lambda \in (0,+\infty), \tag{33}$$

where $\beta \in C^4(\overline{\Omega})$ is nonnegative and has no critical point, i.e.

$$\beta(x) \geq 0, \quad |\nabla\beta(x)| \geq c_0 > 0,\ \forall x \in \Omega, \tag{34}$$

where $K := 2\sup_{x\in\Omega}\beta(x)$ and $\ell \in C^1[0,T]$ is nonnegative, attains its maximum at the origin and vanishes at T, i.e.,

$$\ell(T) = 0, \qquad \ell(0) > \ell(t) \geq 0, \quad \forall t \in (0,T]. \tag{35}$$

We assume in addition that β is pseudo-convex condition with respect to the Laplace operator, in the sense that there exist two constants $\lambda_0 \in (0,+\infty)$ and $\epsilon \in (0,+\infty)$ such that we have

$$\lambda|\nabla\beta\cdot\xi|^2 + D^2\beta(\xi,\xi) \geq \epsilon|\xi|^2,\ \xi \in \mathbb{R}^n,\ \lambda \in [\lambda_0,+\infty), \tag{36}$$

with $D^2\beta(\xi,\xi) := \sum_{i,j=1}^n(\partial_{ij}^2\beta)\xi_i\xi_j$. For the sake of convenience we assume without loss of generality in the sequel that

$$\lambda_0 \geq \max(1, 2(\ln 2)K^{-1}). \tag{37}$$

Next we define the observation zone where the Neumann data used by the analysis of the inverse problems examined in this text, are measured, as the sub-boundary

$$\Gamma_0 := \{x \in \partial\Omega : \nabla\beta(x)\cdot\nu(x) \geq 0\}. \tag{38}$$

Remark 3.1. *1) It is worth mentioning that there exist functions β and ℓ fulfilling the conditions* (34)*,* (35) *and* (36)*. As a matter of fact, for any fixed $x_0 \notin \overline{\Omega}$, we may choose $\beta(x) := |x-x_0|^2$ for all $x \in \overline{\Omega}$ and $\ell(t) := (T+t)(T-t)$ for all $t \in [0,T]$, as in [2]. In this case,* (36) *holds with $\epsilon = 2$ whenever λ_0 is nonnegative, and the observation zone Γ_0 coincides with the x_0-shadowed face of the boundary $\partial\Omega$, i.e. $\Gamma_0 = \{x \in \partial\Omega :\ (x-x_0)\cdot\nu(x) \geq 0\}$.*
2) In a similar way to [2, Proposition 1] and [36, Proposition 2.1], for the purpose of centering the information around the initial state of (1)*, the function α defined by* (33)*,* (34) *and* (35)*, is maximal at $t = 0$. Actually, α is inspired by the weight function introduced in [2]. However in contrast to this one, which blows up at both ends $\pm T$ of the time interval, the weight α that we use in this paper, is bounded at the left endpoint of $(0,T)$. This will translate into the presence of the trace term $s\mathcal{I}_s(u(\cdot,0))$ on the right hand side of the Carleman estimate* (42)*, see Theorem 3.1 below, while the weighted energy traces at $t = \pm T$ vanish in [2, Proposition 1]. The same is true for [36, Proposition 2.1], but this is due to the vanishing of $u(\cdot,\pm T)$. As a matter of fact, the weight function of [36, Proposition 2.1] is quadratic in time, hence it is bounded everywhere in $(-T,T)$.*

From the very definition of α, we see that $\lim_{t\to T}(\varphi w)(\cdot,t) = 0$, and for all $i,j = 1,\ldots,n$, that

$$\nabla\alpha = \nabla\varphi = \lambda\varphi\nabla\beta, \qquad \partial_i\alpha = \partial_i\varphi = \lambda\varphi\partial_i\beta, \qquad \partial_{ij}^2\alpha = \partial_{ij}^2\varphi = \lambda^2\varphi(\partial_i\beta)\partial_j\beta + \lambda\varphi\partial_{ij}^2\beta \tag{39}$$

$$|\partial_t\alpha| = \left|-\frac{2\ell'(e^{\lambda\beta} - e^{\lambda K})}{\ell^3}\right| \leq C_\lambda\varphi^{\frac{3}{2}}, \qquad |\partial_t\nabla\alpha| = |\lambda(\partial_t\varphi)\nabla\beta| \leq C_\lambda\varphi^{\frac{3}{2}}, \qquad |\partial_t\Delta\alpha| \leq C_\lambda\varphi^{\frac{3}{2}}. \tag{40}$$

Here and henceforth, C (resp., C_λ) denotes a generic constant that depends only on ϵ, c_0 and $\ell(0)$ (resp., ϵ, c_0, $\|\beta\|_{L^\infty(\Omega)}$, l and λ). In any case, C and C_λ are independent of s.

3.3 Completion of the proof

In light of (36) and (39)-(40) we have

$$\begin{aligned}
\mathrm{Main}_1 &= \int_Q 4s \sum_{i,j=1}^n (\partial_{ij}^2\alpha)\big[(\partial_i w_{\mathrm{re}})\partial_j w_{\mathrm{re}} + (\partial_i w_{\mathrm{im}})\partial_j w_{\mathrm{im}}\big] dxdt \\
&= \int_Q 4s\lambda\varphi \left[\lambda|\nabla\beta\cdot\nabla w_{\mathrm{re}}|^2 + \lambda|\nabla\beta\cdot\nabla w_{\mathrm{im}}|^2 + D^2\beta(\nabla w_{\mathrm{re}},\nabla w_{\mathrm{re}}) + D^2\beta(\nabla w_{\mathrm{im}},\nabla w_{\mathrm{im}})\right] dxdt \\
&\geq 4\epsilon \int_Q s\lambda\varphi|\nabla w|^2 dxdt, \\
\mathrm{Main}_2 &= \int_Q 2s^3 \sum_{i,j=1}^n (\partial_{ij}^2\alpha)(\partial_i\alpha)(\partial_j\alpha)|w|^2 dxdt \\
&= \int_Q 2s^3\lambda^3\varphi^3 \left[\lambda|\nabla\beta|^4 + D^2\beta(\nabla\beta,\nabla\beta)\right] |w|^2 dxdt \\
&\geq 2\epsilon c_0^2 \int_Q s^3\lambda^3\varphi^3|w|^2 dxdt, \\
|\mathrm{Lower}| &= \left| \int_Q 4s(\partial_t\nabla\alpha)\cdot\nabla w_{\mathrm{im}})w_{\mathrm{re}} dxdt - \int_Q s(\Delta^2\alpha)|w|^2 dxdt + \int_Q 2s(\partial_t\Delta\alpha)w_{\mathrm{im}}w_{\mathrm{re}} dxdt \right| \\
&\leq C_\lambda s^{-\frac{1}{2}} \int_Q (s\varphi|\nabla w|^2 + s^3\varphi^3|w|^2) dxdt, \\
\mathrm{Bndry} &= -\int_\Sigma 2s(\partial_\nu\alpha)\left(|\partial_\nu w_{\mathrm{re}}|^2 + |\partial_\nu w_{\mathrm{im}}|^2\right) d\Sigma + \int_\Omega 2s\left[(\nabla\alpha\cdot\nabla w_{\mathrm{re}})w_{\mathrm{im}} - (\nabla\alpha\cdot\nabla w_{\mathrm{im}})w_{\mathrm{re}}\right] dx\Big|_{t=0}^{t=T} \\
&= -\int_\Sigma 2s\lambda\varphi(\partial_\nu\beta)\left|\partial_\nu w\right|^2 d\Sigma - \int_\Omega 2s\lambda\varphi\left[(\nabla\beta\cdot\nabla u_{\mathrm{re}})u_{\mathrm{im}} - (\nabla\beta\cdot\nabla u_{\mathrm{im}})u_{\mathrm{re}}\right] e^{2s\alpha} dx\Big|_{t=0} \\
&\geq -\int_{\Sigma_0} 2s\lambda\varphi(\partial_\nu\beta)\left|\partial_\nu w\right|^2 d\Sigma - \int_\Omega 2s\lambda\varphi\left[(\nabla\beta\cdot\nabla u_{\mathrm{re}})u_{\mathrm{im}} - (\nabla\beta\cdot\nabla u_{\mathrm{im}})u_{\mathrm{re}}\right] e^{2s\alpha} dx\Big|_{t=0},
\end{aligned}$$

where $\Sigma_0 := (0,T)\times\Gamma_0$. This and (32) imply

$$\begin{aligned}
&\|R_1 w\|_{L^2(Q)}^2 + \|R_2 w\|_{L^2(Q)}^2 + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}\nabla w\|_{L^2(Q)}^2 + \|s^{\frac{3}{2}}\varphi^{\frac{3}{2}} w\|_{L^2(Q)}^2 \leq C\|R_3 w\|_{L^2(Q)}^2 + C_\lambda s \left\|\varphi^{\frac{1}{2}}\partial_\nu w\right\|_{L^2(\Sigma_0)}^2 \\
&+ C_\lambda s^{-\frac{1}{2}} \left(\|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}\nabla w\|_{L^2(Q)}^2 + \|s^{\frac{3}{2}}\varphi^{\frac{3}{2}} w\|_{L^2(Q)}^2\right) + C\left|\int_\Omega 2s\lambda\varphi\left[(\nabla\beta\cdot\nabla u_{\mathrm{re}})u_{\mathrm{im}} - (\nabla\beta\cdot\nabla u_{\mathrm{im}})u_{\mathrm{re}}\right] e^{2s\alpha} dx\Big|_{t=0}\right|
\end{aligned}$$

for all $\lambda \geq \lambda_0 \geq 1$ and all $s \in (0,+\infty)$. Further, bearing in mind that $R_3 w = Rw - is(\partial_t\alpha)w$ and $Rw = e^{s\alpha}Lu$, we infer from the above inequality that

$$\begin{aligned}
&\|R_1 w\|_{L^2(Q)}^2 + \|R_2 w\|_{L^2(Q)}^2 + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}\nabla w\|_{L^2(Q)}^2 + \|s^{\frac{3}{2}}\varphi^{\frac{3}{2}} w\|_{L^2(Q)}^2 \leq C\|Rw\|_{L^2(Q)}^2 + C_\lambda s \left\|\varphi^{\frac{1}{2}}\partial_\nu w\right\|_{L^2(\Sigma_0)}^2 \\
&+ C_\lambda s^{-\frac{1}{2}} \left(\|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}\nabla w\|_{L^2(Q)}^2 + \|s^{\frac{3}{2}}\varphi^{\frac{3}{2}} w\|_{L^2(Q)}^2\right) + C\int_\Omega 2s\lambda\varphi\left|(\nabla\beta\cdot\nabla u_{\mathrm{re}})u_{\mathrm{im}} - (\nabla\beta\cdot\nabla u_{\mathrm{im}})u_{\mathrm{re}}\right| e^{2s\alpha} dx\Big|_{t=0}.
\end{aligned}$$

Thus, going back to $u = e^{-s\alpha}w$ and taking $\lambda \geq \lambda_0$ and $s \geq s_0(\lambda) := 4C_\lambda^2 > 0$ in such a way that the low order term $C_\lambda s^{-\frac{1}{2}}\left(\|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}\nabla w\|_{L^2(Q)}^2 + \|s^{\frac{3}{2}}\varphi^{\frac{3}{2}} w\|_{L^2(Q)}^2\right)$ in the right-hand side of the above inequality is absorbed by $\|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}\nabla w\|_{L^2(Q)}^2 + \|s^{\frac{3}{2}}\varphi^{\frac{3}{2}} w\|_{L^2(Q)}^2$ in the left-hand side, we get for all $s \in [s_0,+\infty)$ that

$$\|R_1(e^{s\alpha}u)\|_{L^2(Q)}^2 + \|R_2(e^{s\alpha}u)\|_{L^2(Q)}^2 + s\|e^{s\alpha}\varphi^{\frac{1}{2}}\nabla u\|_{L^2(Q)}^2 + s^3\|\varphi^{\frac{3}{2}}e^{s\alpha}u\|_{L^2(Q)}^2$$

$$\leq \quad C\|e^{s\alpha}Lu\|^2_{L^2(Q)} + C_\lambda s \left\|e^{s\alpha}\varphi^{\frac{1}{2}}\partial_\nu u\right\|^2_{L^2(\Sigma_0)} + 2sC_\lambda \int_\Omega \varphi e^{2s\alpha} \left|(\nabla\beta\cdot\nabla u_{\rm re})u_{\rm im} - (\nabla\beta\cdot\nabla u_{\rm im})u_{\rm re}\right| dx\Big|_{t=0}.$$

Finally, taking into account that $\ell^{-2}(0) \leq \varphi(x,t) \leq e^{\frac{\lambda K}{2}}\ell^{-2}(0)$ for all $(x,t) \in \Omega \times [0,T]$, we obtain that

$$\begin{aligned}&\|R_1(e^{s\alpha}u)\|^2_{L^2(Q)} + \|R_2(e^{s\alpha}u)\|^2_{L^2(Q)} + s\|e^{s\alpha}\nabla u\|^2_{L^2(Q)} + s^3\|e^{s\alpha}u\|^2_{L^2(Q)} \\ \leq \quad & C\|e^{s\alpha}Lu\|^2_{L^2(Q)} + C_\lambda s\left\|e^{s\alpha}\varphi^{\frac{1}{2}}\partial_\nu u\right\|^2_{L^2(\Sigma_0)} + C_\lambda s\mathcal{I}_s(u(\cdot,0)),\ s \in [s_0,+\infty),\end{aligned}$$

where

$$\mathcal{I}_s(u(\cdot,0)) := \int_\Omega e^{2s\alpha(x,0)}|\nabla\beta(x)\cdot(\overline{u}\nabla u - u\nabla\overline{u})(x,0)|dx. \tag{41}$$

3.4 Statement of the Carleman estimate and brief comments

Summerizing, we have proved the following:

Theorem 3.1. *Let α and φ be defined by* (33) *where $\beta \in C^4(\overline{\Omega})$ and $\ell \in C^1[0,T]$ fulfill* (34)-(36) *for some fixed $\lambda \in [\lambda_0,+\infty)$, λ_0 obeying the condition* (37)*. Then there exist two positive constants s_0 and C, both of them depending only on ϵ, c_0, λ, $\ell(0)$, $\|\beta\|_{L^\infty(\Omega)}$ and $\|\ell'\|_{L^\infty(\Omega)}$, such that the estimate*

$$\begin{aligned}&\|R_1(e^{s\alpha}u)\|^2_{L^2(Q)} + \|R_2(e^{s\alpha}u)\|^2_{L^2(Q)} + s\|e^{s\alpha}\nabla u\|^2_{L^2(Q)} + s^3\|e^{s\alpha}u\|^2_{L^2(Q)} \\ \leq \quad & C\left(\|e^{s\alpha}Lu\|^2_{L^2(Q)} + s\left\|\varphi^{\frac{1}{2}}e^{s\alpha}\partial_\nu u\right\|^2_{L^2(\Sigma_0)} + s\mathcal{I}_s(u(\cdot,0))\right)\end{aligned} \tag{42}$$

holds for all $s \in [s_0,+\infty)$ and any function $u \in L^2(0,T;H^1_0(\Omega))$ satisfying $Lu \in L^2(\Omega\times(0,T))$ and $\partial_\nu u \in L^2(0,T;L^2(\partial\Omega))$. Here the operators R_1, R_2 and L are defined in (30)-(31) *and $\mathcal{I}_s(u(\cdot,0))$ is given by* (41)*.*

The specific novelty of the Carleman estimate of Theorem 3.1 as opposed to those in [2, Proposition 1] or [36, Proposition 2.1], is the term $s\mathcal{I}_s(u(\cdot,0))$, where $\mathcal{I}_s(u(\cdot,0))$ is defined by (41), that cancels if $u(\cdot,0) \in \mathbb{R}$ or in $i\mathbb{R}$. The presence of the integral $s\mathcal{I}_s(u(\cdot,0))$ in the Carleman estimate (42) boils down to (35), and more specifically to the assumption $\ell(0) > 0$, which guarantees that the weight function $t \mapsto \alpha(\cdot,t)$ remains bounded at $t=0$ (i.e. $\alpha(x,0) \in (-\ell^{-2}(0)e^{-\lambda K},0)$ for all $x \in \Omega$). As a matter of fact the corresponding term vanishes at final time since α blows up at the endpoint of the time interval $(0,T)$ (i.e. $\lim_{t\to T-}\alpha(x,t) = -\infty$ for all $x \in \Omega$) as we have $\ell(T) = 0$. This situation is quite reminiscent of the one in [2], where $\ell(t) = (T-t)(T+t)$ vanishes at both ends of the time interval $(-T,T)$ on which the Carleman inequality is established. Similarly there is no such thing as $\mathcal{I}_s$ in the Carleman estimate of [36, Proposition 2.1], which is designed for functions with identically zero initial and final states.

Remark 3.2. *We notice from* (33), (34), (35) *and* (37) *that the estimate*

$$s\varphi(x,t)e^{2s\alpha(x,t)} \leq s\ell^{-2}(t)e^{\frac{\lambda K}{2}}e^{-2s\ell^{-2}(t)e^{\frac{\lambda K}{2}}} \leq \sup_{r\in(0,+\infty)} re^{-2r} \in (0,1),\ \lambda \in [\lambda_0,+\infty),$$

holds uniformly in $s \in [s_0,+\infty)$ and $(x,t) \in \Omega\times[0,T]$. As a consequence the second term in the right hand side of (42) *is bounded as*

$$s\left\|\varphi^{\frac{1}{2}}e^{s\alpha}\partial_\nu u\right\|^2_{L^2(\Sigma_0)} \leq \|\partial_\nu u\|^2_{L^2(\Sigma_0)},\ s \in [s_0,+\infty). \tag{43}$$

4 Proof of Theorems 1.2, 1.3 and 1.4

In the entire section, we shall denote by C a generic constant that may change from line to line, but is independent of the parameter s introduced in the Carleman estimate stated in Theorem 3.1. As a matter of fact it can be checked that the various constants C that will appear in the remaining part of this text depend only on Ω, T, M and g.

4.1 Preliminary estimate

Let us recall from Theorem 1.1 that u_j, $j = 1, 2$, is the $H^2(0,T; L^2(\Omega)) \cap H^1(0,T; H^2(\Omega))$-solution to the IBVP (1) where (A, ρ) is replaced by (A_j, ρ_j). Thus, taking the difference of the two systems, we get that $u := u_1 - u_2$ solves

$$\begin{cases} (-i\partial_t - \Delta_{A_1} + \rho_1)u = 2iA\cdot\nabla u_2 - (\rho + S\cdot A - i\nabla\cdot A)\, u_2 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(\cdot, 0) = 0 & \text{on } \Omega, \end{cases}$$

with $A := A_1 - A_2$, $S := A_1 + A_2$ and $\rho := \rho_1 - \rho_2$. Further, differentiating the above system w.r.t. the time variable t, yields

$$\begin{cases} (-i\partial_t - \Delta_{A_1} + \rho_1)v = 2iA\cdot\nabla\partial_t u_2 - (\rho + S\cdot A - i\nabla\cdot A)\, \partial_t u_2 & \text{in } Q \\ v = 0 & \text{on } \Sigma \\ v(\cdot, 0) = -2A\cdot\nabla u_0 - i\,(\rho + S\cdot A - i\nabla\cdot A)\, u_0 & \text{in } \Omega, \end{cases} \tag{44}$$

with $v := \partial_t u$. Notice that all the above computations make sense as we have $u \in H^2(0,T; L^2(\Omega)) \cap H^1(0,T; H^2(\Omega) \cap H^1_0(\Omega))$, and hence

$$v \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega)). \tag{45}$$

Moreover, it holds true for all $s \in (0, +\infty)$ that

$$\left\| e^{s\alpha(\cdot,0)} v(\cdot, 0) \right\|^2_{L^2(\Omega)} \leq s^{-\frac{3}{2}} \left(\|R_1 e^{s\alpha} v\|^2_{L^2(Q)} + s^3 \|e^{s\alpha} v\|^2_{L^2(Q)} \right), \tag{46}$$

where R_1 is defined by (31). This can be seen through direct calculations. Indeed, in light of (33)-(35) we see that $\lim_{t \underset{<}{\to} T} \alpha(x,t) = -\infty$ for all $x \in \Omega$, whence $\lim_{t \underset{<}{\to} T} e^{s\alpha(\cdot,t)} v(\cdot, t) = 0$ in $L^2(\Omega)$. As a consequence we have

$$\left\| e^{s\alpha(\cdot,0)} v(\cdot, 0) \right\|^2_{L^2(\Omega)} = -\int_0^T \frac{d}{dt} \left\| e^{s\alpha(\cdot,t)} v(\cdot, t) \right\|^2_{L^2(\Omega)} dt = -\int_Q \left(w\overline{\partial_t w} + (\partial_t w)\overline{w} \right) dx\ dt,$$

where $w := \mathrm{e}^{s\alpha} v$. Further, as $\partial_t w = i \left(R_1 w + \Delta w + s^2 |\nabla\alpha|^2 w \right)$ from the very definition of R_1, we have

$$\begin{aligned} \left\| e^{s\alpha(\cdot,0)} v(\cdot, 0) \right\|^2_{L^2(\Omega)} &= i \int_Q \left(w(\overline{R_1 w} + \Delta\overline{w} + s^2 |\nabla\alpha|^2 \overline{w}) - (R_1 w + \Delta w + s^2 |\nabla\alpha|^2 w)\overline{w} \right) dx\ dt \\ &= i \int_Q \left(w\overline{R_1 w} - (R_1 w)\overline{w} + w\Delta\overline{w} - \overline{w}\Delta w \right) dx\ dt. \end{aligned}$$

Finally, since $\int_Q (w\Delta\overline{w} - \overline{w}\Delta w)\, dx\ dt = 0$, we end up getting

$$\left\| e^{s\alpha(\cdot,0)} v(\cdot, 0) \right\|^2_{L^2(\Omega)} = 2\mathrm{Re} \int_Q s^{\frac{3}{4}} i w(x,t) s^{-\frac{3}{4}} R_1 w(x,t) dx\ dt \leq s^{-\frac{3}{2}} \|R_1 w\|^2_{L^2(Q)} + s^{\frac{3}{2}} \|w\|^2_{L^2(Q)},$$

with the help of the Cauchy-Schwarz and Hölder inequalities. This immediately leads to (46).

4.2 Proof of Theorem 1.2

Let us rewrite (44) in the context of Theorem 1.2, where $A_1 = A_2 = A_0$ (and hence $A = 0$); We obtain:

$$\begin{cases} (-i\partial_t - \Delta_{A_0} + \rho_1)v = -\rho\partial_t u_2 & \text{in } Q \\ v = 0 & \text{on } \Sigma \\ v(\cdot, 0) = -i\rho u_0 & \text{in } \Omega. \end{cases} \tag{47}$$

Next, with reference to (30), the first line of (47) reads $Lv = -\rho\partial_t u_2 + 2iA_0 \cdot \nabla v + (i\nabla \cdot A_0 - |A_0|^2 - \rho_1)v$, so we have $Lv \in L^2(0,T;L^2(\Omega))$, with $v \in L^2(0,T;H_0^1(\Omega))$ and $\partial_\nu v \in L^2(0,T;L^2(\partial\Omega))$, in virtue of (45). Therefore, (42)-(43) yield

$$\begin{aligned}
&\|R_1(e^{s\alpha}v)\|^2_{L^2(Q)} + \|R_2(e^{s\alpha}v)\|^2_{L^2(Q)} + s\|e^{s\alpha}\nabla v\|^2_{L^2(Q)} + s^3\|e^{s\alpha}v\|^2_{L^2(Q)} \\
&\leq C\left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + \left\|e^{s\alpha}(\rho\partial_t u_2 - 2iA_0\cdot\nabla v - (i\nabla\cdot A_0 - |A_0|^2 - \rho_1)v)\right\|^2_{L^2(Q)} + s\mathcal{I}_s(-i\rho u_0)\right) \\
&\leq C\left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + \|e^{s\alpha}v\|^2_{L^2(Q)} + \|e^{s\alpha}\rho\|^2_{L^2(Q)} + \|e^{s\alpha}\nabla v\|^2_{L^2(Q)} + s\mathcal{I}_s(-i\rho u_0)\right),\ s \in (s_0,+\infty),
\end{aligned}$$

where $\mathcal{I}_s$ is defined in (41). In the last line we used the energy estimate (11) where u_2 is substituted for u. Further, upon taking $s_1 \in [\max(s_0,1),+\infty)$ so large that $s_1 \geq 2C$ (in such a way that $C(\|e^{s\alpha}\nabla v\|^2_{L^2(Q)} + \|e^{s\alpha}v\|^2_{L^2(Q)})$ is absorbed by $s\|e^{s\alpha}\nabla v\|^2_{L^2(Q)} + s^3\|e^{s\alpha}v\|^2_{L^2(Q)}$ for all $s \in (s_1,+\infty)$), we get that

$$\begin{aligned}
&\|R_1(e^{s\alpha}v)\|^2_{L^2(Q)} + \|R_2(e^{s\alpha}v)\|^2_{L^2(Q)} + s\|e^{s\alpha}\nabla v\|^2_{L^2(Q)} + s^3\|e^{s\alpha}v\|^2_{L^2(Q)} \\
&\leq C\left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + \|e^{s\alpha}\rho\|^2_{L^2(Q)} + s\mathcal{I}_s(-i\rho u_0)\right),\ s \in (s_1,+\infty).
\end{aligned}$$

From this and (46), it then follows that

$$\left\|e^{s\alpha(\cdot,0)}v(\cdot,0)\right\|^2_{L^2(\Omega)} \leq C\left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + s^{-\frac{3}{2}}\|e^{s\alpha}\rho\|^2_{L^2(Q)} + s^{-\frac{1}{2}}\mathcal{I}_s(-i\rho u_0)\right),\ s \in (s_1,+\infty).$$

Moreover, we have $\left\|e^{s\alpha(\cdot,0)}v(\cdot,0)\right\|^2_{L^2(\Omega)} = \left\|e^{s\alpha(\cdot,0)}\rho u_0\right\|^2_{L^2(\Omega)} \geq r_0^2\left\|e^{s\alpha(\cdot,0)}\rho\right\|^2_{L^2(\Omega)}$ for all $s \in (0,+\infty)$, from (13) and the third line of (47), hence

$$\left\|e^{s\alpha(\cdot,0)}\rho\right\|^2_{L^2(\Omega)} \leq C\left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + s^{-\frac{3}{2}}\|e^{s\alpha}\rho\|^2_{L^2(Q)} + s^{-\frac{1}{2}}\mathcal{I}_s(-i\rho u_0)\right),\ s \in (s_1,+\infty). \tag{48}$$

We are left with the task of estimating the two last terms appearing in the right-hand side of (48). For the first one, we take advantage of the fact, arising from (33) and (35), that

$$\alpha(x,t) \leq \alpha(x,0),\ (x,t) \in Q. \tag{49}$$

This yields

$$\|e^{s\alpha}\rho\|^2_{L^2(Q)} \leq \left\|e^{s\alpha(\cdot,0)}\rho\right\|^2_{L^2(Q)} = T\left\|e^{s\alpha(\cdot,0)}\rho\right\|^2_{L^2(\Omega)},\ s \in (0,+\infty). \tag{50}$$

For the second term, we infer from (41) and the last line of (47) that

$$\begin{aligned}
\mathcal{I}_s(-i\rho u_0) &= \int_\Omega e^{2s\alpha(x,0)}\left|\nabla\beta\cdot\left(|u_0|^2(\overline{\rho}\nabla\rho - \rho\nabla\overline{\rho}) + |\rho|^2(\overline{u_0}\nabla u_0 - u_0\nabla\overline{u_0})\right)(x)\right|dx \\
&\leq C\left(\int_\Omega e^{2s\alpha(x,0)}|\overline{\rho}\nabla\rho - \rho\nabla\overline{\rho}|(x)dx + \left\|e^{s\alpha(\cdot,0)}\rho\right\|^2_{L^2(\Omega)}\right),\ s \in (0,+\infty).
\end{aligned}$$

Thus, with reference to (14), entailing $|\overline{\rho}\nabla\rho - \rho\nabla\overline{\rho}| \leq C|\rho|^2$ a.e. in Ω, we get that

$$\mathcal{I}_s(-i\rho u_0) \leq C\left\|e^{s\alpha(\cdot,0)}\rho\right\|^2_{L^2(\Omega)},\ s \in (0,+\infty).$$

Now, putting this together with (48)-(50), we obtain

$$\left\|e^{s\alpha(\cdot,0)}\rho\right\|^2_{L^2(\Omega)} \leq C\left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + (s^{-\frac{1}{2}} + s^{-\frac{3}{2}})\left\|e^{s\alpha(\cdot,0)}\rho\right\|^2_{L^2(\Omega)}\right),\ s \in (s_1,+\infty).$$

Thus, taking $s_2 \in (s_1, +\infty)$ so large that $s_2^{-\frac{1}{2}} + s_2^{-\frac{3}{2}}$ is not greater than, say, $\frac{1}{2C}$, then the above estimate immediately yields

$$\left\| e^{s\alpha(\cdot,0)} \rho \right\|^2_{L^2(\Omega)} \leq C \|\partial_\nu v\|^2_{L^2(\Sigma_0)}, \ s \in (s_2, +\infty). \tag{51}$$

Finally, the desired result follows readily from this and the coming estimate, which follows from (33)–(35):

$$e^{s\alpha(x,0)} = e^{s\ell^{-2}(0)(e^{\lambda\beta(x)} - e^{\lambda K})} \geq e^{s\ell^{-2}(0)(1-e^{\lambda K})} \in (0, +\infty), \ x \in \Omega, \ s \in (0, +\infty). \tag{52}$$

4.3 Proof of Theorem 1.3

In light of (2), (30) and the first line of (44), we see that

$$Lv = 2iA\cdot\nabla\partial_t u_2 - (\rho + S\cdot A - i\nabla\cdot A)\,\partial_t u_2 + 2iA_1\cdot\nabla v - (\rho_1 + |A_1|^2 - i\nabla\cdot A_1)v.$$

Therefore we have $Lv \in L^2(0, T; L^2(\Omega))$, by (45), with

$$\|e^{s\alpha} Lv\|^2_{L^2(Q)} \leq C \left(\|e^{s\alpha}\rho\|^2_{L^2(Q)} + \|e^{s\alpha} A\|^2_{L^2(Q)} + \|e^{s\alpha}\nabla\cdot A\|^2_{L^2(Q)} + \|e^{s\alpha} v\|^2_{L^2(Q)} + \|e^{s\alpha}\nabla v\|^2_{L^2(Q)} \right), \ s \in (0, +\infty).$$

Here we used (11) where u is replaced by u_2. Next, since $v \in L^2(0, T; H^1_0(\Omega))$ and $\partial_\nu v \in L^2(0, T; L^2(\partial\Omega))$, we may apply the Carleman estimate of Theorem 3.1 and (43), getting

$$\begin{aligned}
&\|R_1(e^{s\alpha}v)\|^2_{L^2(Q)} + \|R_2(e^{s\alpha}v)\|^2_{L^2(Q)} + s\|e^{s\alpha}\nabla v\|^2_{L^2(Q)} + s^3\|e^{s\alpha}v\|^2_{L^2(Q)} \\
&\leq C \left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + \|e^{s\alpha}\rho\|^2_{L^2(Q)} + \|e^{s\alpha}A\|^2_{L^2(Q)} + \|e^{s\alpha}\nabla\cdot A\|^2_{L^2(Q)} + \|e^{s\alpha}v\|^2_{L^2(Q)} + \|e^{s\alpha}\nabla v\|^2_{L^2(Q)} + s\mathcal{I}_s(v(\cdot,0)) \right),
\end{aligned}$$

for all $s \in (s_0, +\infty)$, where $\mathcal{I}_s(v(\cdot, 0))$ is defined by (41) with $v(\cdot, 0) = -2A\cdot\nabla u_0 - i\,(\rho + S\cdot A - i\nabla\cdot A)\,u_0$, in virtue of the third line of (44). Taking $s_1 \in (s_0, +\infty)$ so large that $\min(s_1, s_1^3) \geq 2C$ then yields

$$\begin{aligned}
&\|R_1(e^{s\alpha}v)\|^2_{L^2(Q)} + \|R_2(e^{s\alpha}v)\|^2_{L^2(Q)} + s\|e^{s\alpha}\nabla v\|^2_{L^2(Q)} + s^3\|e^{s\alpha}v\|^2_{L^2(Q)} \\
&\leq C \left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + \|e^{s\alpha}\rho\|^2_{L^2(Q)} + \|e^{s\alpha}A\|^2_{L^2(Q)} + \|e^{s\alpha}\nabla\cdot A\|^2_{L^2(Q)} + s\mathcal{I}_s(v(\cdot,0)) \right), \ s \in (s_1, +\infty).
\end{aligned}$$

This and (46) imply

$$\begin{aligned}
\left\| e^{s\alpha(\cdot,0)} v(\cdot,0) \right\|^2_{L^2(\Omega)} &\leq Cs^{-\frac{3}{2}} \left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + \|e^{s\alpha}\rho\|^2_{L^2(Q)} + \|e^{s\alpha}A\|^2_{L^2(Q)} + \|e^{s\alpha}\nabla\cdot A\|^2_{L^2(Q)} + s\mathcal{I}_s(v(\cdot,0)) \right) \\
&\leq Cs^{-\frac{3}{2}} \left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + M_{A,\rho}(s) + s\mathcal{I}_s(v(\cdot,0)) \right), \ (s_1, +\infty),
\end{aligned} \tag{53}$$

with

$$M_{A,\rho}(s) := \left\| e^{s\alpha(\cdot,0)}\rho \right\|^2_{L^2(\Omega)} + \left\| e^{s\alpha(\cdot,0)}A \right\|^2_{L^2(\Omega)} + \left\| e^{s\alpha(\cdot,0)}\nabla\cdot A \right\|^2_{L^2(\Omega)}. \tag{54}$$

Here we applied the estimate $\|e^{s\alpha}Y\|_{L^2(Q)} \leq T^{\frac{1}{2}} \left\| e^{s\alpha(\cdot,0)}Y \right\|_{L^2(\Omega)}$, which follows from (49), to $Y = \rho, A$ and $\nabla\cdot A$, successively.

The next step of the proof is to show that

$$\mathcal{I}_s(v(\cdot,0)) \leq CM_{A,\rho}(s), \ s \in (0, +\infty). \tag{55}$$

To this end we start by noticing from the third line of (44) that

$$\left\| e^{s\alpha(\cdot,0)} v(\cdot,0) \right\|^2_{L^2(\Omega)} \leq CM_{A,\rho}(s), \ s \in (0, +\infty), \tag{56}$$

and that

$$\nabla v(\cdot,0) = -(w_0 + z_0), \tag{57}$$

with

$$\begin{aligned} w_0 &:= 2\mathbb{D}^2_{u_0} A + (\nabla \cdot A)\nabla u_0 + i\left(\rho + S \cdot A\right)\nabla u_0 + iu_0 \mathbb{J}_S A, && (58)\\ z_0 &:= 2\mathbb{J}_A \nabla u_0 + u_0 \nabla(\nabla \cdot A) + iu_0\left(\nabla \rho + \mathbb{J}_A S\right), && (59) \end{aligned}$$

where we set $\mathbb{D}^2_{u_0} := \left(\partial^2_{ij} u_0\right)_{1 \leq i,j \leq n}$ and $\mathbb{J}_Y := \left(\partial_i y_j\right)_{1 \leq i,j \leq n}$ for all $Y = (y_j)_{1 \leq j \leq n} \in H^1(\Omega, \mathbb{R}^n)$. Therefore we have $\overline{v(\cdot,0)}\nabla v(\cdot,0) - v(\cdot,0)\nabla\overline{v(\cdot,0)} = 2i\left(\operatorname{Im}\left(v(\cdot,0)\overline{w_0}\right) + \operatorname{Im}\left(v(\cdot,0)\overline{z_0}\right)\right)$ and consequently

$$\mathcal{I}_s((v(\cdot,0)) \leq C\left(\left\|e^{s\alpha(x,0)} v(\cdot,0)\right\|^2_{L^2(\Omega)} + \left\|e^{s\alpha(x,0)} w_0\right\|^2_{L^2(\Omega)} + \int_\Omega e^{2s\alpha(x,0)}\left|\operatorname{Im}\left(v(x,0)\overline{z_0(x)}\right)\right| dx\right),$$

for all $s \in (0,+\infty)$, by (41). Putting this together with (56) and the estimate

$$\left\|e^{s\alpha(\cdot,0)} w_0\right\|^2_{L^2(\Omega)} \leq CM_{A,\rho}(s),\ \ s \in (0,+\infty),$$

arising from (58), we find that

$$\mathcal{I}_s((v(\cdot,0)) \leq C\left(M_{A,\rho}(s) + \int_\Omega e^{2s\alpha(x,0)}\left|\operatorname{Im}\left(v(x,0)\overline{z_0(x)}\right)\right| dx\right),\ \ s \in (0,+\infty). \tag{60}$$

Further, since $\left\|e^{s\alpha(\cdot,0)} z_0\right\|_{L^2(\Omega)} \leq C\left(\left\|e^{s\alpha(\cdot,0)}\nabla\rho\right\|_{L^2(\Omega)} + \left\|e^{s\alpha(\cdot,0)}\mathbb{J}_A\right\|_{L^2(\Omega)} + \left\|e^{s\alpha(\cdot,0)}\nabla(\nabla \cdot A)\right\|_{L^2(\Omega)}\right)$, by (59), then we have

$$\left\|e^{s\alpha(\cdot,0)} z_0\right\|^2_{L^2(\Omega)} \leq CM_{A,\rho}(s),\ \ s \in (0,+\infty),$$

in virtue of the assumptions (15), (16) and (17). This and (56) yield

$$\int_\Omega e^{2s\alpha(x,0)}\left|\operatorname{Im}\left(v(x,0)\overline{z_0(x)}\right)\right| dx \leq CM_{A,\rho}(s),\ \ s \in (0,+\infty),$$

through the Cauchy-Schwarz inequality. In light of (60), we have obtained (55).

Let us now combine (53) with (55). We get that

$$\left\|e^{s\alpha(\cdot,0)} v(\cdot,0)\right\|^2_{L^2(\Omega)} \leq C\left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + s^{-\frac{1}{2}} M_{A,\rho}(s)\right),\ \ s \in (s_1,+\infty). \tag{61}$$

The last part of the proof is to lower estimate $\left\|e^{s\alpha(\cdot,0)} v(\cdot,0)\right\|^2_{L^2(\Omega)}$ by $M_{A,\rho}(s)$, up to some multiplicative constant which is independent of s. This can be done by referring once more to the third line of (44), giving

$$e^{s\alpha(\cdot,0)} v(\cdot,0) = -e^{s\alpha(\cdot,0)}\left(2A\cdot\nabla u_0 + i\left(\rho + S \cdot A - i\nabla\cdot A\right) u_0\right),\ \ s \in (0,+\infty), \tag{62}$$

and choosing $n+1$ times the initial state u_0 suitably, as described below.

First choice. We set $u_0 = u_0^0$, where u_0^0 is a non-zero constant of the complex plane, and we pick $g^0 \in H^{2(N+\frac{7}{4}),N+\frac{7}{4}}(\Sigma)$ such that the pair (u^0, g^0) fulfills (8) with $m = M$. The estimates (61)-(62) then yield

$$\left\|e^{s\alpha(\cdot,0)}(\rho + S\cdot A - i\nabla\cdot A)\right\|^2_{L^2(\Omega)} \leq C\left(\left\|\partial_\nu v^0\right\|^2_{L^2(\Sigma_0)} + s^{-\frac{1}{2}} M_{A,\rho}(s)\right),\ \ s \in (s_1,+\infty), \tag{63}$$

with $v^0 := \partial_t(u_1^0 - u_2^0)$.

Second choice. We are aiming for the same type of estimates as (63), where $\left\|e^{s\alpha(\cdot,0)}(\rho+S\cdot A-i\nabla\cdot A)\right\|_{L^2(\Omega)}$ is replaced by $\left\|e^{s\alpha(\cdot,0)}A\right\|_{L^2(\Omega)}$. It is clear from (62)-(63) that this can be achieved upon building a sufficiently rich set of $m\in\mathbb{N}$ initial states u_0^k, $k=1,\ldots,m$, such that $\left\|e^{s\alpha(\cdot,0)}A\right\|_{L^2(\Omega)}^2$ is upper bounded by $\sum_{k=1}^m \left\|e^{s\alpha(\cdot,0)}A\cdot\nabla u_0^k\right\|_{L^2(\Omega)}^2$, up to some multiplicative constant. Namely, we choose n functions $u_0^k:\Omega\to\mathbb{R}$, for $k=1,\ldots,n$, such that the matrix $U_0^*U_0$, where $U_0:=\left(\partial_l u_0^k\right)_{1\le k,l\le n}$ and U_0^* denotes the Hermitian conjugate matrix to U_0, is strictly positive definite, i.e. such that

$$\exists r_0\in(0,+\infty),\ |U_0\xi|\ge r_0|\xi|,\ \xi\in\mathbb{C}^n. \tag{64}$$

The strict positive definitness imposed by (64) on U_0 is rather standard in the context of multidimensional inverse coefficient problems, see e.g. [6, 11]. Here, bearing in mind that $e^{s\alpha(\cdot,0)}U_0A=U_0(e^{s\alpha(\cdot,0)}A)$ and applying (64) with $\xi=A$, we obtain

$$\sum_{k=1}^n\left\|e^{s\alpha(\cdot,0)}A\cdot\nabla u_0^k\right\|_{L^2(\Omega)}^2=\left\|e^{s\alpha(\cdot,0)}U_0A\right\|_{L^2(\Omega)}^2\ge r_0^2\left\|e^{s\alpha(\cdot,0)}A\right\|_{L^2(\Omega)}^2, \tag{65}$$

from the definition of U_0.

For each $k=1,\ldots,n$, we pick $g^k\in H^{2(N+\frac{7}{4}),N+\frac{7}{4}}(\Sigma)$ in such a way that (u_0^k,g^k) fulfills (8) with $m=N$, and we combine the well-known estimate

$$|\xi+\zeta|^2\ge\frac{1}{2}|\xi|^2-|\zeta|^2,\ \xi,\zeta\in\mathbb{C}^n, \tag{66}$$

where $\xi=2e^{s\alpha(\cdot,0)}A\cdot\nabla u_0^k$ and $\zeta=ie^{s\alpha(\cdot,0)}\left(\rho+S\cdot A-i\nabla\cdot A\right)u_0^k$, with (62). We find that

$$\left\|e^{s\alpha(\cdot,0)}v^k(\cdot,0)\right\|_{L^2(\Omega)}^2\ge2\left\|e^{s\alpha(\cdot,0)}A\cdot\nabla u_0^k\right\|_{L^2(\Omega)}^2-\left\|u_0^k\right\|_{L^\infty(\Omega)}^2\left\|e^{s\alpha(\cdot,0)}(\rho+S\cdot A-i\nabla\cdot A)\right\|_{L^2(\Omega)}^2,\ s\in(0,+\infty),$$

with $v^k:=\partial_t(u_1^k-u_2^k)$. Summing up the above estimate over $k=1,\ldots,n$ then yields

$$\sum_{k=1}^n\left\|e^{s\alpha(\cdot,0)}v^k(\cdot,0)\right\|_{L^2(\Omega)}^2\ge2\left\|e^{s\alpha(\cdot,0)}U_0A\right\|_{L^2(\Omega)}^2-C\left\|e^{s\alpha(\cdot,0)}(\rho+S\cdot A-i\nabla\cdot A)\right\|_{L^2(\Omega)}^2,\ s\in(0,+\infty),$$

by definition of U_0, and consequently

$$\sum_{k=1}^n\left\|e^{s\alpha(\cdot,0)}v^k(\cdot,0)\right\|_{L^2(\Omega)}^2\ge2r_0^2\left\|e^{s\alpha(\cdot,0)}A\right\|_{L^2(\Omega)}^2-C\left\|e^{s\alpha(\cdot,0)}(\rho+S\cdot A-i\nabla\cdot A)\right\|_{L^2(\Omega)}^2,\ s\in(0,+\infty),$$

by (65). This and (61) entail

$$\left\|e^{s\alpha(\cdot,0)}A\right\|_{L^2(\Omega)}^2\le C\left(\sum_{k=1}^n\left\|\partial_\nu v^k\right\|_{L^2(\Sigma_0)}^2+s^{-\frac{1}{2}}M_{A,\rho}(s)+\left\|e^{s\alpha(\cdot,0)}(\rho+S\cdot A-i\nabla\cdot A)\right\|_{L^2(\Omega)}^2\right),$$

for all $s\in(s_1,+\infty)$, and hence

$$\left\|e^{s\alpha(\cdot,0)}A\right\|_{L^2(\Omega)}^2\le C\left(\sum_{k=0}^n\left\|\partial_\nu v^k\right\|_{L^2(\Sigma_0)}^2+s^{-\frac{1}{2}}M_{A,\rho}(s)\right), \tag{67}$$

by (63). Further, in light of (66) with $\xi=e^{s\alpha(\cdot,0)}(\rho-i\nabla\cdot A)$ and $\zeta=e^{s\alpha(\cdot,0)}S\cdot A$, we have for all $s\in(0,+\infty)$,

$$\left\|e^{s\alpha(\cdot,0)}(\rho+S\cdot A-i\nabla\cdot A)\right\|_{L^2(\Omega)}^2\ge\frac{1}{2}\left\|e^{s\alpha(\cdot,0)}(\rho-i\nabla\cdot A)\right\|_{L^2(\Omega)}^2-\left(\|A_1\|_{L^\infty(\Omega)}+\|A_2\|_{L^\infty(\Omega)}\right)^2\left\|e^{s\alpha(\cdot,0)}A\right\|_{L^2(\Omega)}^2.$$

Putting this together with (63) and (67), we get that

$$\left\| e^{s\alpha(\cdot,0)}(\rho - i\nabla\cdot A)\right\|_{L^2(\Omega)}^2 \leq C\left(\sum_{k=0}^n \left\|\partial_\nu v^k\right\|_{L^2(\Sigma_0)}^2 + s^{-\frac{1}{2}} M_{A,\rho}(s)\right),\ s \in (s_1, +\infty). \tag{68}$$

Having established (68), we turn now to estimating $\left\|e^{s\alpha(\cdot,0)}\rho\right\|_{L^2(\Omega)}$ and $\left\|e^{s\alpha(\cdot,0)}\nabla\cdot A\right\|_{L^2(\Omega)}$ in terms of the boundary measurements $\left\|\partial_\nu v^k\right\|_{L^2(\Sigma_0)}^2$, $k = 0, \ldots, n$, and $M_{A,\rho}(s)$. Let us first notice that we have $\left\|e^{s\alpha(\cdot,0)}(\rho - i\nabla\cdot A)\right\|_{L^2(\Omega)}^2 = \left\|e^{s\alpha(\cdot,0)}\rho\right\|_{L^2(\Omega)}^2 + \left\|e^{s\alpha(\cdot,0)}\nabla\cdot A\right\|_{L^2(\Omega)}^2$ whenever the function ρ is real-valued, in which case (68) yields

$$\left\|e^{s\alpha(\cdot,0)}\rho\right\|_{L^2(\Omega)}^2 + \left\|e^{s\alpha(\cdot,0)}\nabla\cdot A\right\|_{L^2(\Omega)}^2 \leq C\left(\sum_{k=0}^n \left\|\partial_\nu v^k\right\|_{L^2(\Sigma_0)}^2 + s^{-\frac{1}{2}} M_{A,\rho}(s)\right),\ s \in (s_1, +\infty). \tag{69}$$

In the general case where $\rho : \Omega \to \mathbb{C}$, we combine the inequality $|\nabla \cdot A| \leq nM|A|$ in Ω, arising from (16), with (67). We obtain that

$$\left\|e^{s\alpha(\cdot,0)}\nabla\cdot A\right\|_{L^2(\Omega)}^2 \leq C\left(\sum_{k=0}^n \left\|\partial_\nu v^k\right\|_{L^2(\Sigma_0)}^2 + s^{-\frac{1}{2}} M_{A,\rho}(s)\right),\ s \in (s_1, +\infty).$$

This, (68) and the estimate $\left\|e^{s\alpha(\cdot,0)}\rho\right\|_{L^2(\Omega)} \leq \left\|e^{s\alpha(\cdot,0)}(\rho - i\nabla\cdot A)\right\|_{L^2(\Omega)} + \left\|e^{s\alpha(\cdot,0)}\nabla\cdot A\right\|_{L^2(\Omega)}$ for all $s \in (0, +\infty)$, yield (69).

Now, putting (67) and (69) together, and recalling (54), we find that

$$M_{A,\rho}(s) \leq C\left(\sum_{k=0}^n \left\|\partial_\nu v^k\right\|_{L^2(\Sigma_0)}^2 + s^{-\frac{1}{2}} M_{A,\rho}(s)\right),\ s \in (s_1, +\infty).$$

Therefore there exists $s_2 \in (s_1, +\infty)$ so large that $M_{A,\rho}(s) \leq C\sum_{k=0}^n \left\|\partial_\nu v^k\right\|_{L^2(\Sigma_0)}^2$ for all $s \in (s_2, +\infty)$, so the stability estimate of Theorem 1.3 follows from this and (52).

4.4 Proof of Theorem 1.4

We stick with the notations of Subsection 4.3 and we follow the same path as in the proof of Theorem 1.3, establishing (53). This shows existence of $s_1 \in (0, +\infty)$ such that the estimate

$$\left\|e^{s\alpha(\cdot,0)}v(\cdot,0)\right\|_{L^2(\Omega)}^2 \leq Cs^{-\frac{3}{2}}\left(\|\partial_\nu v\|_{L^2(\Sigma_0)}^2 + M_A(s) + s\mathcal{I}_s(v(\cdot,0))\right), \tag{70}$$

holds for all $s \in (s_1, +\infty)$, where

$$M_A(s) := M_{A,0}(s) = \left\|e^{s\alpha(\cdot,0)}A\right\|_{L^2(\Omega)}^2 + \left\|e^{s\alpha(\cdot,0)}\nabla \cdot A\right\|_{L^2(\Omega)}^2$$

is obtained by taking $\rho = 0$ in (54). However, in the framework of Theorem 1.4 where none of the three assumptions (15), (16) and (17) required by Theorem 1.3 is fulfilled, a more careful analysis is needed for majorizing (up to some s-independent multiplicative constant) $\mathcal{I}_s(v(\cdot,0))$ by $M_A(s)$.

To this end we recall from (60) that

$$\mathcal{I}_s((v(\cdot,0)) \leq C\left(M_A(s) + \int_\Omega e^{2s\alpha(x,0)}\left|\operatorname{Im}\left(v(x,0)\overline{z_0(x)}\right)\right|dx\right),\ s \in (0, +\infty), \tag{71}$$

where z_0 is defined from (59) with $\rho = 0$, i.e. $z_0 = 2\mathbb{J}_A \nabla u_0 + u_0 \nabla(\nabla \cdot A) + i u_0 \mathbb{J}_A S$. Moreover, we notice that

$$\mathbb{J}_S A + \mathbb{J}_A S = 0,$$

here. This comes from the assumption $|A_1| = |A_2|$ in Ω, entailing

$$0 = \partial_i \left(|A_1(x)|^2 - |A_2(x)|^2 \right) = \partial_i \left(\sum_{j=1}^n s_j(x) a_j(x) \right) = \sum_{j=1}^n (\partial_i s_j(x))\, a_j(x) + \sum_{j=1}^n (\partial_i a_j(x))\, s_j(x),\ x \in \Omega,$$

for each $i = 1, \ldots, n$, where we used the notations $S = A_1 + A_2 = (s_j)_{1 \le j \le n}$ and $A = (a_j)_{1 \le j \le n}$. Thus we have $z_0 = 2\mathbb{J}_A \nabla u_0 + u_0 \nabla(\nabla \cdot A) - i u_0 \mathbb{J}_S A$ and consequently

$$\mathrm{Im}\,(v(\cdot,0)\overline{z_0}) = \mathrm{Im}\,((2A \cdot \nabla u_0 + u_0 \nabla \cdot A)\,(2\mathbb{J}_A \nabla \overline{u_0} + \overline{u_0} \nabla(\nabla \cdot A))) + \mathrm{Re}\,(\overline{u_0}(2A \cdot \nabla u_0 + u_0 \nabla \cdot A)\mathbb{J}_S A)\,, \tag{72}$$

from the third line of (44), as $S \cdot A = |A_1|^2 - |A_2|^2 = 0$.

Next, since $A \in \mathbb{R}^n$ by assumption, we choose u_0 to be either real-valued or purely imaginary in Ω, in such a way that $\mathrm{Im}\,((2A \cdot \nabla u_0 + u_0 \nabla \cdot A)\,(2\mathbb{J}_A \nabla \overline{u_0} + \overline{u_0} \nabla(\nabla \cdot A))) = 0$. From this and (72) it then follows that

$$\int_\Omega e^{2s\alpha(x,0)} |\,\mathrm{Im}\,(v(x,0)\overline{z_0}(x))| dx \le C \int_\Omega e^{2s\alpha(x,0)} |\overline{u_0}(2A \cdot \nabla u_0 + u_0 \nabla \cdot A)\mathbb{J}_S A|(x) dx \le C M_A(s),\ s \in (0, +\infty).$$

Therefore we have $\mathcal{I}_s((v(\cdot,0)) \le C M_A(s)$ by (71), and hence

$$\left\| e^{s\alpha(\cdot,0)} v(\cdot,0) \right\|^2_{L^2(\Omega)} \le C \left(\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + s^{-\frac{1}{2}} M_A(s) \right),\ s \in (s_1, +\infty) \tag{73}$$

from (70).

The rest of the proof follows the same lines as the derivation of (63) and (67). Namely, by choosing$u_0 = u_0^0$ in (62), where $u_0^0(x) = r_0$ for some $r_0 \in \mathbb{R} \setminus \{0\}$ and a.e. $x \in \Omega$, we get $e^{s\alpha(\cdot,0)} v^0(\cdot,0) = i e^{s\alpha(\cdot,0)} r_0 \nabla \cdot A$ for every $s \in (0, +\infty)$. This leads to

$$\left\| e^{s\alpha(\cdot,0)} \nabla \cdot A \right\|^2_{L^2(\Omega)} \le C \left(\left\|\partial_\nu v^0\right\|^2_{L^2(\Sigma_0)} + s^{-\frac{1}{2}} M_A(s) \right),\ s \in (s_1, +\infty), \tag{74}$$

in virtue of (73). Further we consider n real-valued functions u_0^k, $k = 1, \ldots, n$, fulfilling (64), and we take $u_0 = u_0^k$ in (62). Then, by arguing as in the derivation of (67) where (74) is substituted for (63), we get that

$$\left\| e^{s\alpha(\cdot,0)} A \right\|^2_{L^2(\Omega)} \le C \left(\sum_{k=0}^n \left\|\partial_\nu v^k\right\|^2_{L^2(\Sigma_0)} + s^{-\frac{1}{2}} M_A(s) \right),\ s \in (s_1, +\infty),$$

Putting this together with (74) we end up getting that $M_A(s) \le C \left(\sum_{k=0}^n \left\|\partial_\nu v^k\right\|^2_{L^2(\Sigma_0)} + s^{-\frac{1}{2}} M_A(s) \right)$ for every $s \in (s_1, +\infty)$. The desired result follows upon taking $s \in \left(\max\left(s_1, \frac{1}{4C^2}\right), +\infty \right)$ in the above estimate.

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