HÖLDER STABLY DETERMINING THE TIME-DEPENDENT ELECTROMAGNETIC POTENTIAL OF THE SCHRÖDINGER EQUATION

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ABSTRACT. We consider the inverse problem of determining the time- and space-dependent electromagnetic potential of the Schrödinger equation in a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, by boundary observation of the solution over the entire time span. In contrast with published mathematical results on the recovery of unknown coefficients depending on all variables of the Cauchy problem under consideration, which, to our best knowledge, are of logarithmic type, we establish upon assuming that the divergence of the magnetic potential is fixed, that the electric potential and the magnetic potential can be Hölder stably retrieved from those lateral measurements.

Keywords: Inverse problem, stability estimate, Schrödinger equation, time-dependent electromagnetic potential.

Mathematics subject classification 2010: 35R30, 35Q41.

1. INTRODUCTION

1.1. **Statement of the problem.** Let Ω be a bounded, simply connected domain of $\mathbb{R}^n$, $n \geqslant 2$, with C^3 boundary $\partial\Omega$ (see Section 1.4 below, for a brief justification of this smoothness assumption). For $T \in (0,+\infty)$, we consider the initial boundary value problem (IBVP)

$$\begin{cases} (i\partial_t + \Delta_A + q)\, u = 0 & \text{in } Q := (0,T) \times \Omega, \\ u(0,\cdot) = 0 & \text{in } \Omega, \\ u = g & \text{on } \Sigma := (0,T) \times \Gamma, \end{cases} \tag{1.1}$$

where Δ_A is the Laplace operator $(\nabla + iA(t,x)) \cdot (\nabla + iA(t,x))$, associated with the real-valued magnetic potential $A := (a_j)_{1 \leqslant j \leqslant n} \in W^{1,\infty}(Q;\mathbb{R})^n$, i.e.

$$\Delta_A := \sum_{j=1}^{n} \left(\partial_{x_j} + ia_j(t,x)\right)^2 = \Delta + 2iA(t,x) \cdot \nabla + i(\nabla \cdot A(t,x)) - |A(t,x)|^2 \tag{1.2}$$

and $q \in L^\infty(Q;\mathbb{R})$ is a real-valued electric potential. Here and in the remaining part of this text, we denote by $\nabla := (\partial_{x_1}, \ldots, \partial_{x_n})^T$ the gradient operator with respect to the spatial variable $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, the symbol $\cdot$ (resp., $|\cdot|$) stands for the Euclidian scalar product (resp., norm) in $\mathbb{R}^n$, and the divergence operator with respect to $x \in \mathbb{R}^n$ is represented by the notation $\nabla\cdot$.

For all $s, r \in (0,+\infty)$ and for X being either Ω or $\partial\Omega$, we equip the functional spaces $H^{r,s}((0,T) \times X) := H^r(0,T;L^2(X)) \cap L^2(0,T;H^s(X))$ with the following norm

$$\|u\|^2_{H^{r,s}((0,T)\times X)} := \|u\|^2_{H^r(0,T;L^2(X))} + \|u\|^2_{L^2(0,T;H^s(X))},$$

and we write $H^{r,s}(Q)$ (resp., $H^{r,s}(\Sigma)$) instead of $H^{r,s}((0,T) \times \Omega)$ (resp., $H^{r,s}((0,T) \times \partial\Omega)$). Then, for all

$$g \in \mathcal{H}(\Sigma) := \left\{ g \in H^{\frac{5}{2},\frac{5}{2}}(\Sigma);\ g(0,\cdot) = \partial_t g(0,\cdot) = 0 \text{ on } \partial\Omega \right\},$$

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we establish in Proposition 2.1 below, that there exists a unique solution $u_g \in H^{1,2}(Q)$ to (1.1) and that the mapping $g \mapsto u_g$ is continuous. As a corollary the Dirichlet-to-Neumann (DN) operator associated with (1.1), defined by

$$\begin{array}{rccc} \Lambda_{A,q}: & \mathcal{H}(\Sigma) & \to & L^2(\Sigma) \\ & g & \mapsto & (\partial_\nu + iA\cdot\nu)u_g, \end{array} \tag{1.3}$$

where $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at x, is bounded. The main purpose of this paper is to examine the stability issue in the inverse problem of determining the electromagnetic potential (A, q) from the knowledge of $\Lambda_{A,q}$.

However, there is a natural obstruction to uniqueness in this problem, in the sense that the mapping $(A, q) \mapsto \Lambda_{A,q}$ is not injective. This can be seen from the identity

$$i\partial_t + \Delta_A + q = e^{-i\phi}(i\partial_t + \Delta_{\tilde{A}} + \tilde{q})e^{i\phi},\ \phi \in W^{3,\infty}(Q)$$

arising from (1.2) with $(\tilde{A}, \tilde{q}) = G_{A,q}(\phi) := (A - \nabla\phi, q + \partial_t\phi)$, which entails that $(i\partial_t + \Delta_{\tilde{A}} + \tilde{q})e^{i\phi}u_g = e^{i\phi}(i\partial_t + \Delta_A + q)u_g = 0$ for all $g \in \mathcal{H}(\Sigma)$. Therefore, if ϕ vanishes on Σ then it is apparent that $e^{i\phi}u_g$ is the $H^{1,2}(Q)$-solution to (1.1), where $(\tilde{A}, \tilde{q})$ is substituted for (A, q). Consequently, it holds true that

$$\Lambda_{\tilde{A},\tilde{q}}g = (\partial_\nu + i\tilde{A}\cdot\nu)e^{i\phi}u_g = e^{i\phi}(\partial_\nu + iA\cdot\nu)u_g = \Lambda_{A,q}g,$$

despite of the fact that $(\tilde{A}, \tilde{q})$ does not coincide with (A, q) whenever ϕ is not identically zero in Q. Otherwise stated, since the DN map (1.3) is invariant under the gauge transformation $(A, q) \mapsto G_{A,q}(\phi)$ associated with $\phi \in W_*^{3,\infty}(Q) := \{\phi \in W^{3,\infty}(Q);\ \phi_{|\Sigma} = 0\}$, then it is hopeless to retrieve (A, q) through $\Lambda_{A,q}$ and the best we can expect is to determine the *gauge class* $G_{A,q}(W_*^3(Q)) := \{G_{A,q}(\phi),\ \phi \in W_*^{3,\infty}(Q)\}$ of (A, q). Moreover, for any two gauge equivalent electromagnetic potentials (A, q) and $(\tilde{A}, \tilde{q})$, there exists a unique $\phi \in W_*^{3,\infty}(Q)$ such that we have $(\tilde{A}, \tilde{q}) = G_{A,q}(\phi)$ and we notice for each $t \in (0, T)$ that the function $\phi(t, \cdot)$ is solution to the following elliptic system:

$$\begin{cases} -\Delta\phi(t,\cdot) = \nabla\cdot(\tilde{A} - A)(t,\cdot) & \text{in } \Omega, \\ \phi(t,\cdot) = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, if the time-dependent electromagnetic potential (A, q) can be determined modulo gauge invariance by $\Lambda_{A,q}$ then it is actually possible to recover (A, q) itself provided the divergence $\nabla \cdot A$ is known.

1.2. **What is known so far.** Since inverse problems are of great interest in applied sciences, it is no surprise that the determination of coefficients in partial differential equations such as the magnetic Schrödinger equation under study in this article has attracted the attention of numerous mathematicians over the previous decades.

For instance, using the Bukhgeim-Klibanov method [15], Baudouin and Puel [3] proved Lipschitz stable identification of the time independent electric potential in the dynamical (i.e., non stationary) Schrödinger equation from a single boundary observation of the solution. Here the measurement can be performed on any subpart of the boundary fulfilling the geometric control property expressed by Bardos, Lebeau and Rauch in [2]. This condition was removed by Bellassoued and Choulli in [6], provided the electric potential is *a priori* known in a neighborhood of the boundary. We refer to [19] for the Lipschitz stable reconstruction of the magnetic potential in the Coulomb gauge class by a finite number of boundary measurements of the solution to the Schrödinger equation. More recently, in [13], Ben Aïcha and Mejri claimed simultaneous Lipschitz stable determination of the electric potential and the divergence free magnetic potential, from the same type of boundary data.

The above mentioned results involve a finite number of boundary observations of the solution, performed over the entire time span. But this is no longer the case in [7] where the magnetic field was stably recovered from the knowledge of the DN map associated with the dynamic Schrödinger equation. In the same spirit, Bellassoued and Dos Santos Ferreira proved stable identification of the electric potential by the DN map associated with the Schrödinger equation on a Riemannian manifold in [8]. This result was extended in [4] to simultaneous determination of the electric potential and the magnetic field. We also refer to [20, 26] for an extensive treatment of similar inverse problems. We stress out that all the above results were established in a bounded domain and that the analysis carried out in [3] (resp. [6] and [7]) was adapted to the case of unbounded cylindrical domains in [9] (resp., [10] and [29, 30]).

All the above mentioned works are concerned with space-only dependent (i.e. time independent) coefficients. Actually, there is only a very small number of papers available in the mathematical literature, dealing with the inverse problem of determining time-dependent coefficients of the Schrödinger equation. For instance, it was proved in [21] that the DN map uniquely determines the time-dependent electromagnetic potential modulo gauge invariance. The stability issue for the same problem was examined in [18], where the time-dependent electric potential was logarithmic stably recovered from boundary observation for all times and internal measurement at final time, of the solution. More recently, in [12], this approach was adapted to the case of an electromagnetic potential with sufficiently small time independent magnetic part. To the best of our best knowledge, these two last articles are the only mathematical papers studying the stability issue in the inverse problem of determining time-dependent coefficients of the Schrödinger equation. Nevertheless, we point out that similar problems were addressed in [5, 11, 17, 23, 25, 27, 35, 34, 36, 37, 38, 41] for either parabolic or hyperbolic equations. We stress out that the so-called "relativistic Schrödinger equation" under consideration in [36], where the time-dependent electromagnetic potential is logarithmic stably retrieved, up to the natural gauge, by the DN map, is actually of hyperbolic type.

1.3. **Main result.** The main result of this paper is the following Hölder stability estimate of the electromagnetic potential entering the Schrödinger equation in (1.1), with respect to the DN map.

Theorem 1.1. *Fix $M \in [1,+\infty)$ and for $j=1,2$, let $A_j \in W^{5,\infty}(Q)^n \cap H^6(Q)^n$ and $q_j \in W^{4,\infty}(Q)$ satisfy the three following conditions:*

$$\partial_x^\alpha A_1(t,x) = \partial_x^\alpha A_2(t,x),\ (t,x) \in \Sigma,\ \alpha \in \mathbb{N}^n,\ |\alpha| \leqslant 5, \tag{1.4}$$

$$\nabla \cdot A_1(t,x) = \nabla \cdot A_2(t,x),\ (t,x) \in Q \tag{1.5}$$

and

$$\sum_{j=1}^{2} \left(\|A_j\|_{W^{5,\infty}(Q)^n} + \|A_j\|_{H^6(Q)^n} + \|q_j\|_{W^{4,\infty}(Q)} \right) \leqslant M. \tag{1.6}$$

Then, there exist three positive constants, r and s, depending only on n, and C, depending only on n, T, Ω and M, such that we have

$$\|A_1 - A_2\|_{L^2(0,T;H^5(\Omega))} \leqslant C \|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|^r \tag{1.7}$$

and

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leqslant C \|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|^s. \tag{1.8}$$

In Theorem 1.1 and the remaining part of this article, the DN maps Λ_{A_j,q_j}, $j=1,2$, lie in the space $\mathcal{B}(\mathcal{H}(\Sigma), L^2(\Sigma))$ of linear bounded operators from $\mathcal{H}(\Sigma)$ into $L^2(\Sigma)$ and $\|\cdot\|$ denotes the usual norm in $\mathcal{B}(\mathcal{H}(\Sigma), L^2(\Sigma))$.

1.4. **Brief comments and outline.**

1.4.1. *Hölder stability result.* To the authors' best knowledge, Theorem 1.1 is the only mathematical result claiming Hölder stable determination of unknown coefficients depending on all variables of a partial differential equation (PDE). In the framework of evolution PDEs, where the corresponding inverse problem is to stably retrieve space- and time-varying coefficients, only logarithmic type stability inequalities are available in the mathematical literature, see e.g. [11, 12, 17, 18]. As a matter of fact, in [36, 37] the time-dependent vector (resp., scalar) potential of the relativistic Schrödinger equation is log-stably (resp., log-log-stably) determined. Moreover, it was proved in [33] that logarithmic stable recovery of unknown coefficients of an elliptic PDE is the best one can expect from the DN map, and we refer to [1, 16, 40] for the proof of logarithmic stable determination of the electromagnetic coefficients of the stationary Schrödinger operator.

We point out that upgrading the reconstruction of unknown coefficients from logarithmic-stable to Hölder-stable is interesting not only from a mathematical perspective, but from a numerical and computational perspective as well. Actually, this is all the more important because logarithmic stability can be damaging for the numerical solution of many inverse problems, see e.g. [22, Section 10]. Moreover, even if the identification of unknown coefficients depending on both time- and space-variable is of great interest in its own, it is worth mentioning that it can also be linked with the inverse problem of determining a nonlinear perturbation of a PDE. As a matter of fact it was proved in [17, 24] by mean of a linearization process that the semilinear term entering a nonlinear parabolic equation can be identified by solving the inverse problem of determining the time-dependent coefficient of a related linear parabolic equation. From this viewpoint there is no doubt that Theorem 1.1 is a useful tool for adapting this strategy to the case of semilinear Schrödinger equations.

1.4.2. *A priori smoothness assumptions.* As will appear below, the $\mathcal{C}^3$-regularity imposed on the boundary $\partial\Omega$ is needed by the lifting of the non-homogeneous Dirichlet datum g, appearing in (1.1), into a $H^3(Q)$-function. This lifting up is performed in Section 2.3 with the help of [32, Chapter 4, Theorem 2.3]. The $\mathcal{C}^3$ boundary regularity assumption is thus purely technical but it is essential for proving the existence of a suitably smooth solution to (1.1), and consequently of the DN map used in the analysis of the inverse problem under consideration in this article. It is not clear how the analysis carried out in this paper could be adapted to a less smooth boundary $\partial\Omega$ (in any case it is required that $\partial\Omega$ be at least $\mathcal{C}^{1,1}$ in order keep the elliptic regularity property of the Dirichlet Laplacian, needed by the proof of Lemma 2.2), but this would be at the expense of greater regularity imposed on the coefficients of the Schrödinger equation, i.e. on the electric and the magnetic potentials.

We point out that the smoothness assumptions and the boundary condition (1.4) on the magnetic vector potentials are imposed by the construction in Section 3 of a sufficiently rich set of suitable geometric optics (GO) solutions to the magnetic Schrödinger equation in Q. As a matter of fact, our method of derivation of Theorem 1.1 seeks GO solutions whose trace is in $\mathcal{H}(\Sigma)$. Consequently we build these functions in $H^3(Q)$. This is achieved by taking magnetic vector potentials in $W^{5,\infty}(Q)\cap H^6(Q)$ that fulfill (1.4).

1.4.3. *Outline.* The remaining part of this article is organized as follows. In the coming section, Section 2, we study the well-posedness of problem (1.1) and we prove that the DN map (1.3) is well defined as a linear bounded operator from $\mathcal{H}(\Sigma)$ into $L^2(\Sigma)$. In Section **??** we build a set of geometrical optics solutions to (1.1) which are the main tool for deriving Theorem 1.1. Finally, the proof of the stability estimate (1.7) is presented in Section 4, whereas the one of (1.8) is given in Section 5.

2. Analysis of the forward problem

The main result of this section is the following existence and uniqueness result for the IBVP (1.1).

Proposition 2.1. *For $M\in[1,+\infty)$, let $A\in W^{2,\infty}(Q;\mathbb{R})^n$ and $q\in W^{1,\infty}(Q;\mathbb{R})$ satisfy*

$$\|A\|_{W^{2,\infty}(Q)^n}+\|q\|_{W^{1,\infty}(Q)}\leqslant M. \tag{2.1}$$

Then, for every $g\in\mathcal{H}(\Sigma)$, the system (1.1) *admits a unique solution $u\in H^{1,2}(Q)$. Moreover, there exists a positive constant C, depending only on M, T and Ω, such that we have*

$$\|u\|_{H^{1,2}(Q)}\leqslant C\|g\|_{\mathcal{H}(\Sigma)}. \tag{2.2}$$

With reference to (1.3) and the continuity of the trace operator from $H^{1,2}(Q)$ into $L^2(\Sigma)$, Proposition 2.1 immediately entails the following:

Corollary 2.1. *Under the conditions of Proposition 2.1, the DN map $\Lambda_{A,q}$ is well defined by* (1.3) *and acts as a bounded operator from $\mathcal{H}(\Sigma)$ into $L^2(\Sigma)$.*

The proof of Proposition 2.1 can be found in Section 2.3 by mean of an existence and uniqueness result for the IBVP (1.1) with homogeneous Dirichlet boundary condition and suitable source term, stated in Section 2.2. As a preamble, we recall that the sesquilinear form associated with the operator $-\Delta_{A(t,\cdot)}+q(t,\cdot)=-(\nabla+iA(t,\cdot))\cdot(\nabla+iA(t,\cdot))+q(t,\cdot)$ is $H^1(\Omega)$-elliptic with respect to $L^2(\Omega)$, uniformy in $t\in(0,T)$.

2.1. $H^1(\Omega)$-ellipticity with respect to $L^2(\Omega)$.

We define the magnetic gradient ∇_A, associated with $A\in L^\infty(Q)^n$, by

$$\nabla_A u(t,x):=(\nabla+iA(t,x))u(x),\ u\in H_0^1(\Omega),\ (t,x)\in Q, \tag{2.3}$$

and for $q\in L^\infty(Q)$, we introduce the sesquilinear form

$$a(t;u,v):=\int_\Omega \nabla_A u(t,x)\cdot\overline{\nabla_A v(t,x)}dx+\int_\Omega q(t,x)u(x)\overline{v(x)}dx,\ u,v\in H_0^1(\Omega). \tag{2.4}$$

Then, the Hölder inequality yields

$$\|\nabla_A u(t,\cdot)\|_{L^2(\Omega)^n}^2 \geqslant \frac{\|\nabla u\|_{L^2(\Omega)^n}^2}{2} - 2\|A\|_{L^\infty(Q)^n}^2 \|u\|_{L^2(\Omega)}^2,$$

for every $u \in H_0^1(\Omega)$ and $t \in (0,T)$, so we get

$$a(t;u,u) + \lambda \|u\|_{L^2(\Omega)}^2 \geqslant \alpha \|u\|_{H^1(\Omega)}^2,\ u \in H_0^1(\Omega),\ t \in (0,T), \tag{2.5}$$

with $\lambda := \frac{1}{2} + \|q\|_{L^\infty(Q)}^2 + 2\|A\|_{L^\infty(Q)^n}^2$ and $\alpha := \frac{1}{2}$.

2.2. **Existence and uniqueness result.** The proof of Proposition 2.1 essentially boils down to the following existence and uniqueness result for the following IBVP associated with a suitable source term F:

$$\begin{cases} (i\partial_t + \Delta_A + q)v = F & \text{in } Q, \\ v(0,\cdot) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Sigma. \end{cases} \tag{2.6}$$

Lemma 2.2. *Let M, A and q be the same as in Proposition 2.1 and let $F \in H^1(0,T;L^2(\Omega))$ verify $F(0,\cdot) = 0$ a.e. in Ω. Then, the system* (2.6) *admits a unique solution $v \in \mathcal{C}([0,T],H_0^1(\Omega) \cap H^2(\Omega)) \cap \mathcal{C}^1([0,T],L^2(\Omega))$ satisfying*

$$\|v\|_{\mathcal{C}([0,T],H^2(\Omega))} + \|v\|_{\mathcal{C}^1([0,T],L^2(\Omega))} \leqslant C\|F\|_{H^1(0,T;L^2(\Omega))}, \tag{2.7}$$

for some positive constant C depending only on T, Ω and M.

Proof. We proceed as in the derivation of [31, Section 3, Theorem 10.1] by applying the Faedo-Galerkin method. Namely, we pick a Hilbert basis $\{e_k,\ k \in \mathbb{N}^*\}$ of $H_0^1(\Omega)$ and consider an approximated solution of size $m \in \mathbb{N}^* := \{1,2,\dots\}$ of (2.6), of the form

$$v_m(t,x) := \sum_{k=1}^m g_{k,m}(t) e_k(x),\ (t,x) \in Q, \tag{2.8}$$

where the functions $g_{k,m}$ are defined in such a way that we have

$$\begin{cases} i\langle \partial_t v_m(t,\cdot), e_k\rangle_{L^2(\Omega)} - a(t; v_m(t,\cdot), e_k) = \langle F(t,\cdot), e_k\rangle_{L^2(\Omega)}, & t \in (0,T), \\ v_m(0,\cdot) = 0, \end{cases} \tag{2.9}$$

for all $k = 1,\dots,m$. Since $F \in W^{1,1}(0,T;L^2(\Omega))$ then (2.9) admits a unique solution $v_m \in W^{1,\infty}(0,T;H_0^1(\Omega))$ such that the function $w_m := \partial_t v_m$ solves the following Cauchy problem for every $k = 1,\dots,m$:

$$\begin{cases} i\langle \partial_t w_m(t,\cdot), e_k\rangle_{L^2(\Omega)} - a(t; w_m(t,\cdot), e_k) = a'(t; v_m(t,\cdot), e_k) + \langle \partial_t F(t,\cdot), e_k\rangle_{L^2(\Omega)}, & t \in (0,T), \\ w_m(0) = 0. \end{cases} \tag{2.10}$$

Here, for all $t \in (0,T)$ and all $u, v \in H_0^1(\Omega)$, we have set with reference to (2.3)-(2.4)

$$a'(t;u,v) := i\int_\Omega \partial_t A(t,x)\cdot \left(u(x)\overline{\nabla_A v(t,x)} - \nabla_A u(t,x)\overline{v(x)}\right) dx + \int_\Omega \partial_t q(t,x) u(x)\overline{v(x)} dx. \tag{2.11}$$

Notice for further use from (2.11) that we have

$$|a'(t;u,u)| \leqslant 2M^2 \|u\|_{H^1(\Omega)}^2,\ t \in (0,T),\ u \in H_0^1(\Omega). \tag{2.12}$$

1) The first part of the proof is to establish three *a priori* estimates for the functions v_m and w_m.

a) To this end, we fix $t \in (0,T)$ and we multiply for each $k \in \{1,\dots,m\}$ the first line of (2.9) by $\overline{g_{k,m}(t)}$, sum up over $k = 1,\dots,m$, and infer from (2.8) that

$$i\langle \partial_t v_m(t,\cdot), v_m(t,\cdot)\rangle_{L^2(\Omega)} - a(t; v_m(t,\cdot), v_m(t,\cdot)) = \langle F(t,\cdot), v_m(t,\cdot)\rangle_{L^2(\Omega)}.$$

Taking the imaginary part of both sides of the above identity then yields

$$\frac{d}{ds}\|v_m(s,\cdot)\|_{L^2(\Omega)}^2 = 2\mathrm{Im}\langle F(s,\cdot), \partial_t v_m(s,\cdot)\rangle_{L^2(\Omega)},\ s \in (0,T).$$

Since $v_m(0,\cdot)=0$, we get upon integrating the above identity over $(0,t)$ that

$$\|v_m(t,\cdot)\|^2_{L^2(\Omega)}=2\mathrm{Im}\int_0^t\langle F(s,\cdot),v_m(s,\cdot)\rangle_{L^2(\Omega)}ds\leqslant\int_0^t\|F(s,\cdot)\|^2_{L^2(\Omega)}ds+\int_0^t\|v_m(s,\cdot)\|^2_{L^2(\Omega)}ds.$$

Therefore, by Gronwall's lemma, there exists a positive constant c_0, depending only on T, Ω and M such that we have:

$$\|v_m\|_{L^\infty(0,T;L^2(\Omega))}\leqslant c_0\|F\|_{L^2(Q)}. \tag{2.13}$$

b) Similarly, by multiplying the first line of (2.9) by $\overline{g'_{k,m}(t)}$, summing up over $k=1,\ldots,m$, and applying (2.8) once more, we get that

$$i\|\partial_t v_m(t,\cdot)\|^2_{L^2(\Omega)}-a(t;v_m(t,\cdot),\partial_t v_m(t,\cdot))=\langle F(t,\cdot),\partial_t v_m(t,\cdot)\rangle_{L^2(\Omega)},\ t\in(0,T).$$

Upon taking this time the real part in the above identity, we find that

$$a(s;v_m(s,\cdot),\partial_t v_m(s,\cdot))+a(s;\partial_t v_m(s,\cdot),v_m(s,\cdot))=-2\mathrm{Re}\langle F(s,\cdot),\partial_t v_m(s,\cdot)\rangle_{L^2(\Omega)},\ s\in(0,T),$$

which may be equivalently rewritten as

$$\frac{d}{ds}a(s;v_m(s,\cdot),v_m(s,\cdot))=a'(s;v_m(s,\cdot),v_m(s,\cdot))-2\mathrm{Re}\langle F(s,\cdot),\partial_t v_m(s,\cdot)\rangle_{L^2(\Omega)},\ s\in(0,T).$$

Now, by integrating with respect to s over $(0,t)$, we obtain with the aid of (2.13),

$$\begin{aligned}a(t;v_m(t,\cdot),v_m(t,\cdot)) &= \int_0^t a'(s;v_m(s,\cdot),v_m(s,\cdot))ds-2\mathrm{Re}\int_0^t\langle F(s,\cdot),\partial_t v_m(s,\cdot)\rangle_{L^2(\Omega)}ds\\ &\leqslant 2M^2\int_0^t\|v_m(s,\cdot)\|^2_{H^1(\Omega)}ds+2\left|\int_0^t\langle F(s,\cdot),\partial_t v_m(s,\cdot)\rangle_{L^2(\Omega)}ds\right|.\end{aligned} \tag{2.14}$$

Next, as $\int_0^t\langle F(s,\cdot),\partial_t v_m(s,\cdot)\rangle_{L^2(\Omega)}ds=\langle F(t,\cdot),v_m(t,\cdot)\rangle_{L^2(\Omega)}-\int_0^t\langle\partial_t F(s,\cdot),v_m(s,\cdot)\rangle_{L^2(\Omega)}ds$, we have

$$\begin{aligned}&2\left|\int_0^t\langle F(s,\cdot),\partial_t v_m(s,\cdot)\rangle_{L^2(\Omega)}ds\right|\\ \leqslant\ &\|F(t,\cdot)\|^2_{L^2(\Omega)}+\|v_m(t,\cdot)\|^2_{L^2(\Omega)}+\int_0^t\|\partial_t F(s,\cdot)\|^2_{L^2(\Omega)}ds+\int_0^t\|v_m(s,\cdot)\|^2_{L^2(\Omega)}ds.\end{aligned} \tag{2.15}$$

Further, since $\|v_m(t,\cdot)\|^2_{H^1(\Omega)}\leqslant\alpha^{-1}\left(a(t;v_m(t,\cdot),v_m(t,\cdot))+\lambda\|v_m(t,\cdot)\|^2_{L^2(\Omega)}\right)$ by (2.5), we infer from (2.14) -(2.15) that

$$\|v_m(t,\cdot)\|^2_{H^1(\Omega)}\leqslant c\left(\int_0^t\|v_m(s,\cdot)\|^2_{H^1(\Omega)}ds+\int_0^t\|\partial_t F(s,\cdot)\|^2_{L^2(\Omega)}ds+\|F(t,\cdot)\|^2_{L^2(\Omega)}+\|v_m(t,\cdot)\|^2_{L^2(\Omega)}\right).$$

Here and in the remaining part of this proof, c denotes a generic positive constant depending only on T, Ω, M, α and λ, that may change from line to line. From this, the basic inequality

$$\|F(t,\cdot)\|^2_{L^2(\Omega)}=2\mathrm{Re}\int_0^t\langle F(s,\cdot),\partial_t F(s,\cdot)\rangle_{L^2(\Omega)}ds\leqslant\int_0^t\left(\|F(s,\cdot)\|^2_{L^2(\Omega)}+\|\partial_t F(s,\cdot)\|^2_{L^2(\Omega)}\right)ds,$$

and from the estimate (2.13), it then follows that

$$\|v_m(t,\cdot)\|^2_{H^1(\Omega)}\leqslant c\left(\int_0^t\|v_m(s,\cdot)\|^2_{H^1(\Omega)}ds+\|F\|^2_{H^1(0,T;L^2(\Omega))}\right).$$

Thus

$$\|v_m\|_{L^\infty(0,T;H^1(\Omega))}\leqslant c\|F\|_{H^1(0,T;L^2(\Omega))}, \tag{2.16}$$

by Gronwall's lemma.

c) Further, we put $p(t,x):=|A(t,x)|^2+q(t,x)$ for $(t,x)\in Q$, integrate by parts in the first integral of (2.11) and obtain for all $u,v\in H_0^1(\Omega)$ that

$$a'(t;u,v)\ =\ i\int_\Omega\left(u(x)\partial_t A(t,x)\cdot\overline{\nabla v(x)}-\partial_t A(t,x)\cdot\nabla u(x)\overline{v(x)}\right)dx+\int_\Omega\partial_t p(t,x)u(x)\overline{v(x)}dx$$

$$= \int_\Omega \left(-2i\partial_t A(t,x)\cdot\nabla u(x) + (\partial_t p(t,x) - i\nabla\cdot\partial_t A(t,x))u(x)\right)\overline{v(x)}dx.$$

This and (2.10) yield

$$\begin{cases} i\langle \partial_t w_m(t,\cdot), e_k\rangle_{L^2(\Omega)} - a(t; w_m(t,\cdot), e_k) = \langle F_m(t,\cdot), e_k\rangle_{L^2(\Omega)}, & t\in(0,T),\\ w_m(0)=0, \end{cases} \tag{2.17}$$

for all $k=1,\dots,m$, where

$$F_m(t,x) := -2i\partial_t A(t,x)\cdot\nabla v_m(t,x) + (\partial_t p(t,x) - i\nabla\cdot\partial_t A(t,x))v_m(t,x) + \partial_t F(t,x),\ (t,x)\in Q. \tag{2.18}$$

Next, multiplying the first line in (2.17) by $g'_{k,m}(t)$ and summing up over $k=1,\dots,m$, lead to

$$i\langle \partial_t w_m(t,\cdot), w_m(t,\cdot)\rangle_{L^2(\Omega)} - a(t; w_m(t,\cdot), w_m(t,\cdot)) = \langle F_m(t,\cdot), w_m(t,\cdot)\rangle_{L^2(\Omega)},\ t\in(0,T).$$

Therefore, we have

$$\frac{d}{ds}\|w_m(s,\cdot)\|^2_{L^2(\Omega)} = 2\mathrm{Im}\langle F_m(s,\cdot), w_m(s,\cdot)\rangle_{L^2(\Omega)},\ s\in(0,T).$$

Now, for each $t\in(0,T)$, we find upon integrating both sides of the above identity over $(0,t)$ that

$$\|w_m(t,\cdot)\|^2_{L^2(\Omega)} \leqslant \int_0^t \|w_m(s,\cdot)\|^2_{L^2(\Omega)}ds + \int_0^t \|F_m(s,\cdot)\|^2_{L^2(\Omega)}ds,$$

which, combined with (2.18), entails

$$\|w_m(t,\cdot)\|^2_{L^2(\Omega)} \leqslant \int_0^t \|w_m(s,\cdot)\|^2_{L^2(\Omega)}ds + c\left(\|\partial_t F\|^2_{L^2(Q)} + \|v_m\|^2_{L^\infty(0,T;H^1(\Omega))}\right),$$

for some constant $c=c(T,M)\in(0,+\infty)$. Therefore, we have

$$\|w_m(t,\cdot)\|^2_{L^2(\Omega)} \leqslant c\left(\|\partial_t F\|^2_{L^2(Q)} + \|v_m\|^2_{L^\infty(0,T;H^1(\Omega))}\right),\ t\in(0,T),$$

by Gronwall's lemma, and consequently

$$\|w_m\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\|F\|_{H^1(0,T;L^2(\Omega))}, \tag{2.19}$$

by (2.16), where C is another positive constant depending only on T, M and α.

2) Having established (2.16) and (2.19), we turn now to showing the existence of a solution to (2.6). This can be done in accordance with (2.16) by extracting a subsequence $(v_{m'})_{m'}$ of $(v_m)_m$, which converges to $v\in L^\infty(0,T;H^1_0(\Omega))$ in the weak-star topology of $L^\infty(0,T;H^1_0(\Omega))$. By substituting m' for m in (2.9) and sending m' to infinity, we get that

$$\begin{cases} (i\partial_t + \Delta_A + q)v = F & \text{in } Q,\\ v(0,\cdot)=0 & \text{in } \Omega. \end{cases} \tag{2.20}$$

As a consequence, we have $\partial_t v\in L^\infty(0,T;H^{-1}(\Omega))$ and hence $v\in L^\infty(0,T;H^1_0(\Omega))\cap W^{1,\infty}(0,T;H^{-1}(\Omega))$. Further, due to (2.19) and the Banach-Alaoglu theorem, there exists a subsequence of $(w_m)_m$ which converges to $w\in L^\infty(0,T;L^2(\Omega))$ in the weak-star topology of $L^\infty(0,T;L^2(\Omega))$. Since $w_m=\partial_t v_m$ for every $m\in\mathbb{N}^*$ then we necessarily have $\partial_t v = w\in L^\infty(0,T;L^2(\Omega))$ and thus $v\in L^\infty(0,T;H^1_0(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$. Further, by arguing in the exact same way as in the derivation of [31, Theorem 8.3 and Remark 10.2, Chapter 3], we get that $v\in\mathcal{C}([0,T];H^1_0(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$. Moreover, for all fixed $t\in[0,T]$, we deduce from (2.20) that $v(t,\cdot)$ is solution to the elliptic boundary value problem

$$\begin{cases} (\Delta_A + q(t,.))\,v(t,\cdot) = F(t,\cdot) - i\partial_t v(t,\cdot) & \text{in } \Omega,\\ v(t,\cdot)=0, & \text{on } \partial\Omega. \end{cases}$$

As $F - i\partial_t v\in\mathcal{C}([0,T];L^2(\Omega))$ then we have $v\in\mathcal{C}([0,T],H^1_0(\Omega)\cap H^2(\Omega))$ and (2.7) follows directly from this, (2.4), (2.16) and (2.19). $\square$

Remark 2.3. *a) With reference to* (2.13)*, we point out for further use that the solution v to* (2.2) *satisfies the estimate*

$$\|v\|_{L^\infty(0,T;L^2(\Omega))} \leqslant c_0 \|F\|_{L^2(Q)},$$

for some constant c_0 depending only on Ω, T and M.

b) Let M, A and q be the same as in Lemma 2.2, and let $F \in \overline{H^1(0,T;L^2(\Omega))}$ satisfy $F(T,\cdot) = 0$. Putting $\tilde{A}(t,x) := -A(T-t,x)$, $\tilde{q}(t,x) := q(T-t,x)$ and $\tilde{F}(t,x) := \overline{F(T-t,x)}$ for $(t,x) \in Q$, we see that we have

$$\begin{cases} (i\partial_t + \Delta_A + q)v = F & \text{in } Q, \\ v(T,\cdot) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Sigma, \end{cases} \tag{2.21}$$

if and only if $\tilde{v}(t,x) := \overline{v(T-t,x)}$ is a solution to the system (2.6) *where $(\tilde{A},\tilde{q},\tilde{F})$ is substituted for (A,q,F). Therefore, by Lemma 2.2, there exists a unique solution $v \in \mathcal{C}([0,T],H_0^1(\Omega)\cap H^2(\Omega))\cap\mathcal{C}^1([0,T],L^2(\Omega))$ to* (2.21)*, and it is clear that v verifies the estimate* (2.7)*.*

2.3. **Completion of the proof of Proposition 2.1.** In light of [32, Chapter 4, Theorem 2.3] there exists $G \in H^3(Q)$ satisfying

$$G(0,\cdot) = \partial_t G(0,\cdot) = 0 \text{ in } \Omega \text{ and } G = g \text{ on } \Sigma,$$

and

$$\|G\|_{H^3(Q)} \leqslant C\|g\|_{\mathcal{H}(\Sigma)}, \tag{2.22}$$

for some positive constant C, depending only on Ω and T. Therefore, the function

$$F := -(i\partial_t + \Delta_A + q)G \in H^1(0,T;L^2(\Omega)) \tag{2.23}$$

verifies $F(0,\cdot) = 0$ in Ω. Let v be the $\mathcal{C}([0,T],H_0^1(\Omega)\cap H^2(\Omega))\cap\mathcal{C}^1([0,T],L^2(\Omega))$-solution to (2.6), associated with the source term F defined by (2.23), which is given by Lemma 2.2. Then, $u := G + v \in H^{1,2}(Q)$ is a solution to (1.1) and (2.2) follows directly from (2.7) and (2.22). Finally, we get that such a solution is unique by applying (2.2) with $g = 0$.

3. SOLUTIONS

In this section we build appropriate GO solutions to the magnetic Schrödinger equation in Q. These functions are used in the derivation of the stability estimates of Theorem 1.1, presented in Sections 4 and 5.

3.0.1. *What we are aiming for.* Namely, given $A_j \in W^{5,\infty}(Q)^n \cap H^6(Q)^n$ and $q_j \in W^{4,\infty}(Q)$, $j = 1,2$, fulfilling the conditions (1.4)-(1.6), we seek a solution u_j to the magnetic Schrödinger equation

$$(i\partial_t + \Delta_{A_j} + q_j)u_j = 0 \text{ in } Q, \tag{3.1}$$

of the form

$$u_j(t,x) = \varphi_\sigma(t,x)\left(u_{j,1}(t,x) + \sigma^{-1}u_{j,2}(t,x)\right) + r_{j,\sigma}(t,x) \text{ with } \varphi_\sigma(t,x) := e^{i\sigma(-\sigma t + x\cdot\omega)}. \tag{3.2}$$

Here $\sigma \in (1,+\infty)$ and $\omega \in \mathbb{S}^{n-1} := \{y \in \mathbb{R}^n;\ |y| = 1\}$ are arbitrarily fixed and the remainder term $r_{j,\sigma}$ in the asymptotic expansion of u_j with respect to σ^{-1}, scales at most like σ^{-1} as $\sigma \to +\infty$, in a sense that we will make precise below. Moreover, we impose that $u_{j,1}$ and $u_{j,2}$ be in $H^3(Q)$, and that they satisfy

$$\omega\cdot\nabla_{A_j}u_{j,1} = 0 \text{ and } 2i\omega\cdot\nabla_{A_j}u_{j,2} + (i\partial_t + \Delta_{A_j} + q_j)u_{j,1} = 0 \text{ in } Q, \tag{3.3}$$

and

$$\exists\delta \in \left(0,\frac{T}{4}\right),\ (t \in (0,\delta)\cup(T-\delta,T)) \Rightarrow (u_{j,1}(t,x) = u_{j,2}(t,x) = 0,\ x \in \Omega). \tag{3.4}$$

3.0.2. *Extended magnetic potentials.* The first step of the construction of the functions $u_{j,k}$, for $j,k=1,2$, involves extending the two magnetic potentials A_1 and A_2 to $(0,T)\times\mathbb{R}^n$ as follows. First, we refer to [39, Theorem 5 in Section 3] and pick a magnetic potential $\tilde{A}_1\in W^{5,\infty}((0,T)\times\mathbb{R}^n,\mathbb{R})^n\cap H^6((0,T)\times\mathbb{R}^n,\mathbb{R})^n$ which coincides with A_1 in Q and satisfies

$$\exists r\in(0,+\infty),\ \forall t\in[0,T],\ \mathrm{supp}\ (\tilde{A}_1(t,\cdot))\subset\{x\in\mathbb{R}^n,\ |x|\leqslant r\}$$

and the estimate

$$\|\tilde{A}_1\|_{W^{5,\infty}((0,T)\times\mathbb{R}^n)^n}\leqslant C\|A_1\|_{W^{5,\infty}(Q)^n}\text{ and }\|\tilde{A}_1\|_{H^6((0,T)\times\mathbb{R}^n)^n}\leqslant C\|A_1\|_{H^6(Q)^n},\tag{3.5}$$

for some positive constant C depending only on T and Ω. Thus, putting

$$\tilde{A}_2(t,x):=\begin{cases}A_2(t,x) & \text{if } (t,x)\in Q,\\ \tilde{A}_1(t,x) & \text{if } (t,x)\in((0,T)\times\mathbb{R}^n)\setminus Q,\end{cases}\tag{3.6}$$

we infer from (1.4) that $\tilde{A}_2\in W^{5,\infty}((0,T)\times\mathbb{R}^n)^n\cap H^6((0,T)\times\mathbb{R}^n)^n$. Moreover, it is clear from (3.5)-(3.6) upon possibly substituting $\max(1,C)$ for C in (3.5), that

$$\|\tilde{A}_j\|_{W^{5,\infty}((0,T)\times\mathbb{R}^n)^n}\leqslant C\max_{k=1,2}\|A_k\|_{W^{5,\infty}(Q)^n}\text{ and }\|\tilde{A}_j\|_{H^6((0,T)\times\mathbb{R}^n)^n}\leqslant C\max_{k=1,2}\|A_k\|_{H^6(Q)^n},\ j=1,2.\tag{3.7}$$

3.0.3. *Design of* $u_{1,1}$ *and* $u_{2,1}$. The next step is to introduce two functions, the first one $\chi=\chi_\delta\in\mathcal{C}^\infty(\mathbb{R};[0,1])$, being supported in $(\delta,T-\delta)$, satisfies $\chi(t)=1$ if $t\in[2\delta,T-2\delta]$ and fulfills

$$\forall k\in\mathbb{N},\ \exists C_k\in(0,+\infty),\ \|\chi\|_{W^{k,\infty}(\mathbb{R})}\leqslant C_k\delta^{-k},$$

whereas the second one is defined for $\tau\in\mathbb{R}$, $\xi\in\omega^\perp:=\{x\in\mathbb{R}^n:\ x\cdot\omega=0\}$ and $y\in\mathbb{S}^{n-1}$, by

$$\beta(t,x):=y\cdot\nabla\left(e^{-i(t\tau+\xi\cdot x)}\exp\left(-i\int_{\mathbb{R}}A(t,x+s\omega)\cdot\omega ds\right)\right),\ (t,x)\in(0,T)\times\mathbb{R}^n.\tag{3.8}$$

Here we have set $A=\tilde{A}_1-\tilde{A}_2$, i.e.

$$A(t,x)=\begin{cases}(A_1-A_2)(t,x) & \text{if } (t,x)\in Q,\\ 0 & \text{if } (t,x)\in((0,T)\times\mathbb{R}^n)\setminus Q,\end{cases}$$

according to (3.6). Since $A\in W^{5,\infty}((0,T)\times\mathbb{R}^n)^n\cap H^6((0,T)\times\mathbb{R}^n)^n$, by (3.8), we have $\beta\in W^{4,\infty}((0,T)\times\mathbb{R}^n)\cap H^5((0,T)\times\mathbb{R}^n)$. Moreover it holds true that $\omega\cdot\nabla\beta(t,x)=0$ for all $(t,x)(0,T)\times\mathbb{R}^n$. A direct calculation then shows that each of the two following $H^5((0,T)\times\mathbb{R}^n)$-functions

$$u_{1,1}(t,x):=\chi(t)\beta(t,x)\exp\left(i\int_0^{+\infty}\tilde{A}_1(t,x+s\omega)\cdot\omega ds\right)\tag{3.9}$$

and

$$u_{2,1}(t,x):=\chi(t)\exp\left(i\int_0^{+\infty}\tilde{A}_2(t,x+s\omega)\cdot\omega ds\right)\tag{3.10}$$

is a solution to the first equation of (3.3) fulfilling (3.4) and the estimate Moreover, we have

$$\|u_{1,1}\|_{H^3(Q)}\leqslant C\langle\tau,\xi\rangle^4\delta^{-3}\text{ and }\|u_{2,1}\|_{H^3(Q)}\leqslant C\delta^{-3},\tag{3.11}$$

where $\langle\tau,\xi\rangle$ is a shorthand for $(1+\tau^2+\xi^2)^{1/2}$. Here and in the remaining part of this section, C denotes a positive constant depending only Ω, T and M, which may change from line to line.

3.0.4. *Completion of the GO construction.* With reference to (3.2) and (3.9)- (3.10), the construction of the GO solutions described in Section 3.0.1 is achieved by proving the following statement.

Proposition 3.1. *For $j=1,2$, there exist $(u_{j,2},r_{j,\sigma})\in H^3(Q)\times\left(\mathcal{C}([0,T],H_0^1(\Omega)\cap H^2(\Omega))\cap\mathcal{C}^1([0,T],L^2(\Omega))\right)$ such that $u_{j,2}$ satisfies* (3.3)-(3.4)*, and the function u_j defined by* (3.2) *is a solution to* (3.1) *with zero initial and final conditions :*

$$u_1(0,x)=u_2(T,x)=0,\ x\in\Omega. \tag{3.12}$$

Moreover, we have the following estimates :

$$\|u_{1,2}\|_{H^3(Q)}\leqslant C\langle\tau,\xi\rangle^6\delta^{-4} \text{ and } \|u_{2,2}\|_{H^3(Q)}\leqslant C\delta^{-4}, \tag{3.13}$$

$$\|(i\partial_t+\Delta_{A_1}+q_1)u_{1,2}\|_{L^2(Q)}\leqslant C\langle\tau,\xi\rangle^5\delta^{-2} \text{ and } \|(i\partial_t+\Delta_{A_2}+q_2)u_{2,2}\|_{L^2(Q)}\leqslant C\delta^{-2}, \tag{3.14}$$

and

$$\|r_{1,\sigma}\|_{L^2(0,T;H^1(\Omega))}+\sigma\|r_{1,\sigma}\|_{L^2(Q)}\leqslant C\langle\tau,\xi\rangle^6\delta^{-3}, \tag{3.15}$$

$$\|r_{2,\sigma}\|_{L^2(0,T;H^1(\Omega))}+\sigma\|r_{2,\sigma}\|_{L^2(Q)}\leqslant C\delta^{-3}. \tag{3.16}$$

Proof. Let us first remark that the two conditions in (3.3) can be understood from the formal commutator formula $[i\partial_t+\Delta_{A_j},\varphi_\sigma]=i\partial_t\varphi_\sigma+\Delta\varphi_\sigma+2\nabla\varphi_\sigma\cdot\nabla_{A_j}=2i\sigma\varphi_\sigma\omega\cdot\nabla_{A_j}$, entailing

$$\begin{aligned}&(i\partial_t+\Delta_{A_j}+q_j)(u_j-r_{j,\sigma})=(i\partial_t+\Delta_{A_j}+q_j)\varphi_\sigma(u_{j,1}+\sigma^{-1}u_{j,2})\\ =\ &\varphi_\sigma\left(2i\sigma\omega\cdot\nabla_{A_j}u_{j,1}+(i\partial_t+\Delta_{A_j}+q_j)u_{j,1}+2i\omega\cdot\nabla_{A_j}u_{j,2}+\sigma^{-1}(i\partial_t+\Delta_{A_j}+q_j)u_{j,2}\right)\end{aligned}$$

in Q. This and (3.1)-(3.2) lead to defining $r_{1,\sigma}$ by

$$\begin{cases}(i\partial_t+\Delta_{A_1}+q_1)r_{1,\sigma}=-\sigma^{-1}\varphi_\sigma(i\partial_t+\Delta_{A_1}+q_1)u_{1,2} & \text{in } Q,\\ r_{1,\sigma}(0,\cdot)=0 & \text{in } \Omega,\\ r_{1,\sigma}=0 & \text{on } \Sigma,\end{cases} \tag{3.17}$$

and $r_{2,\sigma}$ by

$$\begin{cases}(i\partial_t+\Delta_{A_2}+q_2)r_{2,\sigma}=-\sigma^{-1}\varphi_\sigma(i\partial_t+\Delta_{A_2}+q_2)u_{2,2} & \text{in } Q,\\ r_{2,\sigma}(T,\cdot)=0 & \text{in } \Omega,\\ r_{2,\sigma}=0 & \text{on } \Sigma.\end{cases} \tag{3.18}$$

The initial condition in (3.17) and the final condition in (3.18) are imposed in such a way that we can deduce (3.12) from (3.4), for u_j, $j=1,2$, given by (3.2). This, (1.6) and (3.7)-(3.8) yield (3.11). Similarly, using that any $x\in\mathbb{R}^n$ decomposes into the sum $x=x_\perp+s\omega$ with $s:=x\cdot\omega$ and $x_\perp:=x-s\omega\in\omega^\perp$, it can be checked through standard computations that

$$u_{j,2}(t,x_\perp+s\omega):=-\frac{1}{2i}\int_0^s\exp\left(-i\int_{s_1}^s\tilde{A}_j(t,x_\perp+s_2\omega)\cdot\omega ds_2\right)(i\partial_t+\Delta_{A_j}+\tilde{q}_j)u_{j,1}(t,x_\perp+s_1\omega)ds_1,\ j=1,2, \tag{3.19}$$

is a solution to the second equation of (3.3) obeying the condition (3.4). Here $\tilde{q}_j$, $j=1,2$, is a $W^{4,\infty}((0,T)\times\mathbb{R}^n)$-extension of q_j (i.e. $\tilde{q}_j(t,x)=q_j(t,x)$ for a.e. $(t,x)\in Q$) such that*

Having specified $u_{j,k}$ for $j,k=1,2$, we turn now to examining the remainder term $r_{j,\sigma}$. Firstly, since $u_{j,2}\in H^3(Q)$ for $j=1,2$, it follows from (3.4) that $\varphi_\sigma(i\partial_t+\Delta_{A_j}+q_j)u_{j,2}\in H_0^1(0,T;L^2(\Omega))$. Thus $r_{1,\sigma}$ (resp. $r_{2,\sigma}$) is

*Notice that in contrast to the condition (3.6) imposing $\tilde{A}_1=\tilde{A}_2$ in $(0,T)\times(\mathbb{R}^n\setminus\Omega)$, and consequently the boundary condition (1.4), it is not requested here that $\tilde{q}_1$ and $\tilde{q}_2$ be the same in $(0,T)\times(\mathbb{R}^n\setminus\Omega)$. Therefore it is not needed that q_1 and q_2 coincide on Σ.

$$\exists r\in(0,+\infty),\ \forall t\in[0,T],\ \text{supp}\,(\tilde{q}_j(t,\cdot))\subset\{x\in\mathbb{R}^n,\ |x|\leqslant r\} \text{ and } \|\tilde{q}_j\|_{W^{4,\infty}((0,T)\times\mathbb{R}^n)}\leqslant C\|\tilde{q}_j\|_{W^{4,\infty}(Q)},$$

whose existence is guaranteed by [39, Section3, Theorem 5]. Now, in view of (3.19) we have $u_{j,2}\in H^3(Q)$ since $u_{j,1}\in H^5(Q)$, $A_j\in W^{5,\infty}(Q)$ and $q_j\in W^{4,\infty}(Q)$, and we get (3.13)-(3.14) with the aid of (1.6) and (3.7).

well defined by Lemma 2.2 (resp., Statement b) of Remark 2.3) as the $\mathcal{C}([0,T],H_0^1(\Omega)\cap H^2(\Omega))\cap\mathcal{C}^1([0,T],L^2(\Omega))$-solution to (3.17) (resp., (3.18)). Next, Statement a) in Remark 2.3 and (3.14) yield

$$\|r_{1,\sigma}\|_{L^2(Q)}\leqslant C\|(i\partial_t+\Delta_{A_1}+q_1)u_{1,2}\|_{L^2(Q)}\sigma^{-1}\leqslant C\langle\tau,\xi\rangle^5\delta^{-2}\sigma^{-1}. \tag{3.20}$$

On the other hand, we know from (2.7) and (3.13) that

$$\|r_{1,\sigma}\|_{L^2(0,T;H^2(\Omega))}\leqslant C\|e^{i\sigma(-\sigma t+x\cdot\omega)}(i\partial_t+\Delta_{A_1}+q_1)u_{1,2}\|_{H^1(0,T;L^2(\Omega))}\sigma^{-1}\leqslant C\langle\tau,\xi\rangle^6\delta^{-3}\sigma,$$

Thus, interpolating with (3.20), we have $\|r_{1,\sigma}\|_{L^2(0,T;H^1(\Omega))}\leqslant C\langle\tau,\xi\rangle^6\delta^{-3}$ and hence we find (3.15). Analogously, we establish (3.16). This completes the proof of the proposition. □

Having built $u_{j,k}$ and $r_{j,\sigma}$, for $j,k=1,2$, fulfilling (3.1)–(3.4), we are now in position to derive the stability estimates (1.7)-(1.8) of Therorem 1.1.

4. Proof of the stability estimate (1.7)

We stick with the notations of Section 3. Namely, for $j=1,2$, we consider the solution u_j to (3.1), expressed by (3.2) and described in Proposition 3.1. The proof of (1.7) boils down to a suitable estimate of the Fourier transform of the function $\chi^2 A$, presented in Lemma 4.2.

4.1. Estimation of the Fourier transform of $\chi^2 A$.

We start by proving the following technical estimate.

Lemma 4.1. *There exists a constant $C=C(T,\Omega,M)\in(0,+\infty)$ such that we have*

$$\begin{aligned}&\left|\int_{\mathbb{R}^{n+1}}\chi^2(t)\beta(t,x)e^{i\int_0^{+\infty}A(t,x+s\omega)\cdot\omega ds}A(t,x)\cdot\omega dxdt\right|\\ \leqslant\ & C\left(\langle\tau,\xi\rangle^6\delta^{-8}\sigma^5\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|+\langle\tau,\xi\rangle^8\delta^{-6}\sigma^{-1}\right),\end{aligned} \tag{4.1}$$

uniformly in $\xi\in\omega^\perp$.

Proof. For $j=1,2$, put $\psi_{j,\sigma}:=u_j-r_{j,\sigma}=u_j$ on Σ. Since $u_{1,1}$ and $u_{1,2}$ are in $H^3(Q)$, we have $\psi_{1,\sigma}=\varphi_\sigma(u_{1,1}+\sigma^{-1}u_{1,2})\in\mathcal{H}(\Sigma)$, by (3.4). Therefore, there exists a unique $v_2\in H^{1,2}(Q)$ such that

$$\begin{cases}(i\partial_t+\Delta_{A_2}+q_2)v_2=0 & \text{in } Q,\\ v_2(0,\cdot)=0 & \text{in } \Omega,\\ v_2=\psi_{1,\sigma} & \text{on } \Sigma,\end{cases}$$

according to Proposition 2.1. In light of (3.1) the function $w:=v_2-u_1$ then solves

$$\begin{cases}(i\partial_t+\Delta_{A_2}+q_2)w=2iA\cdot\nabla u_1+Vu_1 & \text{in } Q,\\ w(0,\cdot)=0 & \text{in } \Omega,\\ w=0 & \text{on } \Sigma,\end{cases}$$

with $V:=i\nabla\cdot A-(|A_1|^2-|A_2|^2)+q_1-q_2$. Next, by multiplying the first equation of the above system by $\overline{u_2}$ and integrating by parts over Q, we deduce from (3.1) and (3.12) that

$$\int_Q(2iA\cdot\nabla+V)u_1(t,x)\overline{u_2(t,x)}dxdt=\int_\Sigma\partial_\nu w(t,x)\overline{u_2(t,x)}d\sigma(x)dt. \tag{4.2}$$

Further, since $(\partial_\nu+iA_2\cdot\nu)v_2=\Lambda_{A_2,q_2}\psi_{1,\sigma}$ and $(\partial_\nu+iA_1\cdot\nu)u_1=\Lambda_{A_1,q_1}\psi_{1,\sigma}$, we have $\partial_\nu w=(\Lambda_{A_2,q_2}-\Lambda_{A_1,q_1})\psi_{1,\sigma}$ in virtue of (1.4), and hence

$$\begin{aligned}\left|\int_\Sigma\partial_\nu w(t,x)\overline{u_2(t,x)}d\sigma(x)dt\right| &\leqslant \|(\Lambda_{A_2,q_2}-\Lambda_{A_1,q_1})\psi_{1,\sigma}\|_{L^2(\Sigma)}\|\psi_{2,\sigma}\|_{L^2(\Sigma)}\\ &\leqslant C\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\|\psi_{1,\sigma}\|_{\mathcal{H}(\Sigma)}\|\psi_{2,\sigma}\|_{L^2(\Sigma)}\\ &\leqslant C\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\langle\tau,\xi\rangle^6\delta^{-8}\sigma^6,\end{aligned} \tag{4.3}$$

by Corollary 2.1, the continuity of the trace operator from $H^3(Q)$ into $\mathcal{H}(\Sigma)$, (3.2) and the estimates (3.11)-(3.13).

On the other hand, we know from (3.2) that

$$\begin{aligned}\int_Q (2iA\cdot\nabla+V)u_1\overline{u_2}(t,x)dxdt &= -2\sigma\int_Q (A\cdot\omega)u_{1,1}\overline{u_{2,1}}(t,x)dxdt+r\\ &= -2\sigma\int_Q \chi^2(t)\beta(t,x)e^{i\int_0^{+\infty}A(t,x+s\omega)\cdot\omega ds}A(t,x)\cdot\omega dxdt+r, \qquad (4.4)\end{aligned}$$

where

$$\begin{aligned}r := & -2\sigma\int_Q u_{1,1}\left(\overline{u_{2,2}}\sigma^{-1}+\varphi_\sigma\overline{r_{2,\sigma}}\right)A(t,x)\cdot\omega dxdt\\ & -2\int_Q A\cdot\left(\varphi_\sigma\left(\nabla u_{1,1}+\nabla u_{1,2}\sigma^{-1}\right)+\nabla r_{1,\sigma}\right)\overline{u_2}(t,x)dxdt+\int_Q Vu_1\overline{u_2}(t,x)dxdt.\end{aligned}$$

Since $|r|\leqslant C\langle\tau,\xi\rangle^8\delta^{-6}$ by (3.15)-(3.16), it then follows from (4.4) that

$$\begin{aligned}&\sigma^{-1}\left|\int_Q (2iA\cdot\nabla+V)u_1\overline{u_2}(t,x)dxdt\right|\\ \geqslant\ & 2\left|\int_Q \chi^2(t)\beta(t,x)e^{i\int_0^{+\infty}A(t,x+s\omega)\cdot\omega ds}A(t,x)\cdot\omega dxdt\right|-C\langle\tau,\xi\rangle^8\delta^{-6}\sigma^{-1},\end{aligned}$$

which, combined with (4.2)-(4.3), yields (4.1). □

Having established Lemma 4.1 we may now estimate the Fourier transform of χA. We recall that the Fourier transform of a function $f\in L^1(\mathbb{R}^{1+n})^n$ is defined for all $(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n$ by

$$\hat{f}(\tau,\xi):=(2\pi)^{-\frac{n+1}{2}}\int_{\mathbb{R}^{1+n}}e^{-i(t\tau+x\cdot\xi)}f(t,x)dxdt.$$

Lemma 4.2. *There exists a positive constant C, depending only on T, Ω and M, such that the inequality*

$$|\xi|\left|\widehat{\chi^2 A}(\tau,\xi)\right|\leqslant C\left(\langle\tau,\xi\rangle^6\delta^{-8}\sigma^5\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|+\langle\tau,\xi\rangle^8\delta^{-6}\sigma^{-1}\right), \qquad (4.5)$$

holds for any $(\tau,\xi)\in\mathbb{R}^{1+n}$.

Proof. The estimate (4.5) being obviously true for $\xi=0$, we will solely focus on the case $\xi\neq0$. We fix $\omega\in\xi^\perp\cap\mathbb{S}^{n-1}$ and we use the decomposition $x=x_\perp+\kappa\omega$, where $\kappa:=x\cdot\omega$ and $x_\perp:=x-\kappa\omega$, and recall from (3.8) that we have $\beta(t,x)=\beta(t,x_\perp)$, so we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{1+n}}\chi^2(t)\beta(t,x)e^{i\int_0^{+\infty}A(t,x+s\omega)\cdot\omega ds}A(t,x)\cdot\omega dxdt\\ =\ &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\omega^\perp}\chi^2(t)\beta(t,x_\perp)e^{i\int_\kappa^{+\infty}A(t,x_\perp+s\omega)\cdot\omega ds}A(t,x_\perp+\kappa\omega)\cdot\omega dx_\perp d\kappa dt\\ =\ & i\int_{\mathbb{R}}\int_{\omega^\perp}\chi^2(t)\beta(t,x_\perp)\left(\int_{\mathbb{R}}\partial_\kappa e^{i\int_\kappa^{+\infty}A(t,x_\perp+s\omega)\cdot\omega ds}d\kappa\right)dx_\perp dt\\ =\ & i\int_{\mathbb{R}}\chi^2(t)\left(\int_{\omega^\perp}\beta(t,x_\perp)\left(1-e^{i\int_{\mathbb{R}}A(t,x_\perp+s\omega)\cdot\omega ds}\right)dx_\perp\right)dt\\ =\ & i\int_{\mathbb{R}}\chi^2(t)e^{-it\tau}\left(\int_{\omega^\perp}y\cdot\nabla\left(e^{-i\xi\cdot x_\perp}e^{-i\int_{\mathbb{R}}A(t,x_\perp+s\omega)\cdot\omega ds}\right)\left(1-e^{i\int_{\mathbb{R}}A(t,x_\perp+s\omega)\cdot\omega ds}\right)dx_\perp\right)dt. \qquad (4.6)\end{aligned}$$

Next, we have

$$\begin{aligned}&\int_{\omega^\perp}y\cdot\nabla\left(e^{-i\xi\cdot x_\perp}e^{-i\int_{\mathbb{R}}A(t,x_\perp+s\omega)\cdot\omega ds}\right)\left(1-e^{i\int_{\mathbb{R}}A(t,x_\perp+s\omega)\cdot\omega ds}\right)dx_\perp\\ =\ &\int_{\omega^\perp}\nabla\cdot\left(ye^{-i\xi\cdot x_\perp}e^{-i\int_{\mathbb{R}}A(t,x_\perp+s\omega)\cdot\omega ds}\right)\left(1-e^{i\int_{\mathbb{R}}A(t,x_\perp+s\omega)\cdot\omega ds}\right)dx_\perp\end{aligned}$$

$$= \quad i\int_{\omega^\perp} e^{-i\xi\cdot x_\perp} y\cdot\nabla\left(\int_{\mathbb{R}} A(t,x_\perp+s\omega)\cdot\omega ds\right)dx_\perp,$$

by integrating by parts. Then (4.6) and the Fubini theorem entail

$$\begin{aligned}
&\int_{\mathbb{R}^{1+n}}\chi^2(t)\beta(t,x)e^{i\int_0^{+\infty}A(t,x+s\omega)\cdot\omega ds}A(t,x)\cdot\omega dxdt\\
=&\quad -\int_{\mathbb{R}}\chi^2(t)e^{-it\tau}\left(\int_{\omega^\perp}\left(\int_{\mathbb{R}} y\cdot\nabla(A(t,x_\perp+s\omega)\cdot\omega)ds\right)e^{-i\xi\cdot x_\perp}dx_\perp\right)dt\\
=&\quad -\int_{\mathbb{R}}\chi^2(t)e^{-it\tau}\left(\int_{\mathbb{R}^n} e^{-i\xi\cdot x} y\cdot\nabla(A(t,x)\cdot\omega)dx\right)dt\\
=&\quad -\int_{\mathbb{R}^{n+1}} e^{-i(t\tau+x\cdot\xi)}\chi^2(t)y\cdot\nabla(A(t,x)\cdot\omega)dxdt.
\end{aligned}$$

Therefore, taking $y=\frac{\xi}{|\xi|}$ and applying Stokes formula to the above integral, we obtain

$$\begin{aligned}
&i\int_{\mathbb{R}^{1+n}}\chi^2(t)\beta(t,x)e^{i\int_0^{+\infty}A(t,x+s\omega)\cdot\omega ds}A(t,x)\cdot\omega dxdt\\
=&\quad |\xi|\int_{\mathbb{R}^{n+1}} e^{-i(t\tau+x\cdot\xi)}\chi^2(t)A(t,x)\cdot\omega dxdt\\
=&\quad (2\pi)^{\frac{n+1}{2}}|\xi|\widehat{\chi^2 A}(\tau,\xi)\cdot\omega,
\end{aligned}$$

which yields

$$|\xi|\left|\widehat{\chi^2 A}(\tau,\xi)\cdot\omega\right|\leqslant C\langle\tau,\xi\rangle^6\delta^{-8}\left(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\sigma^5+\langle\tau,\xi\rangle^2\delta^2\sigma^{-1}\right), \tag{4.7}$$

in virtue of (4.1).

Further, since $\nabla\cdot A=0$ in Q, by (1.5), then we have $\widehat{\chi^2 A}(\tau,\xi)\cdot\xi=0$ by direct calculation and hence

$$\widehat{\chi^2 A}(\tau,\xi)=\sum_{k=1}^{n-1}\left(\widehat{\chi^2 A}(\tau,\xi)\cdot e_k\right)e_k,$$

for any orthonormal basis $\{e_1,\ldots,e_{n-1}\}$ of $\xi^\perp$. Finally, (4.5) follows directly from this upon applying (4.7) with $\omega=e_k$ for $k=1,\ldots,n-1$. $\square$

Having established Lemma 4.2, we are now in position to derive the stability estimate (1.7).

4.2. **Completion of the proof.** We start by estimating $\|\chi^2 A\|_{L^2(0,T;H^5(\Omega))^n}$. Recalling from (1.4) that $\chi^2 A$ is supported in $(\delta,T-\delta)\times\overline{\Omega}$, we see that

$$\|\chi^2 A\|^2_{L^2(0,T;H^5(\Omega))^n}=\int_{\mathbb{R}^{1+n}}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau,$$

where, as usual, $\langle\xi\rangle$ denotes $\left(1+|\xi|^2\right)^{\frac{1}{2}}$. Next, for $R\in(1,+\infty)$ fixed, we put $B_R:=\{(\tau,\xi)\in\mathbb{R}^{1+n};\ |(\tau,\xi)|<R\}$, use that $\chi^2 A\in H^1(\mathbb{R};H^5(\mathbb{R}^n))^n\cap L^2(\mathbb{R};H^6(\mathbb{R}^n))^n$, and obtain

$$\begin{aligned}
\int_{\mathbb{R}^{1+n}\setminus B_R}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau &\leqslant R^{-2}\int_{\mathbb{R}^{1+n}\setminus B_R}|(\xi,\tau)|^2\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau\\
&\leqslant R^{-2}\left(\|\chi^2 A\|^2_{H^1(\mathbb{R};H^5(\mathbb{R}^n))^n}+\|\chi^2 A\|^2_{L^2(\mathbb{R};H^6(\mathbb{R}^n))^n}\right).
\end{aligned}$$

Therefore, we have

$$\int_{\mathbb{R}^{1+n}\setminus B_R}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau\leqslant CR^{-2}\delta^{-2}\|A\|^2_{H^6(Q)^n}\leqslant CR^{-2}\delta^{-2}, \tag{4.8}$$

where C is a generic positive constant depending only on Ω, T and M, which may change from line to line. Further, setting $E_R := \{(\tau,\xi)\in\mathbb{R}^{1+n};\ |\xi|\leqslant R^{-\frac{3}{n}}\}$, we get

$$\begin{aligned}\int_{B_R\cap E_R}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau &\leqslant \int_{-R}^{R}\int_{|\xi|\leqslant R^{-\frac{3}{n}}}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau\\ &\leqslant (1+R^{-\frac{3}{n}})^{10}\|\widehat{\chi^2 A}\|^2_{L^\infty(\mathbb{R}^{1+n})^n}\left(\int_{-R}^{R}\int_{|\xi|\leqslant R^{-\frac{3}{n}}} d\xi d\tau\right),\end{aligned}$$

hence

$$\int_{B_R\cap E_R}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau\leqslant 2^{11}\|\widehat{\chi^2 A}\|^2_{L^\infty(\mathbb{R}^{1+n})^n}R^{-2}.$$

Moreover, since A is supported in Q and $\|\chi\|_{L^\infty(0,T)}\leqslant 1$, we have

$$\|\widehat{\chi^2 A}\|_{L^\infty(\mathbb{R}^{1+n})^n}\leqslant (2\pi)^{-\frac{n+1}{2}}\|\chi^2 A\|_{L^1(\mathbb{R}^{1+n})^n}\leqslant (2\pi)^{-\frac{n+1}{2}}T\,|\Omega|\,\|A\|_{L^\infty(Q)^n},$$

so we get

$$\int_{B_R\cap E_R}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau\leqslant (2\pi)^{-(n+1)}2^{11}T^2|\Omega|^2\|A\|^2_{L^\infty(Q)^n}R^{-2}.$$

This and (4.8) yield

$$\int_{(\mathbb{R}^{1+n}\setminus B_R)\cup(B_R\cap E_R)}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau\leqslant CR^{-2}\delta^{-2}. \tag{4.9}$$

On the other hand, putting $\epsilon := \|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|$, we derive from (4.5) for all $(\tau,\xi)\in B_R\setminus E_R$, that

$$\begin{aligned}|\widehat{\chi^2 A}(\tau,\xi)|^2 &\leqslant C(R^{\frac{6}{n}+12}\delta^{-16}\sigma^{10}\epsilon^2+R^{\frac{6}{n}+16}\delta^{-12}\sigma^{-2})\\ &\leqslant C(R^{15}\delta^{-16}\sigma^{10}\epsilon^2+R^{19}\delta^{-12}\sigma^{-2})\end{aligned}$$

which involves

$$\int_{B_R\setminus E_R}\langle\xi\rangle^{10}|\widehat{\chi^2 A}(\tau,\xi)|^2 d\xi d\tau\leqslant C(R^{26+n}\delta^{-16}\sigma^{10}\epsilon^2+R^{30+n}\delta^{-12}\sigma^{-2}).$$

It follows from this and (4.9) that

$$\|\chi^2 A\|^2_{L^2(0,T;H^5(\Omega))^n}\leqslant C(R^{26+n}\delta^{-16}\sigma^{10}\epsilon^2+R^{30+n}\delta^{-12}\sigma^{-2}+R^{-2}\delta^{-2}). \tag{4.10}$$

Further, by noticing that there exists a constant $C_0>0$ such that

$$\|A-\chi^2 A\|_{L^2(0,T;H^5(\Omega))^n}\leqslant C_0|\Omega|^{1/2}\|1-\chi^2\|_{L^2(0,T)}\|A\|_{W^{5,\infty}(Q)}$$

and taking advantage of the fact that the $[0,1]$-valued function $1-\chi$ vanishes in $[2\delta,T-2\delta]$, we get that $\|1-\chi^2\|^2_{L^2(0,T)}=\int_0^{2\delta}(1-\chi^2(t))^2dt+\int_{T-2\delta}^{T}(1-\chi^2(t))^2dt\leqslant 4\delta$. This entails

$$\|A-\chi^2 A\|^2_{L^2(0,T;H^5(\Omega))^n}\leqslant 4C_0^2|\Omega|\|A\|^2_{W^{5,\infty}(Q)}\delta$$

and consequently

$$\|A\|^2_{L^2(0,T;H^5(\Omega))^n}\leqslant C\left(\epsilon^2R^{26+n}\delta^{-16}\sigma^{10}+R^{30+n}\delta^{-12}\sigma^{-2}+R^{-2}\delta^{-2}+\delta\right), \tag{4.11}$$

by invoking (4.10). Now, the strategy is to choose δ as a power of R so that $R^{-2}\delta^{-2}=\delta$, i.e. $\delta=R^{-\frac{2}{3}}$, and to do the same with σ, that is to take $\sigma=R^{\frac{116+3n}{6}}$, in such a way that the three last terms in the right hand side of (4.11) are equal to $R^{-\frac{2}{3}}$. Evidently, as we have $\delta\in\left(0,\frac{T}{4}\right)$, by assumption, this requires that R be fixed in $\left(\left(\frac{T}{4}\right)^{-\frac{3}{2}},\infty\right)$. Summing up, we infer from (4.11) that

$$\|A\|^2_{L^2(0,T;H^5(\Omega))^n}\leqslant C\left(R^{230+6n}\epsilon^2+R^{-\frac{2}{3}}\right). \tag{4.12}$$

Therefore, we get (1.7) with $r:=\frac{1}{346+9n}$ for all $\epsilon\in(0,\epsilon_r)$, where $\epsilon_r:=\left(\frac{T}{4}\right)^{\frac{1}{2r}}$, upon choosing $R=\epsilon^{-3r}$ in (4.12), whereas $\|A\|_{L^2(0,T;H^5(\Omega))^n}\leqslant\frac{M}{\epsilon_r^r}\epsilon^r$ for all $\epsilon\in[\epsilon_r,+\infty)$. This achieves the proof of (1.7).

5. Proof of the stability estimate (1.8)

Here we use the definitions and notations introduced in Sections 3 and 4, except for the function β, which is no longer given by (3.8) but is rather defined for $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, by

$$\beta(t,x) := e^{-i(t\tau + x\cdot\xi)},\ (t,x) \in (0,T)\times\mathbb{R}^n.$$

It is clear that all the estimates derived in Sections 3 and 4 remain valid with this specific choice of β.

In light of (3.11)–(3.16) and (4.2)-(4.3), it holds true that

$$\begin{aligned} &\left|\int_Q V u_{1,1}\overline{u_{2,1}}(t,x)dxdt\right| \\ \leqslant\ & C\left(\|A\|_{L^\infty(Q)^n}\langle\tau,\xi\rangle^8\delta^{-6}\sigma + \|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\langle\tau,\xi\rangle^6\delta^{-8}\sigma^6 + \langle\tau,\xi\rangle^6\delta^{-4}\sigma^{-1}\right). \end{aligned} \tag{5.1}$$

Let q be the function $q_1 - q_2$ extended by zero in $\mathbb{R}^{1+n}\setminus Q$. Then, from the very definition of V, we have

$$\begin{aligned} \int_Q V u_{1,1}\overline{u_{2,1}}(t,x)dxdt &= \int_Q q u_{1,1}\overline{u_{2,1}}(t,x)dxdt - i\int_{\mathbb{R}^{1+n}} A\cdot\nabla(u_{1,1}\overline{u_{2,1}})(t,x)dxdt \\ &\quad - \int_Q A\cdot(A_1+A_2)u_{1,1}\overline{u_{2,1}}(t,x)dxdt, \end{aligned}$$

by applying the Stokes formula. This, (3.11)–(3.16) and (5.1) yield

$$\left|\int_{\mathbb{R}^{1+n}} q u_{1,1}\overline{u_{2,1}}(t,x)dxdt\right| \tag{5.2}$$
$$\leqslant C\left(\|A\|_{L^\infty(Q)^n}\langle\tau,\xi\rangle^8\delta^{-6}\sigma + \|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\langle\tau,\xi\rangle^6\delta^{-8}\sigma^6 + \langle\tau,\xi\rangle^6\delta^{-4}\sigma^{-1}\right). \tag{5.3}$$

On the other hand, we have

$$\int_{\mathbb{R}^{1+n}} q u_{11}\overline{u_{21}}(t,x)dxdt = \int_{\mathbb{R}^{1+n}} \chi^2(t)q(t,x)e^{-i(t\tau+x\cdot\xi)}e^{i\int_0^{+\infty}A(t,x+s\omega)\cdot\omega ds}dxdt,$$

whence

$$\begin{aligned} &\left|\int_{\mathbb{R}^{1+n}}\chi^2(t)q(t,x)e^{-i(t\tau+x\cdot\xi)}dxdt\right| \\ \leqslant\ & \left|\int_{\mathbb{R}^{1+n}} q u_{11}\overline{u_{21}}(t,x)dxdt\right| + \left|\int_{\mathbb{R}^{1+n}} q(t,x)e^{-i(t\tau+x\cdot\xi)}\left(e^{i\int_0^{+\infty}A(t,x+s\omega)\cdot\omega ds}-1\right)dxdt\right|. \end{aligned}$$

Thus, applying the mean value theorem, we get that

$$\left|\int_{\mathbb{R}^{1+n}}\chi^2(t)q(t,x)e^{-i(t\tau+x\cdot\xi)}dxdt\right| \leqslant \left|\int_{\mathbb{R}^{1+n}} q u_{11}\overline{u_{21}}(t,x)dxdt\right| + C\|A\|_{L^\infty(Q)^n}.$$

Plugging (5.3) into the above estimate, we find that

$$\left|\widehat{\chi^2 q}(\tau,\xi)\right| \leqslant C\left(\|A\|_{L^\infty(Q)^n}\langle\tau,\xi\rangle^8\delta^{-6}\sigma + \|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\langle\tau,\xi\rangle^6\delta^{-8}\sigma^6 + \langle\tau,\xi\rangle^6\delta^{-4}\sigma^{-1}\right). \tag{5.4}$$

The next step of the proof is to upper bound $\|A\|_{L^\infty(Q)^n}$ in terms of $\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|$. To this end, we pick $p \in (n+1,+\infty)$, apply the Sobolev embedding theorem (see e.g. [14, Corollary IX.14]) and obtain that $\|A\|_{L^\infty(Q)^n} \leqslant C\|A\|_{W^{1,p}(Q)^n}$. By interpolating, we get $\|A\|_{L^\infty(Q)^n} \leqslant C\|A\|_{W^{2,p}(Q)^n}^{1/2}\|A\|_{L^p(Q)^n}^{1/2} \leqslant C\|A\|_{L^p(Q)^n}^{1/2}$ since $\|A\|_{W^{2,p}(Q)^n} \leqslant (T|\Omega|)^{1/p}\|A\|_{W^{2,\infty}(Q)^n} \leqslant (T|\Omega|)^{1/p}M$, whence $\|A\|_{L^\infty(Q)^n} \leqslant C\|A\|_{L^2(Q)^n}^{1/p}$ upon remembering that $\|A\|_{L^p(Q)^n}^p \leqslant \|A\|_{L^\infty(Q)}^{p-2}\|A\|_{L^2(Q)}^2 \leqslant M^{p-2}\|A\|_{L^2(Q)^n}^2$. As a consequence we have $\|A\|_{L^\infty(Q)^n} \leqslant C\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|^{r/p}$, by (1.7), so it follows from (5.4) that

$$\left|\widehat{\chi^2 q}(\tau,\xi)\right| \leqslant C\left(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|^{r/p}\langle\tau,\xi\rangle^8\delta^{-6}\sigma + \|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\langle\tau,\xi\rangle^6\delta^{-8} + \langle\tau,\xi\rangle^6\delta^{-4}\sigma^{-1}\right). \tag{5.5}$$

Next, q being bounded and supported in Q, we have $q \in L^2(\mathbb{R}^{1+n})$ with $\|\widehat{\chi^2 q}\|_{L^2(\mathbb{R}^{1+n})} = \|\chi^2 q\|_{L^2(\mathbb{R}^{1+n})} \leqslant \|q\|_{L^2(\mathbb{R}^{1+n})} \leqslant M$ since $\|\chi\|_{L^\infty(\mathbb{R})} = 1$, whence :

$$\int_{\mathbb{R}^{1+n}\setminus B_R} (1+\tau^2+|\xi|^2)^{-1} \left|\widehat{\chi^2 q}(\tau,\xi)\right|^2 d\xi d\tau \leqslant R^{-2}\|\widehat{\chi^2 q}\|^2_{L^2(\mathbb{R}^{1+n})} \leqslant M^2 T\,|\Omega|\,R^{-2},\ R\in(1,+\infty).$$

Thus, using that $\|\chi^2 q\|_{H^{-1}(Q)} \leqslant \|\chi^2 q\|_{H^{-1}(\mathbb{R}^{1+n})}$, we obtain by following the same path as in the proof of (4.10) that

$$\|\chi^2 q\|_{H^{-1}(Q)} \leqslant C\left(R^{\frac{1+n}{2}}\left(\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|^{r/p}R^8\delta^{-6}\sigma + \|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|R^6\delta^{-8} + R^6\delta^{-4}\sigma^{-1}\right) + R^{-1}\right),$$

for all $R\in(1,+\infty)$. This and the estimate $\|\chi^2 q - q\|_{H^{-1}(Q)} \leqslant \|\chi^2 q - q\|_{L^2(Q)} \leqslant C\delta$ yield (1.8) upon arguing as in the derivation of (1.7) from (4.10).

ACKNOWLEDGMENTS

The authors are grateful to the anonymous referees for their careful reading and their suggestions that helped improve the paper. This work was partially supported by the Agence Nationale de la Recherche under grant ANR-17-CE40-0029.

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