Determination of source or initial values for acoustic equations with a time-fractional attenuation

Xinchi HUANG[1], Yavar KIAN[2], Éric SOCCORSI[2], Masahiro YAMAMOTO[3]

Dedicated to the memory of Victor Isakov (1947-2021)

Abstract We consider the inverse problems of determining the initial states or the source term of a hyperbolic equation damped by some non-local time-fractional derivative. This framework is relevant to medical imaging such as thermoacoustic or photoacoustic tomography. We prove a stability estimate for each of these two problems, with the aid of a Carleman estimate specifically designed for the governing equation.

Keywords time-fractional wave equation, inverse problem, stability, Carleman inequality.

AMS Subject Classifications 35R11, 35R30.

1 Introduction and main results

1.1 Settings and governing equation

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary (e.g., of $\mathcal{C}^2$-class). Put $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$, where $T > 0$ is arbitrarily fixed. In what follows $\mathcal{A}$ is the differential operator

$$\mathcal{A}u = -\frac{1}{\rho(x)} \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x)\partial_j u\right) + \sum_{j=1}^{n} b_j(x)\partial_j u + c(x)u, \quad x \in \overline{\Omega}$$

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[1]Institute for Innovative Research, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8550, Japan. E-mail: huang.x.ag@m.titech.ac.jp

[2]Aix-Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France. E-mails: yavar.kian@univ-amu.fr, eric.soccorsi@univ-amu.fr

[3]Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan; Honorary Member of Academy of Romanian Scientists, Ilfov, nr. 3, Bucuresti, Romania, Correspondence member of Accademia Peloritana dei Pericolanti, Palazzo Università, Piazza S. Pugliatti 1 98122 Messina, Italy. E-mail: myama@ms.u-tokyo.ac.jp

with real-valued coefficients $a_{ij} = a_{ji} \in \mathcal{C}^2(\overline{\Omega})$ for $i, j = 1, \ldots, n$, $b_j \in L^\infty(\Omega)$ for $j = 1, \ldots, n$, $c \in L^\infty(\Omega)$ and $\rho \in \mathcal{C}^2(\overline{\Omega})$. We assume that there exists $a_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in \overline{\Omega},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \tag{1}$$

and we suppose that $\rho_0 \leq \rho \leq \rho_1$ in Ω for some positive constants ρ_0 and ρ_1.

Let $q \in W^{1,\infty}(\Omega)$, where $W^{k,p}(X)$, $k \in \mathbb{N}$, $p \in \mathbb{N} \cup \{\infty\}$ denotes the usual k-th order Sobolev space of L^p functions on $X = (0, T)$ or $X = \Omega$, and let $\alpha \in W^{1,\infty}(\Omega)$ satisfy

$$0 < \alpha(x) \leq \alpha_1,\ x \in \Omega,$$

for some constant $\alpha_1 \in (0, 1)$. We introduce ∂_t^γ, the Caputo fractional derivative of order $\gamma \in (0, 1)$ with respect to t, as

$$\partial_t^\gamma u(t) := \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \partial_s u(s) ds,\ t > 0,$$

where Γ is the usual Gamma function, and we consider the following initial-boundary value problem for the time-fractional wave equation

$$\begin{cases} \partial_t^2 u(x,t) + q(x)\partial_t^\alpha u(x,t) + \mathcal{A}(x)u(x,t) = F(x,t), & (x,t) \in Q, \\ u(x,t) = 0, & (x,t) \in \Sigma, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{2}$$

where the fractional derivative ∂_t^α is pointwisely defined by

$$\partial_t^\alpha u(x,t) := \partial_t^{\alpha(x)} u(x,t) \quad (x,t) \in Q.$$

In the present paper, assuming that $\alpha(x) \in (0, 1)$, we examine the stability issue in the inverse problems of determining either the source term F or the initial conditions (u_0, u_1) from a single boundary measurement of the solution to (2).

1.2 Motivation and state of the art

The inverse problems under consideration in this paper are inspired by thermoacoustic or photoacoustic tomography (TAT or PAT) problems, such as the determination of an initial pressure generated by microwave or laser excitations from acoustic boundary measurements. These problems appear in several medical imaging applications (see e.g., [38]) and can be regarded as the first inversion in the multiwave imaging modalities (see e.g., [34]). Tomography problems for biological tissues are commonly modeled by acoustic equations with frequency-dependent attenuation terms (see [31, 36]), and sometimes in the form of time-fractional derivatives, see [7] and [37, Chapter 6] where acoustic dissipation obeying arbitrary frequency power law is expressed by a Caputo derivative of order $\alpha(x) \in (1, 3)$. In this paper, the hyperbolic equation (2) describes the time-evolution of the pressure in biological tissues and we investigate the inverse problems from the TAT or PAT inversion in heterogeneous tissues.

The inverse problems of determining source terms and initial conditions in hyperbolic equations have received much attention over the last decade from the mathematical community because of their applicability to seismology and imaging. One of the important approaches for solving these problems is the Bukhgeim-Klibanov method [5] based on so-called Carleman estimates (see also [4, 19, 22, 26]). As for other works on hyperbolic inverse source problems in a bounded domain, we can refer for example to [17, Chapter 7], [39, 40] and especially to [21] which is concerned with hyperbolic equations with time-dependent principal parts. Moreover in [35], the speed or a space-varying part of the source term of the wave equation are recovered with a single measurement. The above-cited articles are concerned with hyperbolic inverse source problems in a bounded spatial domain. Inverse source problems in an unbounded domain were studied in [3, 12, 13] and in [14], where the source term and an obstacle were retrieved. As for the recovery of initial data in the background of the TAT or PAT, we refer the reader to [2, 11, 30, 33].

All the aforementioned papers considered inverse problems for classical partial differential equations, i.e., partial differential equations without non-local terms. Inverse source and/or initial data problems for time-fractional diffusion equations were examined in [20, 23, 24, 25, 29], while such inverse problems were also considered for some other fractional differential operators in [6, 9]. However, to our best knowledge [1] is the only paper available in the mathematical literature dealing with the recovery of the initial values of a hyperbolic equation with a non-local term. In the present paper, we aim to further generalize the results of [1] by showing determination of either the initial states or the source term of a hyperbolic equation with a more general non-local perturbation. As a matter of fact, it is required by the method of [1] that the kernel of the fractional damping term is non-singular, whereas our approach works for singular kernels. With reference to [8], such a model is more relevant to the underlying physical problem.

1.3 Main results and outline

We start with the unique existence and the regularity of the solution to the initial-boundary value problem (2), needed for the analysis of the inverse problems carried out in this article.

Theorem 1.1. *Let $u_0 \in H_0^1(\Omega)$, let $u_1 \in L^2(\Omega)$ and let $F \in L^1(0,T;L^2(\Omega))$. Then, there exists a unique solution $u \in \mathcal{C}([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$ to (2), satisfying*

$$\|u\|_{\mathcal{C}([0,T];H_0^1(\Omega))} + \|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))} \leq C \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|F\|_{L^1(0,T;L^2(\Omega))} \right) \tag{3}$$

for some positive constant C depending only on Ω, T, ρ, α, a_{ij}, b_j, c and q.

Moreover, if $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $F \in H^1(0,T;L^2(\Omega))$ then we have $u \in \mathcal{C}([0,T];H^2(\Omega) \cap$

$H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^2([0,T]; L^2(\Omega))$ *and the estimate*

$$\|u\|_{\mathcal{C}([0,T];H^2(\Omega))} + \|u\|_{\mathcal{C}^1([0,T];H^1(\Omega))} + \|u\|_{\mathcal{C}^2([0,T];L^2(\Omega))} \leq C \left(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|F\|_{H^1(0,T;L^2(\Omega))} \right). \tag{4}$$

The proof the above statement is rather classical but we have not found any reference for this result stated in the mathematical literature. For self-contained description and the convenience of the reader, the proof of Theorem 1.1 is given in Section 2.

The key achievement of this article is the Lipschitz-stable determination of either the initial states u_0 and u_1 or the source term F, appearing in (2), by one local Neumann data. This dual identification result is established through two different stability inequalities that we will make precise below. For the sake of simplicity, we state these estimates in the peculiar framework where

$$\rho = 1 \text{ in } \Omega \quad \text{ and } \quad a_{ij} = \delta_{ij} \text{ in } \Omega,\ 1 \leq i,j \leq n.$$

That is, we consider the following initial-boundary value problem associated with α, q in $W^{1,\infty}(\Omega)$ and $b_j, c \in L^\infty(\Omega)$, $1 \leq j \leq n$,

$$\begin{cases} \partial_t^2 u(x,t) + q(x)\partial_t^\alpha u(x,t) - \Delta u(x,t) + B(x)\cdot\nabla u(x,t) + c(x)u(x,t) = F(x,t), & (x,t) \in Q, \\ u(x,t) = 0, & (x,t) \in \Sigma, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{5}$$

instead of (2). Here and below we use the notation $B := (b_1, \ldots, b_n)^T$ and we suppose that we have

$$\sum_{j=1}^n \|b_j\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} + \|\alpha\|_{W^{1,\infty}(\Omega)} + \|q\|_{W^{1,\infty}(\Omega)} \leq M$$

for some *a-priori* fixed positive constant M. In this context, our first main result establishes that one local Neumann data stably determines either of the two initial states u_0 or u_1, when the other one and the source term F are known.

Theorem 1.2. *There exist $T_0 > 0$ and a sub-boundary $\Gamma_0 \subset \partial\Omega$ such that for all $T \geq T_0$, all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying either*

$$\text{(i)}\ \ u_0 = 0 \ \textit{in}\ \Omega \qquad \textit{or} \qquad \text{(ii)}\ \ u_1 = 0\ \textit{in}\ \Omega, \tag{6}$$

and all $F \in L^2(Q)$, we have

$$\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \leq C \left(\|F\|_{L^2(Q)} + \|\partial_\nu u\|_{L^2(\Gamma_0 \times (0,T))} \right).$$

Here C is a positive constant depending only on Ω, T, Γ_0 and M, and u is the $\mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$-solution to the initial-boundary value problem (5).

We stress out that the sub-boundary Γ_0 is not arbitrary here. Our arguments are based on the hyperbolic equation, that is, $q \equiv 0$, and so necessary data for the inverse problem must be subject to the hyperbolicity of the equation which requires some larger portion Γ_0 of $\partial\Omega$ and long observation time length $T \geq T_0$. As a matter of fact we will see below that it is required that Γ_0 is taken sufficiently large in order to satisfy the geometric condition (30) for some $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, which is the same condition for the case of a classical hyperbolic equation (see e.g., [4, Chapter 4]).

The second main result of this paper claims stable recovery of the spatial varying factor in the source term, by one local boundary observation of the solution. More precisely, if we assume that F is of the form

$$F(x,t) = R(x,t)f(x),\ (x,t) \in Q, \tag{7}$$

where $R \in W^{1,\infty}(Q)$ is known and satisfies

$$|R(x,0)| \geq r_0, \quad x \in \overline{\Omega} \tag{8}$$

for some $r_0 > 0$, then one local boundary observation of the solution to (5) stably determines the unknown function f in $L^2(\Omega)$.

Theorem 1.3. *There exist $T_0 > 0$ and a sub-boundary Γ_0 such that for all $T \geq T_0$, all $f \in L^2(\Omega)$ and all $R \in W^{1,\infty}(Q)$ satisfying* (8)*, the following estimate*

$$\|f\|_{L^2(\Omega)} \leq C\|\partial_t(\partial_\nu u)\|_{L^2(\Gamma_0\times(0,T))}$$

holds for some positive constant C depending only on Ω, T, Γ_0, M and r_0. Here u denotes the $\mathcal{C}^2([0,T];L^2(\Omega))\cap \mathcal{C}^1([0,T];H_0^1(\Omega))$-solution to the initial-boundary value problem (5) *where $u_0 = u_1 = 0$ in Ω and F is defined by* (7)*.*

In both theorems, the constants $C > 0$ involve exponentials of quantities including norms of coefficients of the equation and the weight function of the key Carleman estimate, and so may be quite large. Our problems are not so-called forward problems for partial differential equations, but inverse problems which might be instable and ill-posed. However our main theorems establish the Lipschitz stability without any a-priori information of solutions. It is an important topic to minimize constants in the stability estimates for inverse problems, and in general, it is expected that the constant heavily depends on choices of classes of unknowns and even stability rates, which require independent works.

Theorems 1.2 and 1.3 are proved in Section 4. We leave it to the reader to check that these two statements remain valid when the master equation (5) is replaced by

$$\partial_t^2 u(x,t) + \sum_{k=1}^{N} q_k(x)\partial_t^{\alpha_k} u(x,t) + \mathcal{A}(x)u(x,t) = F(x,t), \quad (x,t) \in \Omega \times (0,T)$$

with suitable multi-order coefficients $0 < \alpha_k < 1$ in Ω, $k = 1, \dots, N$ and appropriate elliptic operator $\mathcal{A}$ (see e.g., [4, 15]). Moreover, we point out that the case of t-dependent coefficients can be handled by adapting the strategy developed in [41] for a classical hyperbolic equation, to the framework of Theorems 1.2 and 1.3.

To our best knowledge, Theorems 1.2 and 1.3 are the only mathematical results on the recovery of the source term and the initial conditions of a hyperbolic equation with a time-fractional damping term. There is a slightly similar result in [1] though, where the initial conditions of a hyperbolic equation with some non-local zeroth order term associated with a $\mathcal{C}^2(\overline{Q})$-kernel, are retrieved. On the other hand, the fractional attenuation that we consider in the present paper is generated by the singular kernel $t \mapsto \frac{t^{-\alpha(x)}}{\Gamma(1-\alpha(x))}$. We refer the reader to [8] for the physical relevance of non-local singular perturbations of hyperbolic systems.

The derivation of Theorems 1.2 and 1.3 boils down to a specific Carleman inequality (and a suitable energy estimate) for (5), stated in Proposition 3.1. More precisely, we prove Theorem 1.2 by applying the Bukhgeim-Klibanov method. When the attenuation is switched off ($q = 0$ in Ω), this is a well-known approach for solving inverse problems associated with (2). However, when $q \neq 0$, this is no longer the case since the fractional damping term $q\partial_t^\alpha u$ is non-local.

We point out that the approach developed in this article can not be applied to $\alpha \in (1, 2)$, for both the forward problem and the inverse problems solved in, respectively, Theorem 1.1 and Theorems 1.2 and 1.3. Indeed, a crucial point in the proofs of these results is the estimate of Lemma 3.2, which ensures that the damping term $q\partial_t^\alpha$ can be treated as a lower-order perturbation of a hyperbolic equation. This allows us to prove the well-posedness of (5) by means of a fixed point argument, and to derive the Carleman estimate in Proposition 3.1 by absorbing the damping term $q\partial_t^\alpha$. Since Lemma 3.2 is no longer valid for $\alpha \in (1, 2)$, this technique does not work in the super-diffusive case and the corresponding forward/inverse problems should be considered differently as a future work.

The results of Theorems 1.1, 1.2 and 1.3 are known to be true for the classical hyperbolic equations corresponding to either $\alpha = 1$ or $q = 0$. Moreover, the Carleman estimate in Proposition 3.1 is similar to the ones for the classical hyperbolic equations. The main contribution of this article is given by the extension of such properties to the hyperbolic equation (5) with the presence of the time-fractional damping term $q\partial_t^\alpha$. The main difference between the governing equation (5) and the classical hyperbolic equations, comes from the presence of the nonlocal term $q\partial_t^\alpha$ for which properties such as integration by parts and the chain rule are no longer valid. For this reason, Theorems 1.2 and 1.3 as well as Proposition 3.1 can not be proved by directly applying the arguments used for the classical hyperbolic equations. To overcome this difficulty, we introduce a new energy estimate in Lemma 2.1 and we prove the Carleman estimate of Proposition 3.1 by estimating the time-fractional damping term in Lemma 3.2.

The remaining part of this paper is organized as follows. In Section 2, we discuss the unique existence of the solution to (2) and establish some suitable energy estimate. In Section 3, we design a Carleman inequality for the master equation (5). The proofs of Theorems 1.2 and 1.3, which boil down to the energy estimate of Section 2 and the Carleman inequality of Section 3, are given in Section 4.

2 Analysis of the forward problem

In this section we prove Theorem 1.1 and we establish an energy estimate for the solution to the initial-boundary value problem (2), needed for the proof of Theorems 1.2 and 1.3.

2.1 Proof of Theorem 1.1

We shall derive Theorem 1.1 from a fixed-point theorem argument and a classical unique existence result for hyperbolic equations. We start by showing existence within the class $\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$ of the solution to (2), depending continuously on the initial values and the source term.

2.1.1 Solving the direct problem in $\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$

For $(u_0,u_1)\in\mathcal{H}:=H_0^1(\Omega)\times L^2(\Omega)$, we consider the $\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$-solution u to (2) with $q=0$ and $F=0$, given by [28, Chap. 1, Lemma 3.6]. Then, for each $t\in[0,T]$, we introduce the operator

$$U_0(t):(u_0,u_1)\to(u(t),\partial_t u(t)) \tag{9}$$

and recall that $t\mapsto U_0(t)\in\mathcal{C}([0,T];\mathcal{B}(\mathcal{H}))$. Here and henceforth, we denote by $\mathcal{B}(X,Y)$ the set of linear bounded operators from the Banach space X to the Banach space Y, and we write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,X)$.

If $q=0$ and $F\in L^1(0,T;L^2(\Omega))$, then it is well known that (2) admits a unique solution u such that $U_1:=(u,\partial_t u)\in\mathcal{C}([0,T];\mathcal{H})$ reads

$$U_1(t)=U_0(t)(u_0,u_1)+\int_0^t U_0(t-s)(0,F(s))ds,\ t\in[0,T], \tag{10}$$

and satisfies the estimate

$$\|U_1\|_{\mathcal{C}([0,T];\mathcal{H})}\leq C_0\left(\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|F\|_{L^1(0,T;L^2(\Omega))}\right), \tag{11}$$

where the constant $C_0:=\|U_0\|_{\mathcal{C}([0,T];\mathcal{B}(\mathcal{H}))}$ depends only on Ω, T, ρ, c, b_j and a_{ij} for $1\leq i,j\leq n$.

If q is not uniformly zero we put

$$Q(t):=-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\begin{pmatrix}0&0\\0&q\end{pmatrix},\ t\in(0,T] \tag{12}$$

and infer from Duhamel's principle that $u\in\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$ solves (2) if and only if $U:=(u,\partial_t u)$ is a $\mathcal{C}([0,T];\mathcal{H})$-solution to the integral equation

$$U(t)=(\mathcal{G}U)(t):=U_1(t)+\int_0^t\int_0^s U_0(t-s)Q(s-\tau)U(\tau)d\tau ds,\ t\in[0,T]. \tag{13}$$

Thus it is enough to show that some iterate $\mathcal{G}^m$ of $\mathcal{G}$ is a contraction mapping on $\mathcal{C}([0,T];\mathcal{H})$. To do that, we start by noticing that for all $t\in(0,T]$ we have $Q(t)\in\mathcal{B}(\mathcal{H})$ fulfilling

$$\|Q(t)\|_{\mathcal{B}(\mathcal{H})}\leq C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)},\tag{14}$$

where $C_1:=\Gamma(1-\alpha_1)\max\{1,T^{\alpha_1}\}\|q\|_{L^\infty(\Omega)}$. Therefore, setting $\tilde{\mathcal{G}}V:=\mathcal{G}V-U_1$ for all $V\in\mathcal{C}([0,T];\mathcal{H})$, i.e.,

$$(\tilde{\mathcal{G}}V)(t):=\int_0^t\int_0^s U_0(t-s)Q(s-\tau)V(\tau)d\tau ds,\ t\in[0,T],\ V\in\mathcal{C}([0,T];\mathcal{H}),$$

we get that $\left\|(\tilde{\mathcal{G}}V)(t)\right\|_{\mathcal{H}}\leq C\int_0^t\int_0^s\frac{(s-\tau)^{-\alpha_1}}{\Gamma(1-\alpha_1)}\|V(\tau)\|_{\mathcal{H}}d\tau ds$ with $C:=C_0C_1$. As a consequence we have

$$\begin{aligned}\left\|(\tilde{\mathcal{G}}V)(t)\right\|_{\mathcal{H}}&\leq C\int_0^t\int_\tau^t\frac{(s-\tau)^{-\alpha_1}}{\Gamma(1-\alpha_1)}\|V(\tau)\|_{\mathcal{H}}dsd\tau\\&\leq C\int_0^t\frac{(t-\tau)^{1-\alpha_1}}{\Gamma(2-\alpha_1)}\|V(\tau)\|_{\mathcal{H}}d\tau\end{aligned}\tag{15}$$

by Fubini's theorem. From this and (11)-(13) we see that $\mathcal{G}$ maps $\mathcal{C}([0,T];\mathcal{H})$ into itself. Next, by iterating (15) we get in the same manner as in the proof of [10, Proposition 1] (see also [32, Section 2.3.6]) that

$$\left\|(\tilde{\mathcal{G}}^mV)(t)\right\|_{\mathcal{H}}\leq C^m\int_0^t\frac{(t-\tau)^{m(2-\alpha_1)-1}}{\Gamma(m(2-\alpha_1))}\|V(\tau)\|_{\mathcal{H}}d\tau,\ t\in(0,T),\ m\in\mathbb{N}.$$

Thus, for all $V,W\in\mathcal{C}([0,T];\mathcal{H})$, all $m\in\mathbb{N}$ and all $t\in[0,T]$ we have

$$\|(\mathcal{G}^mV)(t)-(\mathcal{G}^mW)(t)\|_{\mathcal{H}}=\left\|(\tilde{\mathcal{G}}^m(V-W))(t)\right\|_{\mathcal{H}}\leq C^m\int_0^t\frac{(t-\tau)^{m(2-\alpha_1)-1}}{\Gamma(m(2-\alpha_1))}\|(V-W)(\tau)\|_{\mathcal{H}}d\tau$$

and consequently

$$\begin{aligned}\|\mathcal{G}^mV-\mathcal{G}^mW\|_{\mathcal{C}([0,T];\mathcal{H})}&\leq C^m\left(\int_0^T\frac{t^{m(2-\alpha_1)-1}}{\Gamma(m(2-\alpha_1))}dt\right)\|V-W\|_{\mathcal{C}([0,T];\mathcal{H})}\\&\leq\frac{C^mT^{m(2-\alpha_1)}}{\Gamma(m(2-\alpha_1)+1)}\|V-W\|_{\mathcal{C}([0,T];\mathcal{H})}\end{aligned}\tag{16}$$

by Young's convolution inequality. Taking $m\in\mathbb{N}$ so large that $\frac{C^mT^{m(2-\alpha_1)}}{\Gamma(m(2-\alpha_1)+1)}<1$, which is possible since $\lim_{p\to+\infty}\frac{C^pT^{p(2-\alpha_1)}}{\Gamma(p(2-\alpha_1)+1)}=0$, we obtain that $\mathcal{G}^m$ is contractive on $\mathcal{C}([0,T];\mathcal{H})$. Therefore, $\mathcal{G}$ admits a unique fixed point $U\in\mathcal{C}([0,T];\mathcal{H})$ by the Banach fixed-point theorem.

Further, putting (11), (13) and (15) together, we get that

$$\begin{aligned}\|U(t)\|_{\mathcal{H}}&\leq C\int_0^t\frac{(t-\tau)^{1-\alpha_1}}{\Gamma(2-\alpha_1)}\|U(\tau)\|_{\mathcal{H}}d\tau+C_0\left(\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|F\|_{L^1(0,T;L^2(\Omega))}\right)\\&\leq C\frac{\max\{1,T^{1-\alpha_1}\}}{\Gamma(2-\alpha_1)}\int_0^t\|U(\tau)\|_{\mathcal{H}}d\tau+C_0\left(\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|F\|_{L^1(0,T;L^2(\Omega))}\right).\end{aligned}$$

This and Grönwall's inequality yield (3).

We turn now to proving the second claim of Theorem 1.1.

2.1.2 Improved regularity

We use the same notations and we follow the same path as in Section 2.1.1, the initial values (u_0, u_1) being taken in $\mathcal{H} := (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and the source term F in $H^1(0,T;L^2(\Omega))$. Namely, with reference to [27, Theorem 2.3 in Chapter 4 and Theorem 7.1 in Chapter 5], the mapping $U_0 \in \mathcal{C}([0,T];\mathcal{B}(\mathcal{H}))$ is defined by (9) where u denotes the $\mathcal{C}([0,T];H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([0,T],H_0^1(\Omega))$-solution to (2) with $q=0$ and $F=0$. Then, we define $U_1 \in \mathcal{C}([0,T];\mathcal{H})$ by (10) and we replace the estimate (11) by

$$\|U_1\|_{\mathcal{C}([0,T];\mathcal{H})} \leq C_0 \left(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|F\|_{H^1(0,T;L^2(\Omega))} \right), \tag{17}$$

where the constant $C_0 = \|U_0\|_{\mathcal{C}([0,T];\mathcal{B}(\mathcal{H}))}$ still depends on Ω, T, ρ, c, b_j and a_{ij} for $1 \leq i,j \leq n$. Now, since the $\mathcal{C}([0,T];H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];H_0^1(\Omega))$-solution to (2) is the first component of the $\mathcal{C}([0,T];\mathcal{H})$-solution to the integral equation (13), we are left with the task of showing that some iterate of $\mathcal{G}$ is a contractive map on $\mathcal{C}([0,T];\mathcal{H})$. This essentially boils down to the following estimate

$$\|Q(t)\|_{\mathcal{B}(\mathcal{H})} \leq C_2 \frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)},\ t \in (0,T), \tag{18}$$

where $\alpha_2 := \frac{1+\alpha_1}{2}$ and C_2 is a positive constant depending only on Ω, T, α and q. Indeed, since α and q lie in $W^{1,\infty}(\Omega)$, then it is clear from (12) that $Q(t) \in \mathcal{B}(\mathcal{H})$ for all $t \in (0,T]$. Moreover, we have

$$\begin{aligned} \nabla \left(\frac{q t^{-\alpha}}{\Gamma(1-\alpha)} \right) &= \nabla \left(\frac{q}{\Gamma(1-\alpha)} \right) t^{-\alpha} + \frac{q}{\Gamma(1-\alpha)} \nabla t^{-\alpha} \\ &= \Gamma(1-\alpha_2) t^{\alpha_2-\alpha} \left(\nabla \left(\frac{q}{\Gamma(1-\alpha)} \right) - \frac{q \nabla \alpha}{\Gamma(1-\alpha)} \ln t \right) \frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)} \end{aligned}$$

by direct computation, hence by taking into account that $t \mapsto t^{\alpha_2-\alpha}(1+|\ln t|)$ is uniformly bounded in $(0,T]$, we get that

$$\left\| \nabla \left(\frac{q t^{-\alpha}}{\Gamma(1-\alpha)} \right) \right\|_{\mathcal{B}(H^1(\Omega);L^2(\Omega))} \leq C \frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)},\ t \in (0,T],$$

for some positive constant C depending only on Ω, T, α and q, which may change from line to line. Now, (18) follows readily from this and the estimate $\left\| \frac{q t^{-\alpha}}{\Gamma(1-\alpha)} \right\|_{\mathcal{B}(L^2(\Omega))} \leq C_1 \frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)}$ arising from (14).

Having established (18), we argue as in the derivation of (16) from (14), where (18) is substituted for (14), and find that (13) admits a unique solution $U \in \mathcal{C}([0,T];\mathcal{H})$ satisfying

$$\|U\|_{\mathcal{C}([0,T];\mathcal{H})} \leq C \left(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|F\|_{H^1(0,T;L^2(\Omega))} \right).$$

Here and below, C denotes a positive constant depending only on T, Ω, ρ, α, q, c, b_j and a_{ij} for $1 \leq i,j \leq n$, which may change from line to line. As a consequence, there exists a unique solution $u \in \mathcal{C}([0,T];H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];H_0^1(\Omega))$ to (2), such that

$$\|u\|_{\mathcal{C}([0,T];H^2(\Omega))} + \|u\|_{\mathcal{C}^1([0,T];H^1(\Omega))} \leq C \left(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} + \|F\|_{H^1(0,T;L^2(\Omega))} \right). \tag{19}$$

Since $\partial_t^2 u = F - q\partial_t^\alpha u - \mathcal{A}u \in \mathcal{C}([0,T];L^2(\Omega))$, we get that $u \in \mathcal{C}^2([0,T];L^2(\Omega))$. Moreover, using that $\|u\|_{\mathcal{C}^2([0,T];L^2(\Omega))} = \|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \left\|\partial_t^2 u\right\|_{\mathcal{C}([0,T];L^2(\Omega))}$, we obtain

$$\begin{aligned}\|u\|_{\mathcal{C}^2([0,T];L^2(\Omega))} &\leq \|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|\mathcal{A}u\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|q\|_{L^\infty(\Omega)}\|\partial_t^\alpha u\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|F\|_{\mathcal{C}([0,T];L^2(\Omega))} \\ &\leq C\left(\|u\|_{\mathcal{C}([0,T];H^2(\Omega))} + \|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|F\|_{H^1(0,T;L^2(\Omega))}\right).\end{aligned}$$

This and (19) yield (4), which completes the proof of Theorem 1.1.

2.2 Energy estimate

We turn now to studying the time-evolution of the energy

$$E(t) := \int_\Omega \left(\frac{1}{\rho(x)}\sum_{i,j=1}^n a_{ij}(x)\partial_i u(x,t)\partial_j u(x,t) + |\partial_t u(x,t)|^2\right)dx,\ t \in [0,T], \tag{20}$$

of the time-fractional hyperbolic system (2) with initial data $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and source term $F \in L^2(Q)$. Here, u denotes the $\mathcal{C}([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$-solution to (2), given by Theorem 1.1.

The result we have in mind is as follows.

Lemma 2.1. *There exists a constant $C > 0$, depending on T, Ω, α, ρ, a_{ij}, b_j for $1 \leq i,j \leq n$, c and q, such that the two following estimates hold for all $t \in (0,T]$:*

$$E(t) \leq C\left(E(0) + \|F\|_{L^2(Q)}^2\right), \tag{21}$$

$$E(0) \leq \frac{C}{t}\left(\int_0^t E(s)ds + \|F\|_{L^2(Q)}^2\right). \tag{22}$$

Proof. For all $t \in [0,T]$, we infer from (1) and (20) that

$$\min\left\{1,\frac{a_0}{\rho_1}\right\}\left(\|\nabla u(\cdot,t)\|_{L^2(\Omega)^n}^2 + \|\partial_t u(\cdot,t)\|_{L^2(\Omega)}^2\right) \leq E(t) \leq \max\left\{1,\frac{a_1}{\rho_0}\right\}\left(\|\nabla u(\cdot,t)\|_{L^2(\Omega)^n}^2 + \|\partial_t u(\cdot,t)\|_{L^2(\Omega)}^2\right), \tag{23}$$

where we have set $a_1 := \max_{1\leq i\leq j\leq n}\|a_{ij}\|_{\mathcal{C}(\overline{\Omega})}$, hence (21) follows immediately from this and (3).

We turn now to proving (22). We proceed in two steps.

Step 1. First we establish (22) for $u \in \mathcal{C}^2([0,T];L^2(\Omega)) \cap \mathcal{C}^1([0,T];H_0^1(\Omega)) \cap \mathcal{C}([0,T];H^2(\Omega) \cap H_0^1(\Omega))$. Namely, in accordance with Theorem 1.1 we consider a solution u to (2) with initial states $(u_0,u_1,F) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times H^1(0,T;L^2(\Omega))$ and we multiply both sides of (2) by $\partial_t u$. Then, integrating over Ω we get

$$\int_\Omega \partial_t u\ \partial_t^2 u\ dx + \int_\Omega q\partial_t u\ \partial_t^\alpha u\ dx + \int_\Omega \partial_t u\Big(-\frac{1}{\rho}\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u) + B\cdot\nabla u + cu\Big)dx = \int_\Omega \partial_t u\ F\ dx$$

in $(0,T)$. Bearing in mind that $u=0$ on Σ, we integrate by parts in $\int_\Omega \partial_t u\left(-\frac{1}{\rho}\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u)\right)dx$ and obtain for all $\tau \in (0,T]$ and all $s \in (0,\tau)$ that

$$-\frac{1}{2}\frac{d}{ds}E(s) = \left\langle \partial_s u(\cdot,s), q\partial_s^\alpha u(\cdot,s) + \sum_{i,j=1}^n a_{ij}\partial_i\Big(\frac{1}{\rho}\Big)\partial_j u(\cdot,s) + B\cdot\nabla u(\cdot,s) + cu(\cdot,s) - F(\cdot,s)\right\rangle_{L^2(\Omega)},$$

where the symbol $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ denotes the usual scalar product in $L^2(\Omega)$. Thus, by Hölder's inequality, we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{ds}E(s) &\leq C\left(\|\partial_s u(\cdot,s)\|_{L^2(\Omega)}^2 + \|\partial_s^\alpha u(\cdot,s)\|_{L^2(\Omega)}^2 + \|u(\cdot,s)\|_{H^1(\Omega)}^2 + \|F(\cdot,s)\|_{L^2(\Omega)}^2\right) \\ &\leq C\left(E(s) + \|\partial_s^\alpha u(\cdot,s)\|_{L^2(\Omega)}^2 + \|F(\cdot,s)\|_{L^2(\Omega)}^2\right),\ s \in (0,\tau), \end{aligned} \tag{24}$$

where C is a positive constant depending on Ω, ρ, a_{ij}, b_j, c and q. In the last line, we applied the Poincaré inequality to $u(\cdot,s) \in H_0^1(\Omega)$ and used the left-hand side of (23). Further, integrating (24) over $(0,\tau)$ yields

$$E(0) \leq E(\tau) + C\left(\int_0^\tau E(s)ds + \|\partial_t^\alpha u\|_{L^2(\Omega\times(0,\tau))}^2 + \|F\|_{L^2(\Omega\times(0,\tau))}^2\right),\ \tau \in (0,T]. \tag{25}$$

Next, applying Young's convolution inequality, we get that $\|\partial_t^\alpha u\|_{L^2(\Omega\times(0,\tau))} \leq \kappa \|\partial_t u\|_{L^2(\Omega\times(0,\tau))}$ for some positive constant κ depending only on α and T, hence (25) reads

$$E(0) \leq E(\tau) + C\left(\int_0^\tau E(s)ds + \|F\|_{L^2(\Omega\times(0,\tau))}^2\right),\ \tau \in (0,T], \tag{26}$$

where the constant C depends not only on Ω, ρ, a_{ij}, b_j, c and q, but also on T and α. Now, for $t \in (0,T]$ fixed, we integrate (26) with respect to τ over $(0,t)$ and find that

$$tE(0) \leq (1+CT)\int_0^t E(s)ds + CT\|F\|_{L^2(\Omega\times(0,t))}^2,$$

which yields (22) for $(u_0,u_1) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and $F \in H^1(0,T;L^2(\Omega))$. We turn now to extending this result to the case where $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ and $F \in L^2(Q)$.

Step 2. Let us consider the $\mathcal{C}([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$-solution u to (2) associated with $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ and $F \in L^2(Q)$, given by Theorem 1.1. By density of $\mathcal{C}_c^\infty(\Omega)$ in $H_0^1(\Omega)$ and in $L^2(\Omega)$, and of $\mathcal{C}_c^\infty(Q)$ in $L^2(Q)$, there exist three sequences $\{u_{0,m}\}_{m=1}^\infty \in \mathcal{C}_c^\infty(\Omega)^{\mathbb{N}}$, $\{u_{1,m}\}_{m=1}^\infty \in \mathcal{C}_c^\infty(\Omega)^{\mathbb{N}}$ and $\{F_m\}_{m=1}^\infty \subset \mathcal{C}_c^\infty(Q)^{\mathbb{N}}$ such that

$$\lim_{m\to+\infty}\|u_0 - u_{0,m}\|_{H_0^1(\Omega)} = 0, \quad \lim_{m\to+\infty}\|u_1 - u_{1,m}\|_{L^2(\Omega)} = 0, \quad \lim_{m\to+\infty}\|F - F_m\|_{L^2(Q)} = 0. \tag{27}$$

For $m \in \mathbb{N}$, we denote by u_m the $\mathcal{C}^2([0,T];L^2(\Omega)) \cap \mathcal{C}^1([0,T];H_0^1(\Omega)) \cap \mathcal{C}([0,T];H^2(\Omega)\cap H_0^1(\Omega)))$-solution to the initial-boundary value problem (2) where $(u_{0,m},u_{1,m},F_m)$ is substituted for (u_0,u_1,F). Evidently we have

$$E_m(0) \leq \frac{C}{t}\left(\int_0^t E_m(s)ds + \|F_m\|_{L^2(Q)}^2\right) \tag{28}$$

from *Step 1*, where $E_m(t)$ denotes the energy obtained by substituting u_m for u in (20). Next, since $u - u_m$ is a solution to (2) where (u_0,u_1,F) is replaced by $(u_0 - u_{0,m}, u_1 - u_{1,m}, F - F_m) \in H_0^1(\Omega)\times L^2(\Omega)\times L^2(Q)$, then we have

$$\begin{aligned} &\|u - u_m\|_{\mathcal{C}([0,T];H_0^1(\Omega))} + \|u - u_m\|_{\mathcal{C}^1([0,T];L^2(\Omega))} \\ \leq\ & C\left(\|u_0 - u_{0,m}\|_{H_0^1(\Omega)} + \|u_1 - u_{1,m}\|_{L^2(\Omega)} + \|F - F_m\|_{L^1(0,T;L^2(\Omega))}\right), \end{aligned} \tag{29}$$

by virtue of (3). Further, with reference to (20) we get for all $t \in [0,T]$ that

$$\begin{aligned} & E(t) - E_m(t) \\ = \quad & \int_\Omega \left(\frac{1}{\rho} \sum_{i,j=1}^n a_{ij} \left(\partial_i (u - u_m) \partial_j u + \partial_i u_m \partial_j (u - u_m) \right) - \partial_t (u - u_m) \left(\partial_t (u - u_m) - 2 \partial_t u \right) \right) dx, \end{aligned}$$

hence

$$\begin{aligned} & |E(t) - E_m(t)| \\ \leq \quad & \frac{M}{\rho_0} \left(2 \|u\|_{\mathcal{C}([0,T];H_0^1(\Omega))} + \|u - u_m\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \right) \|u - u_m\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \\ & + \left(2 \|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|u - u_m\|_{\mathcal{C}^1([0,T];L^2(\Omega))} \right) \|u - u_m\|_{\mathcal{C}^1([0,T];L^2(\Omega))}. \end{aligned}$$

Finally, putting this together with (3), (27) and (29), we obtain (22) by sending m to infinity in (28), which terminates the proof of Lemma 2.1. □

3 Carleman estimate

In this section we design a Carleman inequality specifically for the system described by (5). This estimate is the main tool for the analysis of the inverse problems under investigation in this article and we shall derive it from a suitable Carleman inequality for the wave equation, that we establish beforehand in the following subsection. We point out that since there is no zero trace condition at final time $t = T$ in (5), it is more appropriate for the treatment of the inverse problems discussed in Section 4 to follow the same path as in the derivation of [16, Lemma 1] by incorporating the trace term into the right-hand side of the Carleman inequality.

3.1 A hyperbolic Carleman inequality

Let $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. We consider a sub-boundary $\Gamma_0 \subset \partial\Omega$ such that

$$\{x \in \partial\Omega;\ (x - x_0) \cdot \nu \geq 0\} \subset \Gamma_0 \tag{30}$$

and we put $\tilde{\Sigma}_0 := \Gamma_0 \times (-T,T)$.

Next for $\beta \in (0,1)$ and $\lambda \in (0,+\infty)$ we introduce the following quadratic weight function (which is quite usual in the theory of Carleman estimates for the wave equation, see e.g., [4]),

$$\varphi(x,t) := e^{\lambda\psi(x,t)} \text{ where } \psi(x,t) := |x - x_0|^2 - \beta t^2 \quad (x,t) \in \tilde{Q} := \Omega \times (-T,T). \tag{31}$$

Notice that φ depends on the two parameters β and λ, and should rather be denoted $\varphi_{\beta,\lambda}$, but for the sake of notational simplicity, we omit these dependences.

Keeping in mind the application to inverse problems in Section 4, where we want to avoid the inadequate use of additional initial data u_j, $j = 0, 1$, we aim for a hyperbolic Carleman estimate on the time interval $(-T, T)$, rather than $(0, T)$.

Lemma 3.1. *Assume that $B = (b_1, \ldots, b_n)^T \in L^\infty(\Omega)^n$ and that $c \in L^\infty(\Omega)$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, there exist two positive constants $s_0 = s_0(\lambda)$ and $C = C(\lambda)$ such that for all $s \geq s_0$, we have*

$$s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}u\|^2_{L^2(\tilde{Q})} \leq C\left(\|e^{s\varphi}L_0u\|^2_{L^2(\tilde{Q})} + s\|e^{s\varphi}\partial_\nu u\|^2_{L^2(\tilde{\Sigma}_0)} + \sum_{\tau=\pm T}\mathcal{E}_{s,\tau}(u)\right) \tag{32}$$

provided $u \in \mathcal{C}([-T,T];H^1_0(\Omega)) \cap \mathcal{C}^1([-T,T];L^2(\Omega))$ satisfies $L_0u := \partial_t^2 u - \Delta u + B\cdot\nabla u + cu \in L^2(\tilde{Q})$. Here we have set:

$$\mathcal{E}_{s,\tau}(u) := s\left\|e^{s\varphi(\cdot,\tau)}\nabla_{x,t}u(\cdot,\tau)\right\|^2_{L^2(\Omega)} + s^3\left\|e^{s\varphi(\cdot,\tau)}u(\cdot,\tau)\right\|^2_{L^2(\Omega)},\ \tau = \pm T.$$

Here $L_0u \in L^2(\tilde{Q})$ is understood in the sense of $(C_0^\infty(\tilde{Q}))'$ and the lemma requires that it belongs to $L^2(\tilde{Q})$. Notice that unlike [4, Theorem 4.2] where it is assumed that $u(\cdot,\pm T) = \partial_t u(\cdot,\pm T) = 0$, there are actual trace terms at $t = \pm T$ on the right-hand side of (32), which are gathered in the sum $\mathcal{E}_{s,-T}(u)+\mathcal{E}_{s,T}(u)$ (these quantities arise from the integrations by parts performed in the proof of [4, Lemmas 4.2 and 4.3]).

We refer the reader to [18, Theorem 1.2] for a hyperbolic Carleman estimate with interior data and within the same range of regularity for the function u. Here, for the sake of completeness we provide the proof of (32) for u in the H^2-class by approximating u by solutions to the initial-boundary value problems with smooth initial data and non-homogeneous terms, and we conclude with a density argument.

Proof. Let us suppose that $u \in \mathcal{C}^1([-T,T];L^2(\Omega)) \cap \mathcal{C}([-T,T];H^1_0(\Omega))$ satisfies $L_0u \in L^2(\tilde{Q})$. Set $a := u(\cdot,-T) \in H^1_0(\Omega)$, $b := \partial_t u(\cdot,-T) \in L^2(\Omega)$ and $F := L_0u \in L^2(\tilde{Q})$. Then, by density of $\mathcal{C}_c^\infty(\Omega)$ in $H^1_0(\Omega)$ and in $L^2(\Omega)$, and of $\mathcal{C}_c^\infty(\tilde{Q})$ in $L^2(\tilde{Q})$, we pick three sequences $\{a_m\}_{m=1}^\infty \in \mathcal{C}_c^\infty(\Omega)^{\mathbb{N}}$, $\{b_m\}_{m=1}^\infty \in \mathcal{C}_c^\infty(\Omega)^{\mathbb{N}}$ and $\{F_m\}_{m=1}^\infty \subset \mathcal{C}_c^\infty(\tilde{Q})^{\mathbb{N}}$ such that

$$\lim_{m\to+\infty}\|a - a_m\|_{H^1_0(\Omega)} = 0, \quad \lim_{m\to+\infty}\|b - b_m\|_{L^2(\Omega)} = 0, \quad \lim_{m\to+\infty}\|F - F_m\|_{L^2(\tilde{Q})} = 0. \tag{33}$$

For $m \in \mathbb{N}$, we denote by u_m the solution to the following initial-boundary value problem

$$\begin{cases} L_0u_m = F_m & \text{in } \tilde{Q}, \\ u_m = 0 & \text{on } \tilde{\Sigma} := \partial\Omega\times(-T,T), \\ u_m(\cdot,-T) = a_m,\ \partial_t u_m(\cdot,-T) = b_m & \text{in } \Omega. \end{cases} \tag{34}$$

Then we have $u_m \in \mathcal{C}([-T,T];H^2(\Omega)\cap H^1_0(\Omega)) \cap \mathcal{C}^1([-T,T];H^1_0(\Omega)) \cap \mathcal{C}^2([-T,T];L^2(\Omega))$ and consequently

$$s\|e^{s\varphi}\nabla_{x,t}u_m\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}u_m\|^2_{L^2(\tilde{Q})} \leq C\left(\|e^{s\varphi}F_m\|^2_{L^2(\tilde{Q})} + s\|e^{s\varphi}\partial_\nu u_m\|^2_{L^2(\tilde{\Sigma}_0)} + \sum_{\tau=\pm T}\mathcal{E}_{s,\tau}(u_m)\right)$$

for all $s \geq s_0$, from the Carleman inequality for hyperbolic equations. Putting this together with the basic inequality

$$\begin{aligned} & s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}u\|^2_{L^2(\tilde{Q})} \\ \leq\ & 2\left(s\|e^{s\varphi}\nabla_{x,t}u_m\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}u_m\|^2_{L^2(\tilde{Q})} + s\|e^{s\varphi}\nabla_{x,t}(u-u_m)\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}(u-u_m)\|^2_{L^2(\tilde{Q})}\right) \end{aligned}$$

we get that

$$\begin{aligned} & s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}u\|^2_{L^2(\tilde{Q})} \\ \leq\ & C\left(\|e^{s\varphi}F_m\|^2_{L^2(\tilde{Q})} + s\|e^{s\varphi}\partial_\nu u_m\|^2_{L^2(\tilde{\Sigma}_0)} + \mathcal{E}_s(u_m) + s\|e^{s\varphi}\nabla_{x,t}(u-u_m)\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}(u-u_m)\|^2_{L^2(\tilde{Q})}\right) \end{aligned}$$

for all $s \geq s_0$. Here and in the remaining part of this proof, C denotes a generic positive constant, independent of s and m, which may change from line to line, and for the sake of notational simplicity we write $\mathcal{E}_s$ instead of $\sum_{\tau=\pm T}\mathcal{E}_{s,\tau}$. As a consequence we have

$$s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}u\|^2_{L^2(\tilde{Q})} \leq C\left(\|e^{s\varphi}F\|^2_{L^2(\tilde{Q})} + s\|e^{s\varphi}\partial_\nu u\|^2_{L^2(\tilde{\Sigma}_0)} + \mathcal{E}_s(u) + \mathcal{R}_s(m)\right) \tag{35}$$

for all $s \geq s_0$ and all $m \in \mathbb{N}$, where

$$\begin{aligned} \mathcal{R}_s(m) \ :=\ & s\|e^{s\varphi}\nabla_{x,t}(u-u_m)\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}(u-u_m)\|^2_{L^2(\tilde{Q})} + s\|e^{s\varphi}\partial_\nu(u-u_m)\|^2_{L^2(\tilde{\Sigma}_0)} \\ & + \mathcal{E}_s(u-u_m) + \|e^{s\varphi}(F-F_m)\|^2_{L^2(\tilde{Q})}. \end{aligned}$$

Thus we are left with the task of proving that $\mathcal{R}_s(m)$ tends to zero as m goes to $+\infty$. To do that, we use that the weight function φ is bounded on the closure of $\tilde{Q}$ by a positive constant independent of s, and get that

$$\begin{aligned} \mathcal{R}_s(m) \ \leq\ & e^{sC}\Bigg(s\|\nabla_{x,t}(u-u_m)\|^2_{L^2(\tilde{Q})} + s^3\|u-u_m\|^2_{L^2(\tilde{Q})} + s\|\partial_\nu(u-u_m)\|^2_{L^2(\tilde{\Sigma}_0)} \\ & + \sum_{\tau=\pm T}\left(s\|\nabla_{x,t}(u-u_m)(\cdot,\tau)\|^2_{L^2(\Omega)} + s^3\|(u-u_m)(\cdot,\tau)\|^2_{L^2(\Omega)}\right) + \|F-F_m\|^2_{L^2(\tilde{Q})}\Bigg). \end{aligned} \tag{36}$$

Next we have

$$\begin{cases} L_0(u-u_m) = F - F_m & \text{in } \tilde{Q}, \\ u - u_m = 0 & \text{on } \tilde{\Sigma}, \\ (u-u_m)(\cdot,-T) = a - a_m,\ \partial_t(u-u_m)(\cdot,-T) = b - b_m & \text{in } \Omega, \end{cases}$$

according to (34), hence $u - u_m \in \mathcal{C}([-T,T];H^1_0(\Omega)) \cap \mathcal{C}^1([-T,T];L^2(\Omega))$ satisfies

$$\begin{aligned} & \|u-u_m\|_{\mathcal{C}([-T,T];H^1_0(\Omega))} + \|\partial_t(u-u_m)\|_{\mathcal{C}([-T,T];L^2(\Omega))} \\ \leq\ & C\left(\|a-a_m\|_{H^1(\Omega)} + \|b-b_m\|_{L^2(\Omega)} + \|F-F_m\|_{L^2(\tilde{Q})}\right), \end{aligned} \tag{37}$$

by [4, Lemma 3.2]. Moreover, we have $\partial_\nu(u-u_m) \in L^2(\tilde{\Sigma})$ and

$$\|\partial_\nu(u-u_m)\|_{L^2(\tilde{\Sigma})} \leq C\left(\|a-a_m\|_{H^1(\Omega)} + \|b-b_m\|_{L^2(\Omega)} + \|F-F_m\|_{L^2(\tilde{Q})}\right) \tag{38}$$

from [4, Lemma 3.6]. Inserting (37)-(38) into (36) then yields

$$\mathcal{R}_s(m) \leq C(1+s+s^3)e^{sC}\left(\|a-a_m\|^2_{H^1_0(\Omega)} + \|b-b_m\|^2_{L^2(\Omega)} + \|F-F_m\|^2_{L^2(\tilde{Q})}\right),\ n \in \mathbb{N}.$$

This and (33) entail that $\lim_{m\to+\infty}\mathcal{R}_s(m) = 0$ so the desired result follows by sending m to $+\infty$ in (35). □

Remark 3.1. *We have $\partial_\nu u \in L^2(\tilde{\Sigma})$ according to [28, Theorem 4.1], whenever u satisfies the conditions of Lemma 3.1.*

Having established (32), we are now in a position to design a Carleman estimate for (5).

3.2 Taking the time-fractional damping term into account

Our strategy here is to regard the time-fractional damping term as a source term. That is to say that we aim to apply Lemma 3.1 to the first equation of (5) brought into the form $L_0 u = F - q\partial_t^\alpha u$ by moving the time-fractional term $q\partial_t^\alpha u$ to its right-hand side. This requires that the solution u to (5) be extended to the time interval $(-T,T)$ and that the time-fractional term $\partial_t^\alpha u$ be appropriately estimated.

We shall proceed in three steps: The first step is to estimate $\partial_t^\alpha u$ in terms of $\partial_t u$, the second one is to time-symmetrize u, and finally the last step is to deduce the desired Carleman inequality from Lemma 3.1 and the two preceding steps.

3.2.1 Estimation of the time-fractional damping term

The estimate we have in mind is as follows.

Lemma 3.2. *There exists a constant $C = C(T) > 0$ such that for all $u \in H^1(0,T;L^2(\Omega))$, all $\lambda > 0$ and all $s \geq 0$, we have*

$$\|e^{s\varphi}\partial_t^\alpha u\|_{L^2(Q)} \leq C\|e^{s\varphi}\partial_t u\|_{L^2(Q)}.$$

Proof. For all $x \in \Omega$, $t \mapsto \varphi(x,t)$ is a decreasing function in $(0,T)$, hence $e^{2s\varphi(x,t)} \leq e^{2s\varphi(x,\tau)}$ for all $t \in (0,T)$ and all $\tau \in (0,t)$. Thus we have

$$\begin{aligned}\|e^{s\varphi}\partial_t^\alpha u\|^2_{L^2(Q)} &= \int_Q \frac{e^{2s\varphi(x,t)}}{\Gamma(1-\alpha)^2}\left|\int_0^t (t-\tau)^{-\alpha}\partial_\tau u(x,\tau)d\tau\right|^2 dxdt \\ &\leq \int_Q \frac{1}{\Gamma(1-\alpha)^2}\left(\int_0^t (t-\tau)^{-\alpha}|\partial_\tau u(x,\tau)|e^{s\varphi(x,t)}d\tau\right)^2 dxdt \\ &\leq \int_Q \frac{1}{\Gamma(1-\alpha)^2}\left(\int_0^t (t-\tau)^{-\alpha}|\partial_\tau u(x,\tau)|e^{s\varphi(x,\tau)}d\tau\right)^2 dxdt\end{aligned}$$

by direct calculation, and Young's convolution inequality then yields

$$\|e^{s\varphi}\partial_t^\alpha u\|^2_{L^2(Q)} \leq \int_Q \left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 e^{2s\varphi(x,t)}|\partial_t u(x,t)|^2 dxdt$$

$$\leq \left(\frac{\max\{1,T\}}{\min_{y\in(1,2)}\Gamma(y)}\right)^2 \|e^{s\varphi}\partial_t u\|^2_{L^2(Q)}.$$

□

3.2.2 Time-symmetrization

Let u be the $\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$-solution to (5), defined by Theorem 1.1. Depending on whether the assumption (6)(i) or (6)(ii) is fulfilled, we introduce the odd or the even extension of $t\mapsto u(\cdot,t)$ on $(-T,T)$, by setting

$$u(x,t) := \begin{cases} u(x,t) & \text{if } (x,t)\in\Omega\times[0,T), \\ \mp u(x,-t) & \text{if } (x,t)\in\Omega\times(-T,0). \end{cases}$$

Here and below the notation $\mp u$ means $-u$ in the case of (6)(i) and $+u$ in the case of (6)(ii).

Then, in light of (6) it is easy to check that $u\in\mathcal{C}([-T,T];H_0^1(\Omega))\cap\mathcal{C}^1([-T,T];L^2(\Omega))$. Moreover, for a.e. $t\in(-T,0)$ we have

$$L_0u(\cdot,t) = \mp L_0u(\cdot,-t) = \mp\left(F(\cdot,-t) - q\partial_t^\alpha u(\cdot,-t)\right) \quad \text{in } \Omega$$

and

$$u(\cdot,t) = \mp u(\cdot,-t) = 0 \quad \text{on } \partial\Omega.$$

Therefore, putting $G(x,t) := F(x,t) - q(x)\partial_t^\alpha u(x,t)$ for a.e. $x\in\Omega$ and all $t\in[0,T)$, and

$$G(x,t) := \begin{cases} G(x,t) & \text{if } (x,t)\in\Omega\times[0,T), \\ \mp G(x,-t) & \text{if } (x,t)\in\Omega\times(-T,0), \end{cases}$$

we end up getting that

$$\begin{cases} L_0u = G & \text{in } \tilde{Q}, \\ u = 0 & \text{on } \tilde{\Sigma}, \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1 & \text{in } \Omega. \end{cases} \tag{39}$$

Notice that since it is enough to appropriately extend $t\mapsto G(\cdot,t)$ in $(-T,0)$ for the hyperbolic $L_0u=G$ to hold in $\tilde{Q}$, we can entirely leave aside the more delicate problem of defining the time-fractional derivative $\partial_t^\alpha u$ for negative time values.

Remark 3.2. *In a similar fashion to Remark 3.1, we have $\partial_\nu u\in L^2(\tilde{\Sigma})$ here. This can be seen from Lemma 3.2 (with $s=0$) upon arguing as in the proof of [28, Theorem 4.1].*

3.2.3 Completion of the Carleman estimate

Let us apply Lemma 3.1 to the $\mathcal{C}([-T,T];H_0^1(\Omega))\cap\mathcal{C}^1([-T,T];L^2(\Omega))$-solution u of (39). Taking into account that $\|e^{s\varphi}G\|^2_{L^2(\tilde{Q})} = 2\|e^{s\varphi}G\|^2_{L^2(Q)}$, $\|e^{s\varphi}\partial_\nu u\|^2_{L^2(\tilde{\Sigma}_0)} = 2\|e^{s\varphi}\partial_\nu u\|^2_{L^2(\Sigma_0)}$ where we used the notation

$\Sigma_0 := \Gamma_0 \times (0,T)$, and that $\mathcal{E}_{s,-T}(u) = \mathcal{E}_{s,T}(u)$, we get that

$$\begin{aligned} s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(\tilde{Q})} + s^3\|e^{s\varphi}u\|^2_{L^2(\tilde{Q})} &\le C\left(\|e^{s\varphi}G\|^2_{L^2(Q)} + s\|e^{s\varphi}\partial_\nu u\|^2_{L^2(\Sigma_0)} + \mathcal{E}_{s,T}(u)\right) \\ &\le C\left(\|e^{s\varphi}F\|^2_{L^2(Q)} + \|e^{s\varphi}\partial_t^\alpha u\|^2_{L^2(Q)} + s\|e^{s\varphi}\partial_\nu u\|^2_{L^2(\Sigma_0)} + \mathcal{E}_{s,T}(u)\right) \end{aligned}$$

for all $\lambda \ge \lambda_0$ and $s \ge s_0$. Thus we have

$$s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(Q)} + s^3\|e^{s\varphi}u\|^2_{L^2(Q)} \le C\left(\|e^{s\varphi}F\|^2_{L^2(Q)} + \|e^{s\varphi}\partial_t u\|^2_{L^2(Q)} + s\|e^{s\varphi}\partial_\nu u\|^2_{L^2(\Sigma_0)} + \mathcal{E}_{s,T}(u)\right)$$

by Lemma 3.2, and consequently

$$s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(Q)} + s^3\|e^{s\varphi}u\|^2_{L^2(Q)} \le C\left(\|e^{s\varphi}F\|^2_{L^2(Q)} + s\|e^{s\varphi}\partial_\nu u\|^2_{L^2(\Sigma_0)} + \mathcal{E}_{s,T}(u)\right)$$

upon taking $s \ge s_1 := \max(s_0, 2C)$ and possibly enlarging C. Therefore, bearing in mind for all $s \ge s_1$ and all $(x,t) \in Q$ that $se^{2s\varphi(x,t)} \le e^{Cs}$ for some positive constant C which is independent of s, we obtain the following result.

Proposition 3.1. *Assume* (6)*. Then, for all $\lambda \ge \lambda_0$, where λ_0 is the same as that in Lemma 3.1, there exist two constants $s_1 > 0$ and $C > 0$ such that the estimate*

$$\begin{aligned} & s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(Q)} + s^3\|e^{s\varphi}u\|^2_{L^2(Q)} \\ \le\ & C\left(\|e^{s\varphi}F\|^2_{L^2(Q)} + e^{sC}\|\partial_\nu u\|^2_{L^2(\Sigma_0)} + s\left\|e^{s\varphi(\cdot,T)}\nabla_{x,t}u(\cdot,T)\right\|^2_{L^2(\Omega)} + s^3\left\|e^{s\varphi(\cdot,T)}u(\cdot,T)\right\|^2_{L^2(\Omega)}\right) \end{aligned}$$

holds for all $s \ge s_1$. Here, u is the solution to (5) *associated with $(u_0, u_1, F) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(Q)$, defined by Theorem 1.1.*

We remark that the Carleman estimate for (5) is the same as the one for the hyperbolic equation, that is, $q \equiv 0$. In particular, for the inverse problems, the geometric condition on measurement sub-boundary Γ_0 is independent of $q(x)$ in (5).

4 Proof of Theorems 1.2 and 1.3

Inspired by [16], we prove in this section the two main results of this article, stated in Theorems 1.2 and 1.3.

Prior to doing that we recall that $\Gamma_0 \subset \Gamma$ satisfies the condition (30) for some $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Next we introduce

$$d_0 := \min_{x \in \overline{\Omega}} |x - x_0| \text{ and } d_1 := \max_{x \in \overline{\Omega}} |x - x_0|, \tag{40}$$

set

$$T_0 := \sqrt{2(d_1^2 - d_0^2)}$$

and we pick β so large in $(0,1)$ that the following inequality

$$\beta T^2 \geq 2(d_1^2 - d_0^2) \tag{41}$$

holds whenever $T > T_0$.

We start by showing Theorem 1.2.

4.1 Proof of Theorem 1.2

Let us apply Proposition 3.1 with $\lambda = \lambda_0$ to the solution u of (5). We find for all $s \geq s_1$ that

$$\begin{aligned} & s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(Q)} \\ \leq \; & C\left(\|e^{s\varphi}F\|^2_{L^2(Q)} + e^{Cs}\|\partial_\nu u\|^2_{L^2(\Sigma_0)} + s\left\|e^{s\varphi(\cdot,T)}\nabla_{x,t}u(\cdot,T)\right\|^2_{L^2(\Omega)} + s^3\left\|e^{s\varphi(\cdot,T)}u(\cdot,T)\right\|^2_{L^2(\Omega)}\right) \\ \leq \; & C\left(e^{Cs}\left(\|F\|^2_{L^2(Q)} + \|\partial_\nu u\|^2_{L^2(\Sigma_0)}\right) + e^{M_1 s}\left(s\|\nabla_{x,t}u(\cdot,T)\|^2_{L^2(\Omega)} + s^3\|u(\cdot,T)\|^2_{L^2(\Omega)}\right)\right). \end{aligned} \tag{42}$$

Here, C denotes a generic positive constant which is independent of s and $M_1 := 2e^{\lambda_0(d_1^2 - \beta T^2)}$ where d_1 is defined in (40). Next, bearing in mind that $u(\cdot,T) \in H_0^1(\Omega)$, we get by combining the Poincaré inequality with the energy estimate (21) at final time $t = T$, $\|\nabla_{x,t}u(\cdot,T)\|^2_{L^2(\Omega)} \leq C(E(0) + \|F\|^2_{L^2(Q)})$, that

$$s\|\nabla_{x,t}u(\cdot,T)\|^2_{L^2(\Omega)} + s^3\|u(\cdot,T)\|^2_{L^2(\Omega)} \leq Cs^3(E(0) + \|F\|^2_{L^2(Q)}) \tag{43}$$

for all $s \geq 1$. Here we used that $E(T) = \|\nabla_{x,t}u(\cdot,T)\|^2_{L^2(\Omega)}$ in the framework of Theorem 1.2. Thus, upon possibly enlarging C in such a way that $s^3 e^{M_1 s} \leq e^{Cs}$ for $s \geq s_1$, we infer from (42)-(43) that

$$s\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(Q)} \leq C\left(e^{Cs}\left(\|F\|^2_{L^2(Q)} + \|\partial_\nu u\|^2_{L^2(\Sigma_0)}\right) + s^3 e^{M_1 s}E(0)\right), \; s \geq s_2 := \max\{1, s_1\}. \tag{44}$$

On the other hand, we have

$$\|e^{s\varphi}\nabla_{x,t}u\|^2_{L^2(Q)} \geq \int_0^{\frac{T}{2}}\int_\Omega e^{2s\varphi(x,t)}|\nabla_{x,t}u(x,t)|^2 dxdt \geq e^{M_0 s}\int_0^{\frac{T}{2}} E(t)dt, \tag{45}$$

where $M_0 := 2e^{\lambda_0\left(d_0^2 - \beta\frac{T^2}{4}\right)}$, d_0 defined in (40). Therefore, in light of (22), giving $\int_0^{\frac{T}{2}} E(t)dt \geq CE(0) - \|F\|^2_{L^2(Q)}$, we deduce from (44)-(45) that

$$se^{sM_0}\left(1 - Cs^2 e^{-s(M_0 - M_1)}\right)E(0) \leq Ce^{Cs}\left(\|F\|^2_{L^2(Q)} + \|\partial_\nu u\|^2_{L^2(\Sigma_0)}\right), \; s \geq s_2. \tag{46}$$

Further, since $M_0 - M_1 = M_0\left(1 - e^{-\lambda_0\left(3\beta\frac{T^2}{4} - (d_1^2 - d_0^2)\right)}\right) \geq M_0\left(1 - e^{-\lambda_0\beta\frac{T^2}{4}}\right) > 0$ from (41), we may choose $s \in [s_2, +\infty)$ so large in (46) that

$$E(0) \leq C\left(\|F\|^2_{L^2(Q)} + \|\partial_\nu u\|^2_{L^2(\Sigma_0)}\right).$$

This and the Poincaré inequality yield Theorem 1.2.

4.2 Proof of Theorem 1.3

We turn now to showing Theorem 1.3 by means of the Carleman estimate in Proposition 3.1. We proceed by applying the Bukhgeim-Klibanov method, see [5], which requires the following technical result.

Lemma 4.1. *Let φ be defined by* (31).*Then there exists a constant $C > 0$, depending only on Ω, T and φ, such that for all $\lambda \geq 0$ and all $s \geq 0$, the following estimate*

$$\left\| e^{s\varphi(\cdot,0)} \partial_t v(\cdot,0) \right\|_{L^2(\Omega)}^2 \leq C \left(\| e^{s\varphi} \Box v \|_{L^2(Q)}^2 + s\lambda \| e^{s\varphi} \nabla_{x,t} v \|_{L^2(Q)}^2 + \left\| e^{s\varphi(\cdot,T)} \nabla_{x,t} v(\cdot,T) \right\|_{L^2(\Omega)}^2 \right) \tag{47}$$

holds whenever $v \in \mathcal{C}^1([0,T];L^2(\Omega)) \cap \mathcal{C}([0,T];H_0^1(\Omega))$ satisfies $\Box v := \partial_t^2 v - \Delta v \in L^2(Q)$.

Proof. Let us prove (47) for $v \in \mathcal{C}^2([0,T];L^2(\Omega)) \cap \mathcal{C}([0,T];H^2(\Omega) \cap H_0^1(\Omega))$. In this case we get

$$\left\| e^{s\varphi(\cdot,0)} \partial_t v(\cdot,0) \right\|_{L^2(\Omega)}^2 = \left\| e^{s\varphi(\cdot,T)} \partial_t v(\cdot,T) \right\|_{L^2(\Omega)}^2 - \int_Q \partial_t \left| e^{s\varphi} \partial_t v \right|^2 \, dxdt \tag{48}$$

and

$$\begin{aligned} \int_Q \partial_t \left| e^{s\varphi} \partial_t v \right|^2 \, dxdt &= 2 \int_Q e^{2s\varphi} \left(s\lambda\varphi(\partial_t \psi) |\partial_t v|^2 + (\partial_t^2 v) \partial_t v \right) dxdt \\ &= 2 \int_Q e^{2s\varphi} \left(s\lambda\varphi(\partial_t \psi) |\partial_t v|^2 + (\Box v + \Delta v) \partial_t v \right) dxdt \end{aligned} \tag{49}$$

through direct computation. Next, since $v = 0$ on Σ, we have

$$\int_Q e^{2s\varphi} (\Delta v) \partial_t v \, dxdt = -\int_Q e^{2s\varphi} \left(\nabla v \cdot \nabla(\partial_t v) + 2s\lambda\varphi(\nabla\psi \cdot \nabla v) \partial_t v \right) dxdt$$

by integrating by parts over Ω. Thus, in light of the straightforward identity

$$\begin{aligned} 2 \int_Q e^{2s\varphi} \nabla v \cdot \nabla(\partial_t v) dxdt &= \int_Q e^{2s\varphi} \partial_t |\nabla v|^2 dxdt \\ &= \left\| e^{s\varphi(\cdot,T)} \nabla v(\cdot,T) \right\|_{L^2(\Omega)}^2 - \left\| e^{s\varphi(\cdot,0)} \nabla v(\cdot,0) \right\|_{L^2(\Omega)}^2 - 2s\lambda \int_Q e^{2s\varphi} \varphi(\partial_t \psi) |\nabla v|^2 dxdt, \end{aligned}$$

we find that

$$\begin{aligned} & -2 \int_Q e^{2s\varphi} (\Delta v) \partial_t v \, dxdt \\ = & \left\| e^{s\varphi(\cdot,T)} \nabla v(\cdot,T) \right\|_{L^2(\Omega)}^2 - \left\| e^{s\varphi(\cdot,0)} \nabla v(\cdot,0) \right\|_{L^2(\Omega)}^2 - 2s\lambda \int_Q e^{2s\varphi} \varphi \left((\partial_t \psi) |\nabla v|^2 - 2(\nabla\psi \cdot \nabla v) \partial_t v \right) dxdt \\ \leq & \left\| e^{s\varphi(\cdot,T)} \nabla v(\cdot,T) \right\|_{L^2(\Omega)}^2 + 2s\lambda \int_Q e^{2s\varphi} \varphi \left(|\partial_t \psi| |\nabla v|^2 + 2 |\nabla\psi \cdot \nabla v| |\partial_t v| \right) dxdt. \end{aligned}$$

Putting the above estimate together with (48)-(49) and using that the two functions ψ and φ are in $W^{1,\infty}(Q)$, we obtain (47).

Now, the claim of Lemma 4.1 follows readily from this and the density of $\mathcal{C}^2([0,T];L^2(\Omega)) \cap \mathcal{C}([0,T];H^2(\Omega) \cap H_0^1(\Omega))$ in the space $\{v \in \mathcal{C}^1([0,T];L^2(\Omega)) \cap \mathcal{C}([0,T];H_0^1(\Omega));\ \Box v \in L^2(Q)\}$ endowed with the norm $\|v\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|v\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\Box v\|_{L^2(Q)}$. $\Box$

Armed with Lemma 4.1 we are now in a position to build the proof of Theorem 1.3.

The first step of the Bukhgeim-Klibanov method is to differentiate once with respect to t on both sides of (5). Setting $v := \partial_t u$, we get that

$$\begin{cases} \partial_t^2 v + q\partial_t^\alpha v - \Delta v + B\cdot\nabla v + cv = (\partial_t R)f & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(\cdot,0) = \partial_t u(\cdot,0) = 0, \quad \partial_t v(\cdot,0) = \partial_t^2 u(\cdot,0) = R(\cdot,0)f & \text{in } \Omega. \end{cases} \tag{50}$$

Here we used the fact that q is independent of t and that $u(\cdot,0) = \partial_t u(\cdot,0) = 0$ in Ω, in order to write

$$\partial_t(q\partial_t^\alpha u)(t) = q\partial_t(\partial_t^\alpha u)(t) = q\partial_t^\alpha(\partial_t u)(t) = q\partial_t^\alpha v(t),\ t\in(0,T),$$

see e.g., [32, Section 2.2.5]. Next, we apply Proposition 3.1 with $\lambda = \lambda_0$ to (50). We obtain that for all $s \geq s_1$,

$$\begin{aligned} & s\|e^{s\varphi}\nabla_{x,t}v\|^2_{L^2(Q)} + s^3\|e^{s\varphi}v\|^2_{L^2(Q)} \\ \leq\ & C\left(\|e^{s\varphi}f\|^2_{L^2(Q)} + e^{Cs}\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + s\left\|e^{s\varphi(\cdot,T)}\nabla_{x,t}v(\cdot,T)\right\|^2_{L^2(\Omega)} + s^3\left\|e^{s\varphi(\cdot,T)}v(\cdot,T)\right\|^2_{L^2(\Omega)}\right), \end{aligned} \tag{51}$$

where C denotes a generic positive constant that is independent of s.

In view of retrieving the unknown function f from the (second) initial condition of (50),

$$\partial_t v(\cdot,0) = R(\cdot,0)f \text{ in } \Omega, \tag{52}$$

we apply Lemma 4.1 to v. We get for all $s \geq 1$ that

$$\begin{aligned} & \left\|e^{s\varphi(\cdot,0)}\partial_t v(\cdot,0)\right\|^2_{L^2(\Omega)} \\ \leq\ & C\left(\|e^{s\varphi}((\partial_t R)f - q\partial_t^\alpha v - B\cdot\nabla v - cv)\|^2_{L^2(Q)} + s\|e^{s\varphi}\nabla_{x,t}v\|^2_{L^2(Q)} + \left\|e^{s\varphi(\cdot,T)}\nabla_{x,t}v(\cdot,T)\right\|^2_{L^2(\Omega)}\right) \\ \leq\ & C\left(\|e^{s\varphi}f\|^2_{L^2(Q)} + s\|e^{s\varphi}\nabla_{x,t}v\|^2_{L^2(Q)} + s^3\|e^{s\varphi}v\|^2_{L^2(Q)} + \left\|e^{s\varphi(\cdot,T)}\nabla_{x,t}v(\cdot,T)\right\|^2_{L^2(\Omega)}\right), \end{aligned}$$

where we used Lemma 3.2 in the last line. Thus, with reference to the positivity condition (8), it follows from (51)-(52) that

$$\left\|e^{s\varphi(\cdot,0)}f\right\|^2_{L^2(\Omega)} \leq C\left(\|e^{s\varphi}f\|^2_{L^2(Q)} + e^{Cs}\|\partial_\nu v\|^2_{L^2(\Sigma_0)} + s\left\|e^{s\varphi(\cdot,T)}\nabla_{x,t}v(\cdot,T)\right\|^2_{L^2(\Omega)} + s^3\left\|e^{s\varphi(\cdot,T)}v(\cdot,T)\right\|^2_{L^2(\Omega)}\right) \tag{53}$$

for all $s \geq s_2 = \max\{1, s_1\}$.

Further, since $t \mapsto \varphi(x,t)$ attains its maximum at $t = 0$ for all $x \in \Omega$, then $\|e^{s\varphi}f\|_{L^2(Q)}$ can be made arbitrarily small relative to $\left\|e^{s\varphi(\cdot,0)}f\right\|_{L^2(\Omega)}$ by taking s sufficiently large. This can be seen by following the same path as in the derivation of [4, Lemma 5.3], that is to say by writing

$$\|e^{s\varphi}f\|^2_{L^2(Q)} = \int_\Omega e^{2s\varphi(x,0)}|f(x)|^2\left(\int_0^T e^{-2s(\varphi(x,0)-\varphi(x,t))}dt\right)dx,\ s\geq 0, \tag{54}$$

in accordance with Lebesgue's theorem, and by noticing for all $(x,t) \in Q$ that

$$\begin{aligned} \varphi(x,0) - \varphi(x,t) &= e^{\lambda_0 \psi(x,0)} \left(1 - e^{\lambda_0(\psi(x,t)-\psi(x,0))}\right) \\ &= e^{\lambda_0 |x-x_0|^2} \left(1 - e^{-\lambda_0 \beta t^2}\right) \\ &\geq e^{\lambda_0 d_0^2} \left(1 - e^{-\lambda_0 \beta t^2}\right) := \zeta(t), \end{aligned} \tag{55}$$

where d_0 is defined in (40). Indeed, putting (54) together with (55), we obtain that

$$\|e^{s\varphi} f\|_{L^2(Q)}^2 \leq \kappa(s) \left\| e^{s\varphi(\cdot,0)} f \right\|_{L^2(\Omega)}^2, \ s \geq 0,$$

where the non-negative function $\kappa(s) := \int_0^T e^{-2s\zeta(t)} dt$ converges to zero as s tends to $+\infty$. Therefore, taking s so large that $\kappa(s) \leq (2C)^{-1}$, we get that $2(\|e^{s\varphi(\cdot,0)} f\|_{L^2(\Omega)}^2 - C\|e^{s\varphi} f\|_{L^2(Q)}^2) \geq \|e^{s\varphi(\cdot,0)} f\|_{L^2(\Omega)}^2$ according to the above estimate, which establishes that the first term on the right-hand side of (53) can be absorbed into its left-hand side. As a consequence, there exists $s_3 \geq s_2$ such that

$$\left\| e^{s\varphi(\cdot,0)} f \right\|_{L^2(\Omega)}^2 \leq C \left(e^{Cs} \|\partial_\nu v\|_{L^2(\Sigma_0)}^2 + s \left\| e^{s\varphi(\cdot,T)} \nabla_{x,t} v(\cdot,T) \right\|_{L^2(\Omega)}^2 + s^3 \left\| e^{s\varphi(\cdot,T)} v(\cdot,T) \right\|_{L^2(\Omega)}^2 \right), \ s \geq s_3. \tag{56}$$

We are thus left with the task of showing that the two last terms on the right-hand side of (56) are dominated by $\|e^{s\varphi(\cdot,0)} f\|_{L^2(\Omega)}^2$, provided s is large enough. This can be done by combining the Poincaré inequality, giving

$$\left\| e^{s\varphi(\cdot,T)} \nabla_{x,t} v(\cdot,T) \right\|_{L^2(\Omega)}^2 + \left\| e^{s\varphi(\cdot,T)} v(\cdot,T) \right\|_{L^2(\Omega)}^2 \leq C e^{2se^{\lambda_0 \left(d_1^2 - \beta T^2\right)}} \|\nabla_{x,t} v(\cdot,T)\|_{L^2(\Omega)}^2,$$

with the energy estimate (21) at $t = T$,

$$\|\nabla_{x,t} v(\cdot,T)\|_{L^2(\Omega)}^2 \leq C \left(\|R(\cdot,0) f\|_{L^2(\Omega)}^2 + \|(\partial_t R) f\|_{L^2(Q)}^2 \right) \leq C \|f\|_{L^2(\Omega)}^2$$

and

$$\left\| e^{s\varphi(\cdot,0)} f \right\|_{L^2(\Omega)}^2 \geq e^{2se^{\lambda_0 d_0^2}} \|f\|_{L^2(\Omega)}^2, \ s \geq 0,$$

where we recall that d_0 and d_1 are defined in (40). This entails that

$$\left\| e^{s\varphi(\cdot,T)} \nabla_{x,t} v(\cdot,T) \right\|_{L^2(\Omega)}^2 + \left\| e^{s\varphi(\cdot,T)} v(\cdot,T) \right\|_{L^2(\Omega)}^2 \leq C e^{-2s\delta_0 e^{\lambda_0 d_0^2}} \left\| e^{s\varphi(\cdot,0)} f \right\|_{L^2(\Omega)}^2, \ s \geq 0,$$

where $\delta_0 := 1 - e^{-\lambda_0 \left(\beta T^2 - (d_1^2 - d_0^2)\right)}$, and consequently that

$$\left(1 - Cs^3 e^{-2s\delta_0 e^{\lambda_0 d_0^2}}\right) \left\| e^{s\varphi(\cdot,0)} f \right\|_{L^2(\Omega)}^2 \leq C e^{Cs} \|\partial_\nu v\|_{L^2(\Sigma_0)}^2, \ s \geq s_3,$$

by (56). Finally, since $\delta_0 \geq 1 - e^{-\lambda_0 \beta \frac{T^2}{2}} > 0$ from (41), the desired result follows upon taking s sufficiently large in the above estimate.

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