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The inverse problem of two-state quantum systems with non-adiabatic static linear coupling

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We consider the inverse problem of determining the coupling coefficients in a two-state Schrödinger system. We prove a Lipschitz stability inequality for the zeroth- and first-order coupling terms by finitely many partial lateral measurements of the solution to the coupled Schrödinger equations.

Keywords: Inverse problem; stability estimate; coupled Schrödinger equations.

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1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, with smooth boundary $\Gamma = \partial\Omega$. Given $T \in (0, +\infty)$, we consider the following initial-boundary value

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problem (IBVP) for the coupled two-state Schrödinger equations in the unknowns $u^\pm = u^\pm(x,t)$:

$$\begin{cases} -i\partial_t u^+ - \Delta u^+ + q^+ u^+ + A\cdot\nabla u^- + pu^- = 0 & \text{in } Q = \Omega\times(0,T), \\ -i\partial_t u^- - \Delta u^- + q^- u^- - A\cdot\nabla u^+ + pu^+ = 0 & \text{in } Q, \\ u^+(\cdot,0) = u_0^+, \quad u^-(\cdot,0) = u_0^- & \text{in } \Omega, \\ u^+ = g^+, \quad u^- = g^- & \text{on } \Sigma = \Gamma\times(0,T), \end{cases} \tag{1.1}$$

where $u_0^\pm$ and $g^\pm$ are suitable initial states and Dirichlet boundary conditions, respectively. Here, $A:\Omega\to\mathbb{R}^n$, $p:\Omega\to\mathbb{R}$ and $q^\pm:\Omega\to\mathbb{R}$ are all real-valued. In this paper, we are concerned by the stability issue in the inverse problem of determining the unknown functions A, p and $q^\pm$ from a finite number of local boundary measurements of the solution to (1.1).

The IBVP (1.1) describes the dynamics of a two-state (or two-level) quantum system. This terminology is justified by the fact that the quantum system modeled by (1.1) can exist in any superposition of the two independent (in the sense that they can be physically distinguished) states $u^\pm$. As a matter of fact, particles such as electrons, neutrinos or protons, are fermions, and they have a two-state quantum mechanical label called spin. In Quantum Mechanics, the spin is an intrinsic form of angular momentum carried by elementary particles and the spin of fermions is half-integer. Namely, the electron is a spin-1/2 particle, i.e. the spin of the electron can have values $\hbar/2$ (spin up) or $-\hbar/2$ (spin down), where $\hbar$ is the reduced Planck constant. Note that for the sake of simplicity, the various physical constants appearing in (1.1), such as $\hbar$, the mass of the particle or its charge, are all taken equal to 1 in this text. In (1.1), the dynamics of the two states $u^\pm$ are bound together through non-adiabatic linear coupling $pu^\mp \pm A\cdot\nabla u^\mp$, see [17] and the references therein for the relevance of non-adiabatic processes in physics or reactive chemistry. Gradient coupling appears also naturally in quantum fields theory (see [1, 21]) or quantum cosmology (see [11, 7]), and it can sometimes be seen as a first-order approximation of nonlinear coupling (see [24]).

1.1. *What is known so far*: *A short bibliography*

There is a wide mathematical literature on inverse coefficient problems for the dynamic Schrödinger equation. Without trying to be exhaustive, one may mention [4, 3, 9, 6, 15]. In all these papers, an infinite number of boundary observations of the solution is required, but in [2, 25], the real-valued electric potential of the Schrödinger equation is Lipschitz stably retrieved from a single partial boundary measurement. This result was improved in [20] to smaller partial measurements and extended in [12] to complex-valued electric potentials. The method used in [2, 25, 20, 12] is based essentially on an appropriate Carleman estimate. We refer to [12, 23, 25] for actual examples of this inequality for the Schrödinger equation.

The idea of using a Carleman estimate for solving inverse problems first appeared in Bukhgeim and Klibanov [8]. Since its inception in 1981, this technique has then been widely and successfully applied by numerous authors to parabolic or hyperbolic systems, to the dynamic Schrödinger equation, and even to coupled systems of PDEs. See [16] and references therein, for a complete review of multidimensional inverse problems solved by the Bukhgeim–Klibanov method.

Note that in [2, 25, 20, 12], the data are measured on a part of the boundary that fulfills a geometric condition related to geometric optics condition insuring observability. This condition was relaxed in [4] for a real-valued electric potential, under the assumption that the potential is known in the vicinity of the boundary. We refer to [13, 14, 5] for the same type of inverse problems but stated in an infinite cylindrical domain. The problem of stably determining the space varying part (respectively, static) magnetic potential of the autonomous (respectively, non-autonomous) Schrödinger equation is treated in [10] (respectively, [12]). In both cases, the nth-dimensional unknown magnetic vector potential, $n \geq 1$, is recovered from n partial Neumann data, obtained by n-times suitably changing the initial condition attached at the magnetic Schrödinger equation. All the above-mentioned papers are concerned with the "one state" Schrödinger equation. In [19], the authors show unique determination of the static electric coupling potential in a two state magnetic Schrödinger equation, by one partial measurement of the solution. Otherwise stated, assuming that the gradient coupling potential is known, [19] claims that knowledge of one partial Neumann data uniquely determines the scalar coupling potential. In this paper, the framework is the same as in [19] but with uniformly zero magnetic field, and we investigate the stability issue in the inverse problem of identifying both the electric and the gradient coupling potentials, by finitely many partial boundary observations of the solution.

1.2. *Notations*

Throughout the entire text, $x = (x_1, \ldots, x_n)$ is a generic point of Ω and we set $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial^2_{i,j} = \frac{\partial^2}{\partial x_i \partial x_j}$ for all $i, j = 1, \ldots, n$. We write ∂_i^2 instead of $\frac{\partial^2}{\partial x_i^2}$ and as usual, Δ denotes the Laplace operator, i.e. $\Delta = \partial_1^2 + \cdots + \partial_n^2$. Next, for any multi-index $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we put $|k| = k_1 + \cdots + k_n$ and $\partial_x^k = \partial_1^{k_1} \ldots \partial_n^{k_n}$. Similarly, we write $\partial_t = \frac{\partial}{\partial t}$.

Further, the symbol $\cdot$ denotes the scalar product in $\mathbb{C}^m$, $m \in \mathbb{N}$, and $|\zeta| = \sqrt{\zeta \cdot \zeta}$ for all $\zeta \in \mathbb{C}^m$. For any row vector $a = (a_1, a_2, \ldots, a_m)$ we write a^T for the transpose of a in such a way that $\nabla = (\partial_1, \ldots, \partial_n)^T$ is the gradient operator with respect to x. Further, $\nabla \cdot$ denotes the divergence operator and we set $\partial_\nu u = \nabla u \cdot \nu$, where ν is the outward normal vector to the boundary Γ.

Let us now introduce the following functional spaces. For any manifold X, we set

$$H^{r,s}(X \times (0,T)) = L^2(0,T; H^r(X)) \cap H^s(0,T; L^2(X)), \quad r, s > 0,$$

where $H^s(X)$ denotes the usual Sobolev space on X of order s. For the sake of notational simplicity, we write $H^{r,s}(Q)$ (respectively, $H^{r,s}(\Sigma)$) instead of $H^{r,s}(\Omega \times (0,T))$ (respectively, $H^{r,s}(\Gamma \times (0,T))$). The space $H^{r,s}(X \times (0,T))$ is endowed with the norm $\| \cdot \|_{H^{r,s}(X\times(0,T))} = \| \cdot \|_{L^2(0,T;H^r(X))} + \| \cdot \|_{H^s(0,T;L^2(X))}$ and we recall from [18, Sec. 4, Theorem 2.1] that for all $u \in H^{r,s}(X \times (0,T))$, $r, s > 0$ and all $(j,k) \in \mathbb{N}_0^n \times \mathbb{N}_0$ such that $1 - \frac{|j|}{r} - \frac{k}{s} > 0$, we have

$$\partial_x^j \partial_t^k u \in H^{\mu,\nu}(X \times (0,T)) \quad \text{with } \frac{\mu}{r} = \frac{\nu}{s} = 1 - \frac{|j|}{s} - \frac{k}{s}, \tag{1.2}$$

and the estimate

$$\|\partial_x^j \partial_t^k u\|_{H^{\mu,\nu}(X\times(0,T))} \leq \|u\|_{H^{r,s}(X\times(0,T))}. \tag{1.3}$$

1.3. *Main results*

Prior to examining the inverse problem under consideration in this paper, we treat the well-posedness issue for the IBVP (1.1). Let N be the unique natural number satisfying

$$N \in \mathbb{N} \cap \left(\frac{n+2}{4} + 1, \frac{n+2}{4} + 2 \right].$$

Then we have the following existence, uniqueness and regularity result for the solution to the IBVP (1.1).

Proposition 1.1. *Assume that Γ is $C^{2(N+1)}$ and pick $M \in (0,+\infty)$. Let $A \in W^{2N+1,\infty}(\Omega, \mathbb{R}^n)$ verify $\nabla \cdot A = 0$ a.e. in Ω and let $p \in W^{2N+1,\infty}(\Omega, \mathbb{R})$ and $q^\pm \in W^{2N+1,\infty}(\Omega, \mathbb{R})$ be such that*

$$\|A\|_{W^{2N+1,\infty}(\Omega)^n} + \|p\|_{W^{2N+1,\infty}(\Omega)} + \|q^+\|_{W^{2N+1,\infty}(\Omega)} + \|q^-\|_{W^{2N+1,\infty}(\Omega)} \leq M.$$

Then, for all $g = (g^+, g^-)^T \in H^{2(N+7/4),N+7/4}(\Sigma)^2$ and all $u_0 = (u_0^+, u_0^-)^T \in H^{2N+3}(\Omega)^2$, fulfilling the following compatibility conditions

$$\partial_t^\ell g(\cdot, 0) = (-i)^\ell \begin{pmatrix} -\Delta + q^+ & A \cdot \nabla + p \\ -A \cdot \nabla + p & -\Delta + q^- \end{pmatrix}^\ell u_0 \quad \text{on } \Gamma, \ell = 0, \dots, N, \tag{1.4}$$

the IBVP (1.1) admits a unique solution $u = (u^+, u^-)^T \in W^{1,\infty}(0,T; W^{1,\infty}(\Omega)^2)$. Moreover, there exists a positive constant C, depending only on Ω, T, M, u_0 and g, such that

$$\|u\|_{W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^2)} \leq C. \tag{1.5}$$

One can see that rather strong regularities are assumed in Proposition 1.1, on the data u_0 and g, and on the coefficients A, p and $q^\pm$, of the IBVP (1.1). These hypotheses (and the corresponding compatibility conditions (1.4)) are only sufficient conditions for getting a suitably smooth weak-solution to (1.1), as required by the main result of this paper, stated in Theorem 1.2. Of course, Theorem 1.2

could be established independently of Proposition 1.1, under the indirect assumption that the solution to (1.1) is sufficiently regular. But in doing so we could not guarantee that the result of Theorem 1.2 is applicable to some two-level quantum system.

As a preamble, we introduce the sets of admissible unknown coefficients A, p and $q^{\pm}$. To this purpose, for $M \in (0,+\infty)$, $A_0 \in W^{2N+1,\infty}(\Omega,\mathbb{R}^n)$ and $q_0 \in W^{2N+1,\infty}(\Omega,\mathbb{R})$, we define

(i) the set of admissible unknown gradient vector potentials as

$$\mathcal{A}_M(A_0) = \{A \in W^{2N+1,\infty}(\Omega,\mathbb{R}^n), \|A\|_{W^{2N+1,\infty}(\Omega)^n} \le M, \\ \nabla \cdot A = 0 \text{ and } \partial_x^k A = \partial_x^k A_0 \text{ on } \Gamma, |k| \le 2(N-1)\},$$

(ii) and the set of admissible unknown electric potentials as

$$\mathcal{Q}_M(q_0) = \{q \in W^{2N+1,\infty}(\Omega,\mathbb{R}), \|q\|_{W^{2N+1,\infty}(\Omega)} \le M \text{ and} \\ \partial_x^k q = \partial_x^k q_0 \text{ on } \Gamma, |k| \le 2(N-1)\}.$$

Next, p_0 and $q_0^{\pm}$ being fixed in $W^{2N+1,\infty}(\Omega,\mathbb{R})$, the main result of this paper is as follows.

Theorem 1.2. *Assume that* Γ *is* $C^{2(N+1)}$ *and for* $j=1,2$, *let* $A_j \in \mathcal{A}_M(A_0)$, $p_j \in \mathcal{Q}_M(p_0)$ *and* $q_j^{\pm} \in \mathcal{Q}_M(q_0^{\pm})$. *Then, there exist a sub-boundary* $\Gamma_* \subset \Gamma$, *and* $n+2$ *initial states* $u_0^k = (u_0^{+,k}, u_0^{-,k})^T \in H^{2N+3}(\Omega)^2$ *and boundary conditions* $g^k = (g^{+,k}, g^{-,k})^T \in H^{2(N+7/4),N+7/4}(\Sigma)^2$, $k=1,\dots,n+2$, *fulfilling the compatibility conditions*

$$\partial_t^\ell g^k(\cdot,0) = (-i)^\ell \begin{pmatrix} -\Delta + q_0^+ & A_0\cdot\nabla + p_0 \\ -A_0\cdot\nabla + p_0 & -\Delta + q_0^- \end{pmatrix}^\ell u_0^k \quad \text{on } \Gamma,\ \ \ell = 0,\dots,N, \tag{1.6}$$

such that we have

$$\|A_1 - A_2\|_{L^2(\Omega)^n}^2 + \|p_1 - p_2\|_{L^2(\Omega)}^2 + \|q_1^+ - q_2^+\|_{L^2(\Omega)}^2 + \|q_1^- - q_2^-\|_{L^2(\Omega)}^2 \\ \le C \sum_{k=1}^{n+2} (\|\partial_\nu\partial_t u_1^{-,k} - \partial_\nu\partial_t u_2^{-,k}\|_{L^2(\Sigma_*)}^2 + \|\partial_\nu\partial_t u_1^{+,k} - \partial_\nu\partial_t u_2^{+,k}\|_{L^2(\Sigma_*)}^2),$$

for some positive constant C, *depending only on* Ω, T, M *and* $(u_0^{\pm,k}, g^{\pm,k})$ *for* $k=1,\dots,n+2$. *Here, we have set* $\Sigma_* = \Gamma_* \times (0,T)$ *and* $u_j^k = (u_j^{+,k}, u_j^{-,k})^T$, *for* $j=1,2$ *and* $k=1,\dots,n+2$, *is the solution to* (1.1) *given by Proposition* 1.1, *where* $(A_j, p_j, q_j^{\pm}, u_0^{\pm,k}, g^{\pm,k})$ *is substituted for* $(A, p, q^{\pm}, u_0^{\pm}, g^{\pm})$.

Theorem 1.2 claims Lipschitz stable recovery of $n+3$ unknown functions (p and $q^{\pm}$ and the n components of A) by $n+2$ local boundary measurements of the solution $u = (u^+, u^-)$ to (1.1). Bearing in mind that A is divergence free and that its trace on Γ is prescribed, this amounts to saying that $n+2$ unknown scalar functions can be stably retrieved by the same number of local Neumann data. From this viewpoint, the result of Theorem 1.2 is thus optimal.

Further, we point out that Theorem 1.2 applies in particular for any sub-boundary $\Gamma_* \subset \Gamma$ obeying the geometric condition

$$\{x \in \Gamma, (x - x_0) \cdot \nu(x) \geq 0\} \subset \Gamma_*$$

for an arbitrary $x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$. We refer to Assumption 3.1 and Remark 3.2 in Sec. 3.1 for more details on this specific matter.

1.4. *Outline*

The derivation of Proposition 1.1 can be found in Sec. 2 while Sec. 3 contains the proof of Theorem 1.2. Finally, several technical results used for establishing that the elliptic part of the Schrödinger equation (1.1) is self-adjoint in $L^2(\Omega)^2$, are collected in the appendix.

2. Analysis of the Direct Problem

In this section, we prove Proposition 1.1.

2.1. *Preliminaries: Self-adjointness and basic regularity*

2.1.1. *Self-adjointness*

In this section, we assume that Γ is C^2, that $A \in L^\infty(\Omega, \mathbb{R}^n)$ is gradient free, i.e. that $\nabla \cdot A = 0$ a.e. in Ω, and that $p \in L^\infty(\Omega, \mathbb{R})$ and $q^\pm \in L^\infty(\Omega, \mathbb{R})$.

Let Δ^D denote the Dirichlet-Laplacian in $L^2(\Omega)$, with domain $\mathcal{D}_0 = H_0^1(\Omega) \cap H^2(\Omega)$. Since Γ is C^2 then it is well known that Δ^D is self-adjoint in $L^2(\Omega)$. As a consequence, the operator

$$\Delta^D u = (\Delta^D u^+, \Delta^D u^-)^T, \quad u = (u^+, u^-)^T \in \mathcal{D}_0^2,$$

is self-adjoint in $L^2(\Omega)^2$. Put

$$\tilde{A} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix} \quad \text{and} \quad \tilde{q} = \begin{pmatrix} q^+ & 0 \\ 0 & q^- \end{pmatrix}.$$

Since $p \in L^\infty(\Omega, \mathbb{R})$ (respectively, $q^\pm L^\infty(\Omega, \mathbb{R})$) then the multiplication operator by $\tilde{p}$ (respectively, $\tilde{q}$), defined by $\tilde{p}u = (pu^-, pu^+)^T$ (respectively, $\tilde{q}u = (q^- u^+, q^- u^-)^T$) for all $u = (u^+, u^-)^T \in L^2(\Omega)^2$, and denoted by $\tilde{p}$ (respectively, $\tilde{q}$) is symmetric in $L^2(\Omega)^2$. Similarly, since $\nabla \cdot A = 0$, then we infer from the Stokes formula that the operator

$$\tilde{A} \cdot \nabla u = (A \cdot \nabla u^-, -A \cdot \nabla u^+)^T, \quad u = (u^+, u^-)^T \in H_0^1(\Omega)^2,$$

is symmetric in $L^2(\Omega)^2$ as well (see Appendix A). As a consequence, the operator $\tilde{A} \cdot \nabla + \tilde{p} + \tilde{q}$, with domain $H_0^1(\Omega)^2$, is symmetric in $L^2(\Omega)^2$. Moreover, we know from Appendix B that $\tilde{A} \cdot \nabla + \tilde{p} + \tilde{q}$ is Δ^D-bounded in $L^2(\Omega)^2$, with relative bound zero: For any $\epsilon \in (0, 1)$, there exists $C_\epsilon > 0$, depending only on ϵ, $\|A\|_{L^\infty(\Omega)}$,

$\|p\|_{L^\infty(\Omega)}$ and $\|q^\pm\|_{L^\infty(\Omega)}$, such that

$$\|(\tilde{A}\cdot\nabla+\tilde{p}+\tilde{q})u\|_{L^2(\Omega)^2} \le \epsilon\|\Delta^D u\|_{L^2(\Omega)^2} + C_\epsilon\|u\|_{L^2(\Omega)^2}, \quad u\in\mathcal{D}_0^2.$$

Therefore, the Kato–Rellich Theorem (see [22, Theorem X.12]) yields the following.

Lemma 2.1. *Assume that* Γ *is* C^2, *that* $A\in L^\infty(\Omega,\mathbb{R}^n)$ *fulfills* $\nabla\cdot A=0$ *a.e. in* Ω, *and that* $p\in L^\infty(\Omega,\mathbb{R})$ *and* $q^\pm\in L^\infty(\Omega,\mathbb{R})$. *Then the operator* $H(A,q^\pm,p) = -\Delta^D+\tilde{q}+\tilde{A}\cdot\nabla+\tilde{p}$, *with domain* $D(H(A,q^\pm,p))=\mathcal{D}_0^2$, *is self-adjoint in* $L^2(\Omega)^2$.

Otherwise stated, $H(A,q^\pm,p)$ is the self-adjoint realization in $L^2(\Omega)^2$ of the formal operator acting in $(C_0^\infty(\Omega)^2)'$, $\mathcal{H}(A,q^\pm,p)=-\Delta+\tilde{q}+\tilde{A}\cdot\nabla+\tilde{p}$, endowed with homogeneous boundary conditions on Γ. We point out for further use that $u=(u^+,u^-)^T$ solves (1.1) may be equivalently rewritten as u is solution to the IBVP

$$\begin{cases} -i\partial_t u+\mathcal{H}(A,q^\pm,p)u=0 & \text{in } Q,\\ u=g & \text{on } \Sigma,\\ u(\cdot,0)=u_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

2.1.2. *Existence, uniqueness and basic regularity result*

In this section, we establish the following existence and uniqueness result by adapting the analysis carried out in [12, Sec. 2] to the coupled system (1.1).

Lemma 2.2. *Assume that* Γ, A, p *and* $q^\pm$ *are the same as in Lemma* 2.1. *Then, for all* $g=(g^+,g^-)^T\in H^{7/2,7/4}(\Sigma)^2$ *and all* $u_0=(u_0^+,u_0^-)^T\in H^3(\Omega)^2$ *such that*

$$g(\cdot,0)=u_0 \quad \text{on } \Gamma, \tag{2.2}$$

the IBVP (1.1) *admits a unique solution* $u=(u^+,u^-)^T\in H^{2,1}(Q)^2$ *to* (1.1). *Moreover, there exists a constant* C, *depending only on* Ω, T *and* M, *such that*

$$\|u\|_{H^{2,1}(Q)^2}\le C(\|u_0\|_{H^3(\Omega)^2}+\|g\|_{H^{7/2,7/4}(\Sigma)^2}). \tag{2.3}$$

Proof. Since $g\in H^{7/2,7/4}(\Sigma)^2$ and $u_0\in H^3(\Omega)^2$ fulfill (2.2), then, by virtue of [18, Sec. 4, Theorem 2.3], there exists $G=(G^+,G^-)^T\in H^{4,2}(Q)^2$ such that $G=g$ on Σ and $G(\cdot,0)=u_0$ in Ω. Moreover, we have

$$\|G\|_{H^{4,2}(Q)^2}\le C(\|u_0\|_{H^3(\Omega)^2}+\|g\|_{H^{7/2,7/4}(\Sigma)^2}), \tag{2.4}$$

for some constant $C>0$, depending only on Ω, T and M.

Evidently, $u=(u^+,u^-)^T$ is solution to (1.1) if and only if $v=u-G=(u^+-G^+,u^--G^-)^T$ is solution to the following Cauchy problem

$$\begin{cases} -i\partial_t v+H(A,q^\pm,p)v=f & \text{in } Q,\\ v(\cdot,0)=0 & \text{in } \Omega, \end{cases} \tag{2.5}$$

where $f = (f^+, f^-)^T = -(-i\partial_t - \Delta + \tilde{q})G$. Further, with reference to [18, Sec. 4, Proposition 2.3] we have $\partial_t G \in H^{2,1}(Q)^2$ with $\|\partial_t G\|_{H^{2,1}(Q)^2} \leq C\|G\|_{H^{4,2}(Q)^2}$, whence $f \in H^1(0,T;L^2(\Omega)^2)$ and

$$\|f\|_{H^1(0,T;L^2(\Omega)^2)} \leq C(\|\partial_t G\|_{H^{2,1}(Q)^2} + \|G\|_{H^{2,1}(Q)^2}) \leq C\|G\|_{H^{4,2}(Q)^2}. \quad (2.6)$$

Moreover, since the operator $-iH(A, q^\pm, p)$ is m-dissipative in $L^2(\Omega)^2$, by Lemma 2.1, we deduce from (2.5) upon applying [9, Lemma 2.1] (with $X = L^2(\Omega)^2$, $U = -iH(A, q^\pm, p)$ and $B = 0$) that there exists a unique solution $v \in H^{2,1}(Q)^2$ to (2.5), such that

$$\|v\|_{H^{2,1}(Q)^2} \leq C\|f\|_{H^1(0,T;L^2(\Omega)^2)}.$$

Finally, bearing in mind that $u = v + G$, we obtain (2.3) by combining the above estimate with (2.4) and (2.6). □

Armed with Lemma 2.1, we may now seek higher regularity for the solution to the IBVP (1.1) upon imposing more restrictive conditions on Γ, A, p, $q^\pm$, u_0 and g.

2.2. *Improved regularity and proof of Proposition* 1.1

2.2.1. *Improved regularity result*

The statement we are aiming for can be formulated as follows.

Lemma 2.3. *Fix $m \in \mathbb{N}_0$ and assume that Γ is $C^{2(m+1)}$. Let $A \in W^{2m+1,\infty}(\Omega, \mathbb{R}^n)$ verify $\nabla \cdot A = 0$ a.e. in Ω and pick $p \in W^{2m+1,\infty}(\Omega, \mathbb{R})$ and $q^\pm \in W^{2m+1,\infty}(\Omega, \mathbb{R})$ in such a way that*

$$\|A\|_{W^{2m+1,\infty}(\Omega)^n} + \|p\|_{W^{2m+1,\infty}(\Omega)} + \|q^+\|_{W^{2m+1,\infty}(\Omega)} + \|q^-\|_{W^{2m+1,\infty}(\Omega)} \leq M,$$

for some a priori fixed positive constant M. Then for all $g = (g^+, g^-)^T \in H^{2(m+7/4),m+7/4}(\Sigma)^2$ and all $u_0 = (u_0^+, u_0^-)^T \in H^{2m+3}(\Omega)^2$ fulfilling the compatibility conditions

$$\partial_t^\ell g(\cdot, 0) = (-i)^\ell \mathcal{H}(A, q^\pm, p)^\ell u_0 \quad \text{on } \Gamma, \ \ \ell = 0, \ldots, m, \quad (2.7)$$

there exists a unique solution $u \in \bigcap_{\ell=0}^{m+1} H^{m+1-\ell}(0,T;H^{2\ell}(\Omega)^2)$ to (2.5). Moreover u satisfies the estimate

$$\sum_{\ell=0}^{m+1} \|u\|_{H^{m+1-\ell}(0,T;H^{2\ell}(\Omega)^2)} \leq C(\|u_0\|_{H^{2m+3}(\Omega)^2} + \|g\|_{H^{2(m+7/4),m+7/4}(\Sigma)^2}), \quad (2.8)$$

where C is a positive constant depending only on Ω, T and M.

Proof. We will prove the result by induction on $m \in \mathbb{N}_0$.

(1) *Base case.* We first consider the case $m = 0$. Since the compatibility conditions (2.7) with $m = 0$ reduce to (2.2) (as we have $(-i)^0 = 1$, $\partial_t^0 g(\cdot, 0) = g(\cdot, 0)$ and

$\mathcal{H}(A, q^{\pm}, p)^0$ is the identity operator in $L^2(\Omega)^2$, by convention), then the claim of Lemma 2.3 follows readily from Lemma 2.2.

(2) *Induction step.* Let us suppose that the claim of Lemma 2.3 holds for some $m \in \mathbb{N}_0$. We shall prove that it is still true for $m+1$, provided Γ is C^{2m+3}, $A \in W^{2m+3,\infty}(\Omega, \mathbb{R}^n)$, $q^{\pm} \in W^{2m+3,\infty}(\Omega, \mathbb{R})$, $p \in W^{2m+3,\infty}(\Omega, \mathbb{R})$ and $(g, u_0) \in H^{2(m+11/4),m+11/4}(\Sigma)^2 \times H^{2m+5}(\Omega)^2$ verify the compatibility condition (2.7) where $m+1$ is substituted for m, i.e.

$$\partial_t^{\ell} g(\cdot, 0) = (-i)^{\ell} \mathcal{H}(A, q^{\pm}, p)^{\ell} u_0 \quad \text{on } \Gamma, \ \ \ell = 0, \ldots, m+1. \tag{2.9}$$

Let u denote the $\bigcap_{\ell=0}^{m+1} H^{m+1-\ell}(0, T; H^{2\ell}(\Omega)^2)$-solution to (2.1), satisfying (2.8). Then upon differentiating (2.1) with respect to t, we see that $z = \partial_t u$ solves

$$\begin{cases} -i\partial_t z + \mathcal{H}(A, q^{\pm}, p) z = 0 & \text{in } Q, \\ z = \partial_t g & \text{on } \Sigma, \\ z(\cdot, 0) = z_0 & \text{in } \Omega, \end{cases} \tag{2.10}$$

with $z_0 = -i\mathcal{H}(A, q^{\pm}, p) u_0 \in H^{2m+3}(\Omega)^2$. Moreover, we know from (1.2) that $\partial_t g \in H^{2(m+7/4),m+7/4}(\Sigma)^2$ and from (2.9) that

$$\begin{aligned} \partial_t^{\ell}(\partial_t g)(\cdot, 0) &= \partial_t^{\ell+1} g(\cdot, 0) = (-i)^{\ell+1} \mathcal{H}(A, q^{\pm}, p)^{\ell+1} u_0 \\ &= (-i)^{\ell} \mathcal{H}(A, q^{\pm}, p)^{\ell} z_0, \quad \ell = 0, \ldots, m. \end{aligned}$$

Therefore, we have $z \in \bigcap_{\ell=0}^{m+1} H^{m+1-\ell}(0, T; H^{2\ell}(\Omega)^2)$ and

$$\begin{aligned} \sum_{\ell=0}^{m+1} \|z\|_{H^{m+1-\ell}(0,T;H^{2\ell}(\Omega)^2)} &\leq C(\|z_0\|_{H^{2m+3}(\Omega)^2} + \|\partial_t g\|_{H^{2(m+7/4),m+7/4}(\Sigma)^2}) \\ &\leq C(\|u_0\|_{H^{2m+5}(\Omega)^2} + \|g\|_{H^{2(m+11/4),m+11/4}(\Sigma)^2}), \end{aligned} \tag{2.11}$$

by induction hypothesis (and from the estimate $\|\partial_t g\|_{H^{2(m+7/4),m+7/4}(\Sigma)^2} \leq \|g\|_{H^{2(m+11/4),m+11/4}(\Sigma)^2}$, arising from (1.3)). Since $u \in \bigcap_{\ell=0}^{m+1} H^{m+1-\ell}(0, T; H^{2\ell}(\Omega)^2)$, then this may be equivalently rewritten as

$$u \in \bigcap_{\ell=0}^{m+1} H^{m+2-\ell}(0, T; H^{2\ell}(\Omega)^2) \tag{2.12}$$

and we infer from (2.8) and (2.11) that

$$\sum_{\ell=0}^{m+1} \|u\|_{H^{m+2-\ell}(0,T;H^{2\ell}(\Omega)^2)} \leq C(\|u_0\|_{H^{2m+5}(\Omega)^2} + \|g\|_{H^{2(m+11/4),m+11/4}(\Sigma)^2}). \tag{2.13}$$

Thus, we are left with the task of showing that $u \in L^2(0, T; H^{2(m+2)}(\Omega)^2)$ and that

$$\|u\|_{L^2(0,T;H^{2(m+2)}(\Omega)^2)} \leq C(\|u_0\|_{H^{2m+5}(\Omega)^2} + \|g\|_{H^{2(m+11/4),m+11/4}(\Sigma)^2}).$$

This can be done by applying the elliptic regularity theorem twice. Indeed, for a.e. $t \in (0,T)$ we infer from (2.1) that $u(\cdot,t)$ is solution to the following elliptic system:

$$\begin{cases} \Delta u(\cdot,t) = h(\cdot,t) & \text{in } \Omega, \\ u(\cdot,t) = g(\cdot,t) & \text{on } \Gamma, \end{cases} \tag{2.14}$$

where $h(\cdot,t) = -iz(\cdot,t) - \tilde{q}u(\cdot,t) - \tilde{A}\cdot\nabla u(\cdot,t) + \tilde{p}u(\cdot,t)$. As $h(\cdot,t) \in H^{2m+1}(\Omega)^2$ and $g(\cdot,t) \in H^{2(m+11/4)}(\Gamma)^2 \subset H^{2m+5/2}(\Gamma)^2$, then we have $u(\cdot,t) \in H^{2m+3}(\Omega)^2$ by the elliptic regularity theorem, with

$$\begin{aligned} \|u(\cdot,t)\|_{H^{2m+3}(\Omega)^2} &\leq C(\|h(\cdot,t)\|_{H^{2m+1}(\Omega)^2} + \|g(\cdot,t)\|_{H^{2m+5/2}(\Gamma)^2}) \\ &\leq C(\|z(\cdot,t)\|_{H^{2m+1}(\Omega)^2} + \|u(\cdot,t)\|_{H^{2m+2}(\Omega)^2} \\ &\quad + \|g(\cdot,t)\|_{H^{2(m+11/4)}(\Gamma)^2}). \end{aligned} \tag{2.15}$$

Therefore, it holds true that $h(\cdot,t) \in H^{2(m+1)}(\Omega)^2$ for a.e. $t \in (0,T)$. Moreover, as we have $g(\cdot,t) \in H^{2(m+11/4)}(\Gamma)^2 \subset H^{2m+7/2}(\Gamma)^2$, then another application of the elliptic regularity theorem yields $u(\cdot,t) \in H^{2(m+2)}(\Omega)^2$ and

$$\begin{aligned} \|u(\cdot,t)\|_{H^{2(m+2)}(\Omega)^2} &\leq C(\|h(\cdot,t)\|_{H^{2(m+1)}(\Omega)^2} + \|g(\cdot,t)\|_{H^{2m+7/2}(\Gamma)^2}) \\ &\leq C(\|z(\cdot,t)\|_{H^{2(m+1)}(\Omega)^2} + \|u(\cdot,t)\|_{H^{2m+3}(\Omega)^2} \\ &\quad + \|g(\cdot,t)\|_{H^{2(m+11/4)}(\Gamma)^2}). \end{aligned}$$

Putting the above estimate with (2.15), we end up getting that

$$\begin{aligned} \|u(\cdot,t)\|_{H^{2(m+2)}(\Omega)^2} &\leq C(\|z(\cdot,t)\|_{H^{2(m+1)}(\Omega)^2} + \|u(\cdot,t)\|_{H^{2(m+1)}(\Omega)^2} \\ &\quad + \|g(\cdot,t)\|_{H^{2(m+11/4)}(\Gamma)^2}). \end{aligned} \tag{2.16}$$

Further, since u and z are in $L^2(0,T;H^{2(m+1)}(\Omega)^2)$ and since $g \in L^2(0,T;H^{2(m+11/4)}(\Gamma)^2)$, we infer from (2.16) that $u \in L^2(0,T;H^{2(m+2)}(\Omega)^2)$ verifies

$$\|u\|_{L^2(0,T;H^{2(m+2)}(\Omega)^2)} \leq C(\|u\|_{H^1(0,T;H^{2(m+1)}(\Omega)^2)} + \|g\|_{L^2(0,T;H^{2(m+11/4)}(\Gamma)^2)}).$$

In view of (2.13) this entails that

$$\|u\|_{L^2(0,T;H^{2(m+2)}(\Omega)^2)} \leq C(\|u_0\|_{H^{2m+5}(\Omega)^2} + \|g\|_{H^{2(m+11/4),m+11/4}(\Sigma)^2}),$$

which, together with (2.12) and (2.13) yield the statement of Lemma 2.3 for $m+1$. □

Having established Lemma 2.3, we turn now to proving Proposition 1.1.

2.2.2. *Proof of Proposition* 1.1

We apply Lemma 2.3 with $m = N$ and get a unique solution u to (1.1) within the space $H^2(0,T;H^{2(N-1)}(\Omega)^2)$. Since $2(N-1) > \frac{n}{2} + 1$ from the very definition of

N, then $u \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^2)$ by the Sobolev embedding theorem. Moreover, (2.8) yields

$$\begin{aligned}\|u\|_{W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^2)} &\le \|u\|_{H^2(0,T;H^{2(N-1)}(\Omega)^2)} \\ &\le C(\|u_0\|_{H^{2N+3}(\Omega)^2} + \|g\|_{H^{2(N+7/4),N+7/4}(\Sigma)^2}),\end{aligned}$$

for some constant C depending only on Ω, T and M. This proves the desired result.

3. Analysis of the Inverse Problem

This section contains the proof of Theorem 1.2.

3.1. *Preliminaries: Carleman estimate and all that*

In this section, we establish in Corollary 3.4 a weighted energy estimate for the Schrödinger equation, which is the main tool used in the derivation of Theorem 1.2. This inequality is a byproduct of the global Carleman estimate for the Schrödinger operator of [2, Proposition 1], that we recall in Proposition 3.3. In order to state this inequality, we consider a function $\tilde{\beta} \in C^4(\overline{\Omega}, \mathbb{R}_+)$ and an open subset $\Gamma_* \subset \Gamma$ fulfilling the following conditions:

Assumption 3.1. (1) There exists a constant $c > 0$ such that the estimate $|\nabla\tilde{\beta}(x)| \ge c$ holds for all $x \in \Omega$;

(2) $\partial_\nu \tilde{\beta}(x) = \nabla\tilde{\beta}(x) \cdot \nu(x) < 0$ for all $x \in \partial\Omega \backslash \Gamma_*$, where ν is the outward unit normal vector to Γ;

(3) There exists $\Lambda_1 > 0$ and $\epsilon > 0$ such that $\lambda|\nabla\tilde{\beta}(x) \cdot \zeta|^2 + D^2\tilde{\beta}(x,\zeta,\zeta) \ge \epsilon|\zeta|^2$ for all $\zeta \in \mathbb{R}^n$, $x \in \Omega$ and $\lambda > \Lambda_1$, where $D^2\tilde{\beta}(x) = (\frac{\partial^2\tilde{\beta}(x)}{\partial x_i \partial x_j})_{1\le i,j\le n}$ and $D^2\tilde{\beta}(x,\zeta,\zeta)$ denotes the $\mathbb{R}^n$-scalar product of $D^2\tilde{\beta}(x)\zeta$ with ζ.

Remark 3.2. We stress out that there exist actual $\tilde{\beta}$ and Γ_* satisfying Assumption 3.1. As a matter of fact, for all $x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$ fixed, this is the case of the function $\tilde{\beta}(x) = |x - x_0|^2$ and any open subset $\Gamma_* \subset \Gamma$ containing $\{x \in \Gamma; (x - x_0) \cdot \nu(x) \ge 0\}$.

Next, we put

$$\beta(x) = \tilde{\beta}(x) + r\|\tilde{\beta}\|_{L^\infty(\Omega)}, \quad x \in \overline{\Omega}, \tag{3.1}$$

for some $r > 1$ and $K = \|\beta\|_{L^\infty(\Omega)}$, and we set

$$\begin{aligned}\varphi(x,t) &= \frac{e^{2\lambda\beta(x)}}{(T+t)(T-t)} \quad \text{and} \\ \eta(x,t) &= \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)}, \quad (x,t) \in \tilde{Q} = \Omega \times (-T,T),\end{aligned} \tag{3.2}$$

for some $\lambda > 0$. Further, for all $s > 0$, we introduce the two following operators acting in $(C_0^\infty)'(\tilde{Q})$:

$$M_1 = i\partial_t + \Delta + s^2|\nabla\eta|^2 \quad \text{and} \quad M_2 = is\eta' + 2s\nabla\eta \cdot \nabla + s(\Delta\eta). \tag{3.3}$$

It can be checked that M_1 (respectively, M_2) is the adjoint (respectively, skew-adjoint) part of the operator $e^{-s\eta}Le^{s\eta}$, where $L = i\partial_t + \Delta$. Then the global Carleman estimate borrowed from [2, Proposition 1] is as follows.

Proposition 3.3. *Let $\tilde{\beta}$ and Γ_* fulfill Assumption* 3.1, *let β, φ and η be given by* (3.1)–(3.2), *and let the operators M_j, $j = 1, 2$, be defined by* (3.3). *Then there are two constants $s_0 > 1$ and $C_0 > 0$, depending only on Ω, T and Γ_*, such that the estimate*

$$s\|e^{-s\eta}\nabla w\|^2_{L^2(\tilde{Q})^n} + s^3\|e^{-s\eta}w\|^2_{L^2(\tilde{Q})} + \sum_{j=1,2}\|M_j e^{-s\eta}w\|^2_{L^2(\tilde{Q})}$$
$$\leq C_0(s\|e^{-s\eta}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu w\|^2_{L^2(\tilde{\Sigma}_*)} + \|e^{-s\eta}Lw\|^2_{L^2(\tilde{Q})}),$$

holds for all $s > s_0$ and all $w \in L^2(-T, T; H_0^1(\Omega))$ satisfying $Lw \in L^2(\tilde{Q})$ and $\partial_\nu w \in L^2(\tilde{\Sigma}_*)$, where $\tilde{\Sigma}_* = \Gamma_* \times (-T, T)$.

As a corollary, we have the following technical result. Its proof can be found in [12, Sec. 4.1], but for the sake of self-containedness and for the convenience of the reader, we give it at the end of the section.

Corollary 3.4. *Under the conditions of Proposition* 3.3, *we have*

$$s^{-1/2}(\|e^{-s\eta}z\|^2_{L^2(\tilde{Q})} + \|e^{-s\eta}\nabla z\|^2_{L^2(\tilde{Q})^n}) + \|e^{-s\eta(\cdot,0)}z(\cdot,0)\|^2_{L^2(\Omega)}$$
$$\leq 2C_0 s^{-3/2}(s\|e^{-s\eta}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu z\|^2_{L^2(\tilde{\Sigma}_*)} + \|e^{-s\eta}Lz\|^2_{L^2(\tilde{Q})}), \quad s > s_0,$$

for all $z \in L^2(-T, T; H_0^1(\Omega))$ fulfilling $Lz \in L^2(\tilde{Q})$ and $\partial_\nu z \in L^2(\tilde{\Sigma}_*)$, where the constants C_0 and s_0 are the same as in Proposition 3.3.

Proof. Put $w = e^{-s\eta}z$. Since $\lim_{t\to -T}\eta(x, t) = +\infty$ for all $x \in \Omega$, then it holds true that $\lim_{t\to -T} w(\cdot, t) = 0$ in $L^2(\Omega)$ and hence

$$\|w(\cdot, 0)\|^2_{L^2(\Omega)} = \int_{(-T,0)\times\Omega} \partial_t|w(x, t)|^2 dxdt = 2\Re\left(\int_{(-T,0)\times\Omega}(\partial_t w)\overline{w}(x, t)dxdt\right).$$

On the other hand, we have

$$\Im\left(\int_{(-T,0)\times\Omega}(M_1 w)\overline{w}(x, t)dxdt\right)$$
$$= \Re\left(\int_{(-T,0)\times\Omega}(\partial_t w)\overline{w}(x, t)dxdt\right)$$
$$+ \Im\left(\int_{(-T,0)\times\Omega}(\Delta w + s^2|\nabla\eta|^2 w)\overline{w}(x, t)dxdt\right)$$

$$= \Re\left(\int_{(-T,0)\times\Omega} (\partial_t w)\overline{w}(x,t)dxdt\right)$$
$$- \Im\left(\int_{(-T,0)\times\Omega} (|\nabla w|^2 - s^2|\nabla\eta|^2|w|^2)(x,t)dxdt\right)$$
$$= \Re\left(\int_{(-T,0)\times\Omega} (\partial_t w)\overline{w}(x,t)dxdt\right),$$

whence $\|w(\cdot,0)\|^2_{L^2(\Omega)} = 2\Im(\int_{(-T,0)\times\Omega}(M_1 w)\overline{w}(x,t)dxdt)$. Therefore, we get

$$\|e^{-s\eta(\cdot,0)}z(\cdot,0)\|^2_{L^2(\Omega)} \le 2\|M_1 w\|_{L^2(\tilde{Q})}\|w\|_{L^2(\tilde{Q})}$$
$$\le s^{-3/2}(s^3\|e^{-s\eta}z\|^2_{L^2(\tilde{Q})} + \|M_1 e^{-s\eta}z\|^2_{L^2(\tilde{Q})})$$

with the help of the Cauchy–Schwarz and Hölder inequalities. As a consequence we have

$$s^{-1/2}(\|e^{-s\eta}z\|^2_{L^2(\tilde{Q})} + \|e^{-s\eta}\nabla z\|^2_{L^2(\tilde{Q})^n}) + \|e^{-s\eta(\cdot,0)}z(\cdot,0)\|^2_{L^2(\Omega)}$$
$$\le s^{-3/2}(s\|e^{-s\eta}\nabla z\|^2_{L^2(\tilde{Q})^n} + 2s^3\|e^{-s\eta}z\|^2_{L^2(\tilde{Q})} + \|M_1 e^{-s\eta}z\|^2_{L^2(\tilde{Q})})$$
$$\le 2C_0 s^{-3/2}(s\|e^{-s\eta}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu z\|^2_{L^2(\tilde{\Sigma}_*)} + \|e^{-s\eta}Lz\|^2_{L^2(\tilde{Q})}), \quad s > s_0,$$

by Proposition 3.3, which is the desired result. □

3.2. *Proof of Theorem* 1.2

We start by linearizing the system (1.1). That is, we denote by $u_j = (u_j^+, u_j^-)$, $j = 1,2$, the solution to (1.1), where $(A_j, p_j, q_j^\pm)$ is substituted for $(A, p, q^\pm)$ and we take the difference of the two systems (1.1) associated with $j = 1,2$. Thus, putting $A = A_1 - A_2$, $p = p_1 - p_2$ and $q^\pm = q_1^\pm - q_2^\pm$, we get that $u = (u^+, u^-)^T$, where $u^\pm = u_1^\pm - u_2^\pm$, solves

$$\begin{cases} -i\partial_t u^+ - \Delta u^+ + q_1^+ u^+ = -A_1\cdot\nabla u^- - p_1 u^- & \text{in } Q, \\ \qquad - A\cdot\nabla u_2^- - p u_2^- - q^+ u_2^+ & \\ -i\partial_t u^- - \Delta u^- + q_1^- u^- = A_1\cdot\nabla u^+ - p_1 u^+ & \text{in } Q, \\ \quad + A\cdot\nabla u_2^+ - p u_2^+ - q^- u_2^- & \\ u^+(\cdot,0) = 0, \quad u^-(\cdot,0) = 0 & \text{in } \Omega, \\ u^+ = 0, \quad u^- = 0 & \text{on } \Sigma. \end{cases} \tag{3.4}$$

Since $u^\pm \in H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, we can differentiate (3.4) with respect to the time-variable. We obtain that $v = \partial_t u = (v^+, v_-)$, where

$v^\pm = \partial_t u^\pm \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, is solution to the following coupled system:

$$\begin{cases} -i\partial_t v^+ - \Delta v^+ + q_1^+ v^+ = -A_1\cdot\nabla v^- - p_1 v^- - A & \text{in } Q, \\ \quad \cdot\nabla\partial_t u_2^- - p\partial_t u_2^- - q^+\partial_t u_2^+ & \\ -i\partial_t v^- - \Delta v^- + q_1^- v^- = A_1\cdot\nabla v^+ - p_1 v^+ + A & \text{in } Q, \\ \quad \cdot\nabla\partial_t u_2^+ - p\partial_t u_2^+ - q^-\partial_t u_2^- & \\ v^+(\cdot,0) = -i(A\cdot\nabla u_0^- + pu_0^- + q^+u_0^+) & \text{in } \Omega, \\ v^-(\cdot,0) = -i(-A\cdot\nabla u_0^+ + pu_0^+ + q^-u_0^-) & \text{in } \Omega, \\ v^+ = 0, \quad v^- = 0 & \text{on } \Sigma. \end{cases} \tag{3.5}$$

We extend $u_2^\pm$ on $\tilde{Q}$ by setting $u_2^\pm(x,t) = \overline{u_2^\pm(x,-t)}$ for a.e. $(x,t)\in\Omega\times(-T,0)$. Since $u_0^\pm$, A, p and $q^\pm$ are all real valued, then it is easy to see that the function v, where $v^\pm$ is extended on $\Omega\times(-T,0)$ as $v^\pm(x,t) = -\overline{v^\pm(x,-t)}$, is solution to

$$\begin{cases} -i\partial_t v^+ - \Delta v^+ + q_1^+ v^+ = -A_1\cdot\nabla v^- - p_1 v^- & \text{in } \tilde{Q}, \\ \quad - A\cdot\nabla\partial_t u_2^- - p\partial_t u_2^- - q^+\partial_t u_2^+ & \\ -i\partial_t v^- - \Delta v^- + q_1^- v^- = A_1\cdot\nabla v^+ - p_1 v^+ & \text{in } \tilde{Q}, \\ \quad + A\cdot\nabla\partial_t u_2^+ - p\partial_t u_2^+ - q^-\partial_t u_2^- & \\ v^+(\cdot,0) = -i(A\cdot\nabla u_0^- + pu_0^- + q^+u_0^+) & \text{in } \Omega, \\ v^-(\cdot,0) = -i(-A\cdot\nabla u_0^+ + pu_0^+ + q^-u_0^-) & \text{in } \Omega, \\ v^+ = 0, \quad v^- = 0 & \text{on } \tilde{\Sigma}, \end{cases} \tag{3.6}$$

with $\tilde{\Sigma} = \Gamma\times(-T,T)$.

Let us apply Corollary 3.4 to $v^\pm$, where for the sake of notational simplicity we write $\mu^\pm$ instead of $\|e^{-s\eta}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu v^\pm\|^2_{L^2(\tilde{\Sigma}_*)}$. Then, in light of (3.6), we get for all $s > s_0$ that

$$\begin{aligned} & s^{-1/2}(\|e^{-s\eta}v^\pm\|^2_{L^2(\tilde{Q})} + \|e^{-s\eta}\nabla v^\pm\|^2_{L^2(\tilde{Q})^n}) \\ &\qquad + \|e^{-s\eta(\cdot,0)}(\pm A\cdot\nabla u_0^\mp + pu_0^\mp + q^\pm u_0^\pm)\|^2_{L^2(\Omega)} \\ &\quad \le 2C_0 s^{-3/2}(\|e^{-s\eta}A\cdot\nabla\partial_t u_2^\mp\|^2_{L^2(\tilde{Q})} + \|e^{-s\eta}p\partial_t u_2^\mp\|^2_{L^2(\tilde{Q})} \\ &\qquad + \|e^{-s\eta}q^\pm\partial_t u_2^\pm\|^2_{L^2(\tilde{Q})} + \|e^{-s\eta}A_1\cdot\nabla v^\mp\|^2_{L^2(\tilde{Q})} + \|e^{-s\eta}p_1 v^\mp\|^2_{L^2(\tilde{Q})} + s\mu^\pm) \\ &\quad \le C_1 s^{-3/2}(\|e^{-s\eta}A\|^2_{L^2(\Omega)^n} + \|e^{-s\eta}p\|^2_{L^2(\Omega)} + \|e^{-s\eta}q^\pm\|^2_{L^2(\tilde{Q})} \\ &\qquad + \|e^{-s\eta}v^\mp\|^2_{L^2(\tilde{Q})} + \|e^{-s\eta}\nabla v^\mp\|^2_{L^2(\tilde{Q})} + s\mu^\pm), \end{aligned}$$

with $C_1 = 2C_0 \max(M^2, C^2)$, where C is the constant appearing in Proposition 1.1. Indeed, in the last line we used the energy inequality (1.5), entailing that $\|\partial_t u_2^{\pm}\|_{L^\infty(\tilde{Q})} + \|\nabla \partial_t u_2^{\pm}\|_{L^\infty(\tilde{Q})^n} \leq C$. Summing up the two above estimates and recalling that $e^{-s\eta(x,t)} \leq e^{-s\eta(x,0)}$ for all $(x,t) \in \tilde{Q}$ then leads for all $s > s_1 = \max(s_0, 2C_1)$ to

$$\begin{aligned}
&\|e^{-s\eta(\cdot,0)}(-A \cdot \nabla u_0^+ + pu_0^+ + q^- u_0^-)\|_{L^2(\Omega)}^2 \\
&\quad + \|e^{-s\eta(\cdot,0)}(A \cdot \nabla u_0^- + pu_0^- + q^+ u_0^+)\|_{L^2(\Omega)}^2 \\
&\leq C_1'(s^{-3/2}(\|e^{-s\eta(\cdot,0)}A\|_{L^2(\Omega)^n}^2 + \|e^{-s\eta(\cdot,0)}p\|_{L^2(\Omega)}^2 \\
&\quad + \|e^{-s\eta(\cdot,0)}q^+\|_{L^2(\Omega)}^2 + \|e^{-s\eta(\cdot,0)}q^-\|_{L^2(\Omega)}^2) \\
&\quad + s^{-1/2}(\|e^{-s\eta(\cdot,0)}\varphi^{1/2}(\partial_\nu \beta)^{1/2}\partial_\nu v^-\|_{L^2(\tilde{\Sigma}_*)}^2 \\
&\quad + \|e^{-s\eta(\cdot,0)}\varphi^{1/2}(\partial_\nu \beta)^{1/2}\partial_\nu v^+\|_{L^2(\tilde{\Sigma}_*)}^2)),
\end{aligned} \tag{3.7}$$

where $C_1' = 2C_1$.

Having established (3.7), we shall now specify $u_0^{\pm}$ in order to estimate A, p and $q^{\pm}$. Namely we probe the system (1.1) with $n+2$ initial states $u_0^k = (u_0^{+,k}, u_0^{-,k})$, $k = 1, \ldots, n+2$, that we shall describe in the following, and suitable Dirichlet boundary conditions $g^k = (g^{+,k}, g^{-,k})$ fulfilling the compatibility condition (1.6). We proceed in two steps: In the first one, we describe u_0^k for $k = 1, 2$, while the initial states u_0^k associated with $k = 3, \ldots, n+2$, are defined in the second one. In the sequel, $u^k = (u^{+,k}, u^{-,k})$ denotes the solution to (1.1) associated with $(u_0^{\pm}, g^{\pm}) = (u_0^{\pm,k}, g^{\pm,k})$, we put $v^{\pm,k} = \partial_t u^{\pm,k}$ and $\mu_k^{\pm}(s) = \|e^{-s\eta(\cdot,0)}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu v^{\pm,k}\|_{L^2(\tilde{\Sigma}_*)}^2$, and we set $\mu_k = \mu_k^+ + \mu_k^-$.

Step 1. Choose $(u_0^{+,1}, u_0^{-,1}) = (0, 1)$ and $(u_0^{+,2}, u_0^{-,2}) = (1, 0)$. Then, taking $u_0^{\pm} = u_0^{\pm,1}$ in (3.7), we get for all $s > s_1$ that

$$\begin{aligned}
&\|e^{-s\eta(\cdot,0)}p\|_{L^2(\Omega)}^2 + \|e^{-s\eta(\cdot,0)}q^-\|_{L^2(\Omega)}^2 \\
&\leq C_1' s^{-3/2}(\|e^{-s\eta(\cdot,0)}A\|_{L^2(\Omega)^n}^2 + \|e^{-s\eta(\cdot,0)}p\|_{L^2(\Omega)}^2 \\
&\quad + \|e^{-s\eta(\cdot,0)}q^+\|_{L^2(\Omega)}^2 + \|e^{-s\eta(\cdot,0)}q^-\|_{L^2(\Omega)}^2 + s\mu_1),
\end{aligned}$$

whereas with $u_0^{\pm} = u_0^{\pm,2}$, we obtain

$$\begin{aligned}
&\|e^{-s\eta(\cdot,0)}q^+\|_{L^2(\Omega)}^2 + \|e^{-s\eta(\cdot,0)}p\|_{L^2(\Omega)}^2 \\
&\leq C_1' s^{-3/2}(\|e^{-s\eta(\cdot,0)}A\|_{L^2(\Omega)^n}^2 + \|e^{-s\eta(\cdot,0)}p\|_{L^2(\Omega)}^2 \\
&\quad + \|e^{-s\eta(\cdot,0)}q^+\|_{L^2(\Omega)}^2 + \|e^{-s\eta(\cdot,0)}q^-\|_{L^2(\Omega)}^2 + s\mu_2).
\end{aligned}$$

Summing up the two above estimates then yields

$$(1-2C_1's^{-3/2})(\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)}+\|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)}+\|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)})$$
$$\leq 2C_1'\left(s^{-3/2}\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n}+s^{-1/2}\sum_{k=1}^2\mu_k(s)\right),\quad s>s_1.$$

Therefore, for all $s>s_2=\max(s_1,(4C_1')^{2/3})$ we have

$$\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)}+\|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)}+\|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)}$$
$$\leq C_2\left(s^{-3/2}\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)}+s^{-1/2}\sum_{k=1}^2\mu_k(s)\right),\tag{3.8}$$

with $C_2=4C_1'$.

Step 2. For $k=1,\dots,n$, we choose $(u_0^{+,k+2},u_0^{-,k+2})=(x_k,0)$ in (3.7) and find that

$$\|e^{-s\eta(\cdot,0)}(-A_k+x_kp)\|^2_{L^2(\Omega)}+\|e^{-s\eta(\cdot,0)}x_kq^+\|^2_{L^2(\Omega)}$$
$$\leq C_1's^{-3/2}(\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n}+\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)}$$
$$+\|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)}+\|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)}+s\mu_{k+2}(s)),$$

provided $s>s_1$. Thus, taking into account that

$$|\pm A_k+x_kp|^2\geq\frac{1}{2}|A_k|^2-|x_kp|^2,\quad k=1,\dots,n,$$

we get for all $s>s_1$ that

$$\frac{1}{2}\|e^{-s\eta(\cdot,0)}A_k\|^2_{L^2(\Omega)}\leq C_1's^{-3/2}(\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n}+\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)}$$
$$+\|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)}+\|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)}+s\mu_{k+2}(s))$$
$$+\|e^{-s\eta(\cdot,0)}x_kq^+\|^2_{L^2(\Omega)},$$

whence

$$\|e^{-s\eta(\cdot,0)}A_k\|^2_{L^2(\Omega)}-C_{1,k}s^{-3/2}\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n}$$
$$\leq C_{1,k}(\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)}+\|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)}$$
$$+\|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)}+s^{-1/2}\mu_{k+2}(s)),$$

where $C_{1,k} = 2(C_1' + \|x_k\|^2_{L^\infty(\Omega)})$. Summing up over $k = 1, \ldots, n$, we obtain for all $s > s_1$ that

$$(1 - C_1'' s^{-3/2})\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n} \leq C_1'' \left(\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} + s^{-1/2}\sum_{k=1}^{n} \mu_{k+2}(s) \right),$$

with $C_1'' = \sum_{k=1}^n C_{1,k}$. Thus, taking $s > s_2' = \max(s_1, (2C_1'')^{2/3})$ and setting $C_2' = 2C_1''$, we find that

$$\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n} \leq C_2' \left(\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} + s^{-1/2}\sum_{k=1}^{n} \mu_{k+2}(s) \right).$$

From this and (3.8), it then follows for all $s > s_* = \max(s_2, s_2', (2C_2C_2')^{2/3})$ that

$$\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n} + \|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} \leq C_3 \sum_{k=1}^{n+2} \mu_k(s),$$

with $C_3 = (1 + C_2)C_2'$. Thus, bearing in mind that $0 \leq \eta(x, 0) \leq e^{2\lambda K}/T^2$ for all $x \in \Omega$, we find upon setting $C_4 = e^{(2s_* e^{2\lambda K})/T^2} C_3$ that

$$\|A\|^2_{L^2(\Omega)^n} + \|p\|^2_{L^2(\Omega)} + \|q^+\|^2_{L^2(\Omega)} + \|q^-\|^2_{L^2(\Omega)} \leq C_4 \sum_{k=1}^{n+2} \mu_k(s_*).$$

Now, the result follows from this upon noticing that

$$\begin{aligned}\mu^{\pm,k}(s_*) &= 2\|e^{-s_*\eta(\cdot,0)}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu v^{\pm,k}\|^2_{L^2(\Sigma_*)} \\ &\leq C\|\partial_\nu v^{\pm,k}\|^2_{L^2(\Sigma_*)}, \quad k = 1, \ldots, n+2,\end{aligned}$$

with $C = 2\|e^{-\eta(\cdot,0)}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\|^2_{L^\infty(\Sigma_*)} < +\infty$, since $\varphi(x, t) = \varphi(x, -t)$, $\eta(x, t) = \eta(x, -t)$ and $v^{\pm,k+2}(x, t) = -\overline{v^{\pm,k+2}(x, -t)}$ for all $(x, t) \in \Omega \times (-T, 0)$.

3.3. *Remark: A wider choice of initial conditions*

We point out that Theorem 1.2 works for a much wider choice of initial conditions than the ones we used in the proof presented above. As a matter of fact, it is easy to see that we may as well choose $(u_0^{+,1}, u_0^{-,1}) = (0, z_-)$ and $(u_0^{+,2}, u_0^{-,2}) = (z_+, 0)$ in Step 1 of Sec. 3.2, where $z_\pm$ is any non-zero complex number, and that Step 2 can be rewritten as follows:

Step 2′. We choose $2n$ functions $u_0^{\pm,k+2} : \Omega \to \mathbb{C}$, for $k = 1, \ldots, n$, such that the two[a] matrices $(U_0^{\pm})^* U_0^{\pm}$, where $U_0^{\pm} = (\partial_l u_0^{\pm,k+2})_{1\le k,l\le n}$ and $(U_0^{\pm})^*$ is the Hermitian conjugate of $U_0^{\pm}$, are strictly positive definite, i.e.

$$\exists\ c_0^{\pm} > 0, \quad |U_0^{\pm}\xi| \ge c_0^{\pm}|\xi|, \quad \xi \in \mathbb{C}^n. \tag{3.9}$$

For each $k = 1, \ldots, n$, we substitute $u_0^{\pm,k+2}$ for $u_0^{\pm}$ in (3.7) and find that

$$\begin{aligned} &\|e^{-s\eta(\cdot,0)}(-A\cdot\nabla u_0^{+,k+2} + pu_0^{+,k+2} + q^- u_0^{-,k+2})\|^2_{L^2(\Omega)} \\ &\quad + \|e^{-s\eta(\cdot,0)}(A\cdot\nabla u_0^{-,k+2} + pu_0^{-,k+2} + q^+ u_0^{+,k+2})\|^2_{L^2(\Omega)} \end{aligned}$$

can be upper bounded by

$$\begin{aligned} &C_1' s^{-3/2}(\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n} + \|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)} \\ &\quad + \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} + s\mu_{k+2}(s)), \end{aligned}$$

provided $s > s_1$. Thus, taking into account that

$$\begin{aligned} &|\pm A\cdot\nabla u_0^{\mp,k+2} + pu_0^{\mp,k+2} + q^{\pm}u_0^{\pm,k+2}|^2 \\ &\quad \ge \frac{1}{2}|A\cdot\nabla u_0^{\mp,k+2}|^2 - |pu_0^{\mp,k+2} + q^{\pm}u_0^{\pm,k+2}|^2, \quad k = 1, \ldots, n, \end{aligned}$$

we get for all $s > s_1$ that

$$\begin{aligned} &\frac{1}{2}(\|e^{-s\eta(\cdot,0)}A\cdot\nabla u_0^{+,k+2}\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}A\cdot\nabla u_0^{-,k+2}\|^2_{L^2(\Omega)}) \\ &\quad \le C_1' s^{-3/2}(\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n} + \|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)} \\ &\qquad + \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} + s\mu_{k+2}(s)) + \|e^{-s\eta(\cdot,0)}(pu_0^{+,k+2} + q^- u_0^{-,k+2})\|^2_{L^2(\Omega)} \\ &\qquad + \|e^{-s\eta(\cdot,0)}(pu_0^{-,k+2} + q^+ u_0^{+,k+2})\|^2_{L^2(\Omega)}, \end{aligned}$$

whence

$$\begin{aligned} &\|e^{-s\eta(\cdot,0)}A\cdot\nabla u_0^{+,k+2}\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}A\cdot\nabla u_0^{-,k+2}\|^2_{L^2(\Omega)} \\ &\qquad - C_{1,k}s^{-3/2}\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n} \\ &\quad \le C_{1,k}(\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)} \\ &\qquad + \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} + s^{-1/2}\mu_{k+2}(s)), \end{aligned}$$

[a]Actually, it is enough that either U_0^+ or U_0^- fulfills (3.9) for the result of Theorem 1.2 to hold, but for the sake of simplicity, we assume here that both matrices satisfy this condition.

where $C_{1,k} = 2(C_1' + \|u_0^{+,k+2}\|^2_{L^\infty(\Omega)} + \|u_0^{-,k+2}\|^2_{L^\infty(\Omega)})$. Summing up over $k = 1, \dots, n$, we obtain for all $s > s_1$ that

$$\|e^{-s\eta(\cdot,0)}U_0^+ A\|^2_{L^2(\Omega)^n} + \|e^{-s\eta(\cdot,0)}U_0^- A\|^2_{L^2(\Omega)^n} - C_1'' s^{-3/2}\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n}$$
$$\leq C_1'' \Bigg(\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)}$$
$$+ \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} + s^{-1/2}\sum_{k=1}^n \mu_{k+2}(s)\Bigg),$$

with $C_1'' = \sum_{k=1}^n C_{1,k}$. Consequently we have

$$(c_0^+ + c_0^- - C_1'' s^{-3/2})\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n}$$
$$\leq C_1'' \Bigg(\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)}$$
$$+ \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} + s^{-1/2}\sum_{k=1}^n \mu_{k+2}(s)\Bigg),$$

by virtue of (3.9). Thus, taking $s > s_2' = \max(s_1, (2C_1''/(c_0^- + c_0^+))^{2/3})$ and setting $C_2' = 2C_1''/(c_0^- + c_0^+)$, we find that

$$\|e^{-s\eta(\cdot,0)}A\|^2_{L^2(\Omega)^n} \leq C_2' \Bigg(\|e^{-s\eta(\cdot,0)}p\|^2_{L^2(\Omega)} + \|e^{-s\eta(\cdot,0)}q^+\|^2_{L^2(\Omega)}$$
$$+ \|e^{-s\eta(\cdot,0)}q^-\|^2_{L^2(\Omega)} + s^{-1/2}\sum_{k=1}^n \mu_{k+2}(s)\Bigg).$$

Now, the end of the proof is similar to the one presented in Sec. 3.2.

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Appendix A. Symmetry

To prove that $\tilde{A}\cdot\nabla$ is symmetric it is enough to see for all $u=(u^+,u^-)^T$ and $v=(v^+,v^-)^T$ in $(H_0^1(\Omega))^2$, that we have

$$\begin{aligned}\langle \tilde{A}\cdot\nabla u, v\rangle_{L^2(\Omega)^2} &= \int_\Omega A\cdot\nabla u^-\overline{v^+}dx - \int_\Omega A\cdot\nabla u^+\overline{v^-}dx\\ &= \int_\Omega (\nabla\cdot(Au^-))\overline{v^+}dx - \int_\Omega (\nabla\cdot(A\cdot u^+))\overline{v^-}dx\\ &= -\int_\Omega u^-\cdot A\overline{v^+}dx + \int_\Omega u^+\cdot A\overline{v^-}dx\\ &= \int_\Omega u^+\cdot\overline{Av^-}dx - \int_\Omega u^-\cdot\overline{Av^+}dx\\ &= \langle u, \tilde{A}\cdot\nabla v\rangle_{L^2(\Omega)^2}.\end{aligned}$$

Note that we used the facts that A is real-valued and divergence free, i.e. that $\nabla\cdot A=0$.

Appendix B. Δ^D-bounded perturbation

Each of the three operators $\tilde{p}$, $\tilde{q}$ and $\tilde{A}\cdot\nabla$ is Δ^D-bounded with relative bound zero. Indeed, for all $u=(u^+,u^-)^T\in\mathcal{D}_0^2$, we have

$$\begin{aligned}\|\tilde{p}u\|^2_{L^2(\Omega)^2} &= \|pu^-\|^2_{L^2(\Omega)} + \|pu^+\|^2_{L^2(\Omega)} \le 2\|p\|^2_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)^2},\\ \|\tilde{q}u\|^2_{L^2(\Omega)^2} &= \|q^+u^+\|^2_{L^2(\Omega)} + \|q^-u^-\|^2_{L^2(\Omega)}\\ &\le (\|q^+\|^2_{L^\infty(\Omega)} + \|q^-\|^2_{L^\infty(\Omega)})\|u\|^2_{L^2(\Omega)^2}\end{aligned}$$

and

$$\begin{aligned}\|\tilde{A}\cdot\nabla u\|^2_{L^2(\Omega)^2} &= \|A\cdot u^-\|^2_{L^2(\Omega)} + \|A\cdot u^+\|^2_{L^2(\Omega)}\\ &\le \|A\|^2_{L^\infty(\Omega)^n}(\|\nabla u^+\|^2_{L^2(\Omega)} + \|u^-\|^2_{L^2(\Omega)})\\ &\le \|A\|^2_{L^\infty(\Omega)^n}\int_\Omega(-\Delta u^+\overline{u^+} - \Delta u^-\overline{u^-})dx\\ &\le \|A\|^2_{L^\infty(\Omega)^n}(\|\Delta u^+\|_{L^2(\Omega)}\|u^+\|_{L^2(\Omega)} + \|\Delta u^-\|_{L^2(\Omega)}\|u^-\|_{L^2(\Omega)})\end{aligned}$$

by the Cauchy–Schwarz inequality, with

$$\begin{aligned}&\|\Delta u^+\|_{L^2(\Omega)}\|u^+\|_{L^2(\Omega)} + \|\Delta u^-\|_{L^2(\Omega)}\|u^-\|_{L^2(\Omega)}\\ &\quad\le \|A\|^2_{L^\infty(\Omega)^n}\left(\frac{\epsilon}{\|A\|^2_{L^\infty(\Omega)^n}}(\|\Delta u^+\|^2_{L^2(\Omega)} + \|\Delta u^-\|^2_{L^2(\Omega)})\right.\\ &\qquad\left. + \frac{\|A\|^2_{L^\infty(\Omega)^n}}{\epsilon}(\|u^+\|^2_{L^2(\Omega)} + \|u^-\|^2_{L^2(\Omega)})\right)\end{aligned}$$

uniformly in $\epsilon \in (0,1)$, upon applying the Hölder inequality, whence

$$\|\tilde{A} \cdot \nabla u\|^2_{L^2(\Omega)^2} \leq \epsilon \|\Delta^D u\|^2_{L^2(\Omega)^2} + \frac{\|A\|^4_{L^\infty(\Omega)^n}}{\epsilon} \|u\|^2_{L^2(\Omega)^2}.$$

References

[1] M. Barronco, R. J. Lombard, S. Marcos and S. A. Mozkowski, Multi-lambda matter in a derivative coupling model, *Phys. Rev. C* **44**(1) (1991) 178–183.

[2] L. Baudouin and J.-P. Puel, Uniqueness and stability in an inverse problem for the Schrödinger equation, *Inverse Problems* **18** (2002) 1537–1554.

[3] M. Bellassoued, Stable determination of coefficients in the dynamical Schrödinger equation in a magnetic field, *Inverse Problems* **33** (2017) 055009, 36 pp.

[4] M. Bellassoued and M. Choulli, Stability estimate for an inverse problem for the magnetic Schrödinger equation from the Dirichlet-to-Neumann map, *J. Funct. Anal.* **91**(258) (2010) 161–195.

[5] M. Bellassoued, Y. Kian and E. Soccorsi, An inverse stability result for non compactly supported potentials by one arbitrary lateral Neumann observation, *J. Differential Equations* **260**(10) (2016) 7535–7562.

[6] M. Bellassoued, Y. Kian and É. Soccorsi, An inverse problem for the magnetic Schrödinger equation in infinite cylindrical domains, *Publications of the Research Institute for Mathematical Sciences of Kyoto University* **54**(4) (2018) 679–728.

[7] C. G. Boehmer, N. Tamanini and M. Wright, Interacting quintessence from a variational approach Part I: Algebraic couplings, *Phys. Rev. D* **91**(12) (2015) 123002, 22 pp.

[8] A. L. Bukhgeim and M. V. Klibanov, Global Uniqueness of a class of multidimensional inverse problems, *Sov. Math. Dokl.* **24** (1981) 244–247.

[9] M. Choulli, Y. Kian and E. Soccorsi, Stable determination of time-dependent scalar potential from boundary measurements in a periodic quantum waveguide, *SIAM J. Math. Anal.* **47**(6) (2015) 4536–4558.

[10] M. Cristofol and E. Soccorsi, Stability estimate in an inverse problem for non-autonomous Schrödinger equations, *Appl. Anal.* **90**(10) (2011) 1499–1520.

[11] J. Dutta, W. Khyllep and N. Tamanini, Dark energy with a gradient coupling to the dark matter fluid: Cosmological dynamics and structure formation, *J. Cosmol. Astropart. Phys.* **2018**(1) (2018) 038, front matter+38 pp.

[12] X. Huang, Y. Kian, É. Soccorsi and M. Yamamoto, Carleman estimate for the Schrödinger equation and application to magnetic inverse problems, *J. Math. Anal. Appl.* **474**(1) (2019) 116–142.

[13] Y. Kian, Q. S. Phan and E. Soccorsi, Carleman estimate for infinite cylindrical quantum domains and application to inverse problems, *Inverse Problems* **30**(5) (2014) 055016, 16 pp.

[14] Y. Kian, Q. S. Phan and E. Soccorsi, Hölder stable determination of a quantum scalar potential in unbounded cylindrical domains, *J. Math. Anal. Appl.* **426**(1) (2015) 194–210.

[15] Y. Kian and É. Soccorsi, Hölder stably determining the time-dependent electromagnetic potential of the Schrödinger equation, *SIAM J. Math. Anal.* **51**(2) (2019) 627–647.

[16] M. V. Klibanov, Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems, *J. Inverse Ill-Posed Probl.* **21** (2013) 477–560.

[17] B. H. Lengsfield, P. Saxe and D. R. Yarkony, On the evaluation of nonadiabatic coupling matrix elements using SA MCSCF/CI wave functions and analytic gradient methods. I, *J. Chem. Phys.* **81**(10) (1984) 4549–4553.

[18] J.-L. Lions and E. Magenes, *Problèmes Aux Limites non HomogÈnes Et Applications*, Vol. 2 (Dunod, Paris, 1968).

[19] S. Liu and R. Triggiani, Global uniqueness in determining electric potentials for a system of strongly coupled Schrödinger equations with magnetic potential terms, *J. Inverse Ill-Posed Prob.* **19** (2011) 223–254.

[20] A. Mercado, A. Osses and L. Rosier, Inverse problems for the Schrödinger equation via Carleman in-equalities with degenerate weights, *Inverse Problems* **24**(1) (2008) 015017, 18 pp.

[21] J. Rantaharju, Gradient flow coupling in the SU(2) gauge theory with two adjoint fermions, *Phys. Rev. D* **93** (2016) 094516.

[22] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 2, Fourier Analysis, Self-adjointness (Academic Press, 1975).

[23] D. Tataru, Carleman estimates, unique continuation and controllability for anisotropic PDEs, *Contemp. Math.* **209** (1997) 267–279.

[24] A. Yew, Multipulses of non-linearly coupled Schrödinger equations, *J. Differential Equations* **173** (2001) 92–137.

[25] G. Yuan and M. Yamamoto, Carleman estimates for the Schrödinger equation and applications to an inverse problem and an observability inequality, *Chin. Ann. Math. Ser. B* **31** (2010) 555–578.