SOLVING LINEAR TIME-FRACTIONAL DIFFUSION EQUATIONS WITH A SINGULAR SOURCE TERM

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Abstract. This article deals with linear time-fractional diffusion equations with time-dependent singular source term. Whether the order of the time-fractional derivative is multi-term, distributed or space-dependent, we prove that the system admits a unique weak solution enjoying a Duhamel representation, provided that the time-dependence of the source term is a distribution. As an application, the square integrable space-dependent part and the distributional time-dependent part of the source term of a multi-term time-fractional diffusion equation are simultaneously recovered by partial internal observation of the solution.

1. Introduction and settings

1.1. **Time-fractional derivatives.** In the present article, Ω is a bounded and connected open subset of $\mathbb{R}^d$, $d \geqslant 2$, with Lipschitz boundary $\partial\Omega$. Given $K \in L^1_{\mathrm{loc}}(\mathbb{R}_+, L^\infty(\Omega)) \cap C^\infty(\mathbb{R}_+, L^\infty(\Omega))$, where $\mathbb{R}_+ := (0, +\infty)$, we introduce the integral operator

$$(I_K g)(t,x) := \int_0^t K(t-s,x) g(s,x) ds,\ g \in L^1_{\mathrm{loc}}(\mathbb{R}_+, L^2(\Omega)),\ x \in \Omega,\ t \in \mathbb{R}_+.$$

For any complete locally convex topological vector space X, we denote by $D'_+(\mathbb{R}, X)$ (resp., $\mathcal{S}'_+(\mathbb{R}, X)$) the set of X-valued distributions (resp., tempered distributions) in $D'(\mathbb{R}, X)$ (resp., $\mathcal{S}'(\mathbb{R}, X)$) that are supported in $[0, +\infty)$. Notice that any distribution in $D'_+(\mathbb{R}, X')$, where X' is the topological space dual to X, may be regarded as a continuous linear form in $D_{\mathrm{a}}(\mathbb{R}, X) := \{\psi \in C^\infty(\mathbb{R}, X);\ \exists R > 0,\ \operatorname{supp} \psi \subset (-\infty, R)\}$, endowed with the associated canonical LF-topology. We denote by $\langle \cdot, \cdot \rangle_{\mathrm{a},X}$ or simply by $\langle \cdot, \cdot \rangle$ when there is no ambiguity, the corresponding duality pairing. Thus, bearing in mind that $s \mapsto \int_0^{+\infty} K(t, \cdot)\psi(t+s, \cdot) dt \in D_{\mathrm{a}}(\mathbb{R}, L^2(\Omega))$ when $\psi \in D_{\mathrm{a}}(\mathbb{R}, L^2(\Omega))$, we extend I_K as a continuous linear map from $D'_+(\mathbb{R}, L^2(\Omega))$ to

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$D'_+(\mathbb{R}, L^2(\Omega))$ by setting

$$\langle I_K v, \psi\rangle := \left\langle v(s,\cdot), \int_0^{+\infty} K(t,\cdot)\psi(t+s,\cdot)dt\right\rangle,\ v \in D'_+(\mathbb{R}, L^2(\Omega)),\ \psi \in D_{\mathrm{a}}(\mathbb{R}, L^2(\Omega)),$$

where, as usual, $L^2(\Omega)$ is identified with its dual space. Next, with reference to the definition [32, Chapter 6, Section 5, Eq. 15] of non-integer order distributional derivatives, we introduce the Riemann-Liouville (resp., Caputo) fractional derivative $D_{t,K}$ (resp., $\partial_{t,K}$) with kernel K as $D_{t,K}v := \partial_t I_K v$ (resp., $\partial_{t,K}v := I_K \partial_t v$) for all $v \in D'_+(\mathbb{R}, L^2(\Omega))$.

1.2. **Initial boundary value problem with a singular source term.** We consider the following initial boundary value problem (IBVP) with initial state u_0 and source F,

$$\begin{cases} (\partial_{t,K} + \mathcal{A})u(t,x) &= F(t,x), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u(t,x) &= 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ u(0,x) &= u_0(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

Here and below, we set

$$\mathcal{A}u(x) := -\sum_{i,j=1}^d \partial_{x_i}\left(a_{i,j}(x)\partial_{x_j}u(x)\right) + q(x)u(x),\ x \in \Omega,$$

where $q \in L^{\frac{d}{2}}(\Omega)$ is non-negative and $a := (a_{i,j})_{1\leqslant i,j\leqslant d} \in L^\infty(\Omega, \mathbb{R}^{d^2}) \cap H^1(\Omega, \mathbb{R}^{d^2})$ is symmetric and satisfies the ellipticity condition:

$$\exists c > 0,\ \sum_{i,j=1}^d a_{i,j}(x)\xi_i\xi_j \geqslant c|\xi|^2,\ x \in \Omega,\ \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d. \tag{1.2}$$

Let $u_0 \in L^2(\Omega)$ and $F \in D'_+(\mathbb{R}, L^2(\Omega))$.

Definition 1.1. *A weak-solution to* (1.1) *is any* $u \in \mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$ *satisfying the two following conditions simultaneously:*

- (i) $\langle D_{t,K}u + \mathcal{A}u, \psi\rangle_{\mathrm{a},D(\Omega)} = \langle K_+ u_0 + F, \psi\rangle_{\mathrm{a},L^2(\Omega)}$ *for all* $\psi \in D_{\mathrm{a}}(\mathbb{R}, D(\Omega))$, *where* $K_+(t,x) := K(t,x)\mathbb{1}_{\mathbb{R}_+}(t)$ *and* $\mathbb{1}_{\mathbb{R}_+}$ *is the characteristic function of* $\mathbb{R}_+$.
- (ii) *For all* $p \in \mathbb{C}_+ := \{z \in \mathbb{C};\ \Re z > 0\}$, *the Laplace transform (with respect to* t*) of* u *at* p, *defined for a.e.* $x \in \Omega$ *by* $\widehat{u}(p,x) := \langle u(t,x), e^{-pt}\rangle_{\mathcal{S}'_+(\mathbb{R}),\mathcal{S}_+(\mathbb{R})}$ *where* $\mathcal{S}_+(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R});\ \varphi_{|\mathbb{R}_+} \in \mathcal{S}(\mathbb{R}_+)\}$, *lies in* $H_0^1(\Omega)$.

1.3. **Anomalous diffusion processes.** In this article we study the existence and the uniqueness issues of a weak solution to (1.1) in the presence of a singular source term F, for each of the three following classical types of diffusion models:

1) *Time-fractional diffusion equations of space-dependent variable order* $\alpha \in L^\infty(\Omega)$ fulfilling

$$0 < \alpha_0 \leqslant \alpha(x) \leqslant \alpha_M < 1,\ x \in \Omega, \text{where } \alpha_M < 2\alpha_M.$$

These models are defined by the IBVP (1.1) associated with

$$K(t,x) := \frac{t^{-\alpha(x)}}{\Gamma(1-\alpha(x))},\ t \in \mathbb{R}_+,\ x \in \Omega. \tag{1.3}$$

2) *Distributed order time-fractional diffusion models* with a non-negative weight function $\mu \in L^\infty(0,1)$ obeying

$$\exists \alpha_0 \in (0,1),\ \exists \varepsilon \in (0,\alpha_0),\ \forall \alpha \in (\alpha_0 - \varepsilon, \alpha_0),\ \mu(\alpha) \geq \frac{\mu(\alpha_0)}{2} > 0.$$

They are described by (1.1) when

$$K(t,x) := \int_0^1 \mu(\alpha) \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha, \quad t \in \mathbb{R}_+,\ x \in \Omega. \tag{1.4}$$

3) *Multiterm time-fractional diffusion equations* of orders $\alpha_1 < \ldots < \alpha_N$, where $N \in \mathbb{N} := \{1, 2, \ldots\}$, $\alpha_1 > 0$ and $\alpha_N < 1$, with density functions $\rho_j \in L^\infty(\Omega)$ satisfying

$$0 < c_0 \leqslant \rho_j(x) \leqslant C_0 < +\infty,\ \text{a.e. } x \in \Omega,\ j = 1, \ldots, N.$$

Such models can be expressed by (1.1) with

$$K(t,x) := \sum_{j=1}^N \rho_j(x) \frac{t^{-\alpha_j}}{\Gamma(1-\alpha_j)},\ t \in \mathbb{R}_+,\ x \in \Omega. \tag{1.5}$$

Anomalous diffusion in a heterogeneous medium is a growing issue of scientific research, with numerous applications areas such as geophysics, hydrology or biology, see e.g., [1, 5, 8]. Some typical examples are fluid flow in porous media, propagation of seismic waves, and protein dynamics. In this framework the variations of permeability in different spatial positions caused by the heterogeneities of the medium induce location dependent diffusion phenomena which are correctly described by the space dependent model (1.1) associated with the kernel (1.3). On the other hand, (1.4) is used for modeling ultra slow diffusion processes whose mean square displacement scales like a log with respect to the time variable. For instance, such phenomena were observed in polymer physics or the kinetics of particles moving in quenched random force fields, see e.g. in [28, 33]. Finally, the behavior of viscoelastic fluids and rheological material is commonly described by the multi-term time-fractional diffusion equation (1.1) associated with (1.5), see e.g. [6].

2. Singular sources and the well-posedness issue

2.1. **What we are aiming for.** The well-posedness issue of constant-order fractional diffusion equations (these are equations of the form (1.1) with $K(t,x) := \frac{t^{-\beta}}{\Gamma(1-\beta)}$ for some fixed $\beta \in (0,1)$) has received a great deal of attention from the mathematical community over the last decades, see e.g. [4, 31, 20] and the references therein. Similarly, several techniques were used in [18, 24, 23, 25, 23] to build a solution to variable-order, distributed or multi-term time-fractional processes, and we refer the reader to [13] for a global comparative analysis of these different approaches. All the above mentioned works assume that the source term F is within the class $L^1_{\mathrm{loc}}(\mathbb{R}_+, L^2(\Omega))$ but, recently, the well-posedness of constant-order time-fractional diffusion systems was examined in [35] when $t \mapsto F(t,\cdot)$ lies in a negative order Sobolev space. The aim of this article is to extend the study carried out in [35] in two main directions: Firstly, by adapting the analysis to the wider class of diffusion equations described in Section 1.3 and, secondly, by considering source terms with distributional temporal dependence, that is to say source terms F such that $t \mapsto F(t,\cdot)$ is not necessarily a function but rather a distribution.

2.2. **Statement of the result.** We start with the definition of $\mathcal{R} * v$ when $v \in D'_+(\mathbb{R}, L^2(\Omega))$ and $\mathcal{R} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{B}(L^2(\Omega))) \cap D'_+(\mathbb{R}, \mathcal{B}(L^2(\Omega)))$, where $\mathcal{B}(L^2(\Omega))$ is as usual the set of linear bounded operators in $L^2(\Omega)$. Let us denote the adjoint operator to $\mathcal{R}(t)$ by $\mathcal{R}^*(t)$. Then, bearing in mind that for all $\varphi \in D_{\mathrm{a}}(\mathbb{R}, L^2(\Omega))$,

$$s \mapsto \int_{\mathbb{R}} \mathcal{R}(t)^* \varphi(t+s,\cdot) dt = \int_0^{+\infty} \mathcal{R}(t)^* \varphi(t+s,\cdot) dt \in D_{\mathrm{a}}(\mathbb{R}, L^2(\Omega)),$$

we define $\mathcal{R} * v \in D'_+(\mathbb{R}, L^2(\Omega))$ by

$$\langle \mathcal{R} * v, \varphi \rangle_{\mathrm{a}, L^2(\Omega)} := \left\langle v(s,\cdot), \int_{\mathbb{R}} \mathcal{R}(t)^* \varphi(t+s,\cdot) dt \right\rangle_{\mathrm{a}, L^2(\Omega)}, \ \varphi \in D_{\mathrm{a}}(\mathbb{R}, L^2(\Omega)).$$

As an immediate consequence we obtain that $\partial_t^m (\mathcal{R} * v) = \mathcal{R} * (\partial_t^m v)$ for all $m \in \mathbb{N}$.

This being said, let K be given by either (1.3), (1.4) or (1.5). Next, we fix $\theta \in (\frac{\pi}{2}, \pi)$ and $\delta \in (0, +\infty)$, put

$$\gamma_0(\delta,\theta) := \{\delta\, e^{i\beta},\ \beta \in [-\theta,\theta]\}, \quad \gamma_\pm(\delta,\theta) := \{s\, e^{\pm i\theta},\ s \in [\delta,\infty)\}, \tag{2.1}$$

and introduce the following contour in $\mathbb{C}$,

$$\gamma(\delta,\theta) := \gamma_-(\delta,\theta) \cup \gamma_0(\delta,\theta) \cup \gamma_+(\delta,\theta),$$

oriented counterclockwise. Then, with reference to [13, 18, 23], we set for all $\psi \in L^2(\Omega)$,

$$S_{j,K}(t)\psi := \frac{\mathbb{1}_{\mathbb{R}_+}(t)}{2i\pi} \int_{\gamma(\delta,\theta)} e^{tp} \left(A + p\widehat{K}(p,\cdot)\right)^{-1} \widehat{K}(p,\cdot)^{1-j}\psi dp,\ j=0,1, \tag{2.2}$$

where A is the self-adjoint operator in $L^2(\Omega)$ acting as $\mathcal{A}$ on its domain

$$D(A) := \{h \in H_0^1(\Omega);\ \mathcal{A}h \in L^2(\Omega)\}.$$

Notice that the right hand side on (2.2) is well-defined as $-p\widehat{K}(p,x)$ is in the resolvent set of A for all $p \in \gamma(\delta,\theta)$ and a.e. $x \in \Omega$. Now, applying [16, Lemma 6.1] and [13, Lemmas 2.1, 3.1 and 4.2], we get that

$$S_{j,K} \in L^1_{\mathrm{loc}}(\mathbb{R}, L^2(\Omega)) \cap \mathcal{S}'_+(\mathbb{R}, \mathcal{B}(L^2(\Omega))),\ j=0,1.$$

Moreover, it is clear that for all $F \in D'_+(\mathbb{R}, L^2(\Omega))$,

$$u(t,\cdot) := S_{0,K}(t)u_0 + S_{1,K} * F(t,\cdot),\ t \in \mathbb{R}, \tag{2.3}$$

lies in $D'_+(\mathbb{R}, L^2(\Omega))$.

In the peculiar case where $F \in L^1_{\mathrm{loc}}(\mathbb{R}, L^2(\Omega)) \cap D'_+(\mathbb{R}, L^2(\Omega))$, the identity (2.3) reads

$$u(t,\cdot) = S_{0,K}(t)u_0 + \int_0^t S_{1,K}(t-s)F(s,\cdot)ds,\ t \in \mathbb{R}.$$

Furthermore, by assuming in addition that $(1+t)^{-N}F \in L^1(\mathbb{R}_+; L^2(\Omega))$ for some $N \in \mathbb{N}$, we obtain from [13, Theorems 1.3, 1.4 and 1.5] that the distribution u defined in (2.3) is a weak solution to the IBVP (1.1) in the sense of Definition 1.1. The main achievement of this short article is the following generalization of this result to the case of source terms $F \in \mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$.

Theorem 2.1. *Let $u_0 \in L^2(\Omega)$, let $F \in \mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$ and let K be defined by either* (1.3), (1.4) *or* (1.5). *Assume* (1.2). *Then, the distribution expressed by* (2.3) *is the unique weak solution to* (1.1) *in the sense of Definition 1.1.*

The proof of Theorem 2.1 is given in Section 3.

Notice that Definition 1.1 of a weak solution to diffusion equations with a variable, distributed or multi-term fractional derivative, and a possibly singular source term F in $D'_+(\mathbb{R}, L^2(\Omega))$, generalizes the ones given in [4, 13, 18, 21, 23] and in the references therein, in the context of more specific diffusion processes. Moreover, the representation formula of the weak solution used in [13, 18, 23] is a byproduct of the Duhamel formula (2.3) that is established in Theorem 2.1 for a wider set of source terms lying in $\mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$.

It is worth pointing out that Theorem 2.1 holds provided that the elliptic part $\mathcal{A}$ of the diffusion equation (1.1) is symmetric. This can be explained by the fact that suitable resolvent estimates for symmetric operators established in either [18, Proposition 2.1] or [13, Lemma 4.1], depending on whether K is given by (1.3) or (1.5), are needed by the proof of Theorem 2.1. But, as these estimates extensively rely on the symmetry property of $\mathcal{A}$, it is not clear yet how the statement of Theorem 2.1 could be adapted to elliptic operators with a non-symmetric advection term.

For further reference we also stress out that Theorem 2.1 provides a unique weak solution enjoying the Duhamel representation formula (2.3), to anomalous diffusion processes governed by the IBVP (1.1) with a discrete-in-time source term of the form

$$F(t,x) = \sum_{j=1}^{N} \delta_{t_j}^{(k_j)}(t) f_j(x),\ t \in \mathbb{R},\ x \in \Omega, \tag{2.4}$$

where $N \in \mathbb{N}$, $0 \leqslant t_1 < t_2 < \ldots < t_N < +\infty$, $k_1, \ldots, k_N$ are non-negative integers, and $f_1, \ldots, f_N$ are in $L^2(\Omega)$. The general expression given by (2.4) is inspired from the one that is used in [2, 3] (in the special case where $k_j = 0$ for all $j = 1, \ldots, N$) for modeling rock fractures in an inverse seismic source problem.

2.3. **Target models and related inverse problems.** The Duhamel formula (2.3) is a not only useful for describing and studying the weak-solution to the IBVP (1.1) but it is also a powerful tool for solving numerous inverse problems appearing in the context of anomalous diffusion processes. For instance, such a representation formula was successfully used in [16, 18] for the identification of the weight function K (which is indicative of the nature of the undergoing anomalous diffusion process), in [15, 17] for the determination of unknown physical parameters such as the velocity field or the density, and in [10, 11, 12, 19, 34] for the detection of a source term.

One of the benefits of the Duhamel formula (2.3) of Theorem 2.1 is that it allows the analysis carried out in [10, 11, 12, 19, 34] to be extended to the case of time-singular source terms. As a matter of fact, assuming that the source term F in (1.1) is the tensor product (in the distributional sense)

$$F(t,x) = \sigma(t) \otimes f(x),\ t \in \mathbb{R}_+,\ x \in \Omega, \tag{2.5}$$

where f is square integrable and σ is a compactly supported distribution, we will establish that either of the two unknowns σ or f can be retrieved by partial internal measurement of the solution u to (1.1). The corresponding result can be stated as follows.

Theorem 2.2. *Let K be given by* (1.5) *with $N = 1$ and $\rho_1 \equiv 1$, and let F be defined by* (2.5) *where $f \in L^2(\Omega)$ and $\sigma \in D'(\mathbb{R})$ satisfies $\mathrm{supp}\,(\sigma) \subset (0, T_1)$ for some $T_1 > 0$. Denote by u the unique weak solution in the sense of Definition 1.1 to* (1.1) *with $u_0 \equiv 0$, that is given by Theorem 2.1. Then, for any $T > T_1$ and any (non-empty) open subset ω of Ω, we have the implication:*

$$u_{|(0,T)\times\omega} \equiv 0 \Longrightarrow \sigma \equiv 0 \text{ or } f \equiv 0. \tag{2.6}$$

The inverse problem solved by Theorem 2.1 has important applications in geology, with the detection of hydraulic cracks in fractured rocks through anomalous diffusion, see e.g., [29], or in applied mechanics with the localization of patterns of moisture in viscoelastic polymers, see e.g., [30]. But, as far as we know, this problem was treated only for L^1 time-dependent admissible sources of time-fractional partial differential equations (this the regularity requested by [10, 11, 12, 19] but it is raised to C^1 in [34]), since time integrability of the function F was needed to write a representation formula of the solution to the system under examination. As will appear in the proof of Theorem 2.2, below, the critical tool enabling us to solve the inverse problem of determining a discrete-in-time source of a sub-diffusive equation, is the Duhamel formula given in Theorem 2.1. Moreover, we believe that it should allow us to tackle the same inverse source problem as in Theorem 2.2, where the data are replaced by boundary measurements, and to take advantage of the "memory effect" exhibited by the system (1.1) (see [9, 16]) to downsize the observation window to $(T - \varepsilon, T)$, where $\varepsilon > 0$ is arbitrary small.

2.4. **Outline.** The structure of the remaining part of this article is as follows. Section 3 is devoted to the proof of Theorem 2.1, while Section 4 contains the analysis of the inverse problem and the proof of Theorem 2.2.

3. Proof of Theorem 2.1

Prior to showing Theorem 2.1, we establish a technical result needed by the proof.

3.1. **Preliminaries.** The result is as follows.

Lemma 3.1. *Let K be as in Theorem 2.1. Then, for all $u \in \mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$ we have $I_K u \in \mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$ and it holds true that*

$$\widehat{I_K u}(p, \cdot) = \widehat{K}(p, \cdot)\widehat{u}(p, \cdot),\ p \in \mathbb{C}_+, \tag{3.1}$$

where $\widehat{u}$ still denotes the Laplace transform of u.

Proof. For all $\varphi \in D_{\mathrm{a}}(\mathbb{R}, L^2(\Omega))$, we have

$$\langle I_K u, \varphi \rangle_{\mathrm{a}, L^2(\Omega)} = \langle u, \varphi_K \rangle_{\mathcal{S}'_+(\mathbb{R}, L^2(\Omega)), \mathcal{S}_+(\mathbb{R}, L^2(\Omega))},$$

where $\varphi_K(t, \cdot) := \int_0^{+\infty} K(s, \cdot)\varphi(s + t, \cdot)ds$. Since $\operatorname{supp} u \subset [0, +\infty)$, it follows from this that

$$\left| \langle I_K u, \varphi \rangle_{\mathrm{a}, L^2(\Omega)} \right| \leqslant C \sup_{t \in [0, +\infty)} \|(1 + t)^{m_2} \partial_t^{m_1} \varphi_K(t)\|_{L^2(\Omega)}, \tag{3.2}$$

for some natural numbers m_1 and m_2. Here and in the remaining part of this proof, C denotes a generic positive constant which may change from line to line.

Furthermore, since K is defined by either (1.3), (1.4) or (1.5), we have

$$K \in L^1(0, 1; L^\infty(\Omega)) \cap L^\infty(1, \infty; L^\infty(\Omega)),$$

hence we obtain for all $t \in [0, +\infty)$ that

$$\begin{aligned} & \|(1 + t)^{m_2} \partial_t^{m_1} \varphi_K(t)\|_{L^2(\Omega)} \\ \leqslant & \int_0^{+\infty} \|K(s, \cdot)\|_{L^\infty(\Omega)} \|(1 + t + s)^{m_2} \partial_t^{m_1} \varphi(s + t, \cdot)\|_{L^2(\Omega)} ds \\ \leqslant & C \sup_{t \in [0, +\infty)} \left\|(1 + t)^{m_2 + 2} \partial_t^{m_1} \varphi(t)\right\|_{L^2(\Omega)}, \end{aligned}$$

where C can be taken equal to $\|K\|_{L^1(0,1;L^\infty(\Omega))} + \|K\|_{L^\infty(1,\infty;L^\infty(\Omega))} \int_1^\infty (1 + s)^{-2} ds$ From this and (3.2) it then follows that $\left| \langle I_K u, \varphi \rangle_{\mathrm{a}, L^2(\Omega)} \right| \leqslant C \sup_{t \in [0, +\infty)} \left\|(1 + t)^{m_2 + 2} \partial_t^{m_1} \varphi(t)\right\|_{L^2(\Omega)}$. Therefore, $I_K u$ can be extended by density to some vector in $\mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$.

Moreover, for all $p \in \mathbb{C}_+$ we have

$$\begin{aligned} \widehat{I_K u}(p, \cdot) &= \left\langle u(t, \cdot), \int_0^{+\infty} K(s, \cdot) e^{-p(t+s)} ds \right\rangle_{\mathcal{S}'_+(\mathbb{R}), \mathcal{S}_+(\mathbb{R})} \\ &= \left\langle u(t, \cdot), \left(\int_0^{+\infty} K(s, \cdot) e^{-ps} ds \right) e^{-pt} \right\rangle_{\mathcal{S}'_+(\mathbb{R}), \mathcal{S}_+(\mathbb{R})} \\ &= \left(\int_0^{+\infty} K(s, \cdot) e^{-ps} ds \right) \langle u(t, \cdot), e^{-pt} \rangle_{\mathcal{S}'_+(\mathbb{R}), \mathcal{S}_+(\mathbb{R})}, \end{aligned}$$

which yields (3.1). □

Armed with Lemma 3.1 we are now in position to prove Theorem 2.1.

3.2. **Completion of the proof.** With reference to [13, Theorems 1.3, 1.4 and 1.5], we may assume without loss of generality that $u_0 = 0$ in Ω. We shall examine the uniqueness and the existence problems separately. We start with the uniqueness issue.

Uniqueness. Let u be a weak solution, in the sense of Definition 1.1, to (1.1) associated with $F = 0$ in $\mathbb{R}_+ \times \Omega$ and $u_0 = 0$ in Ω. Then, we have $D_{t,K}u = \partial_t I_K u \in \mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$ by Lemma 3.1 and $\mathcal{A}u \in \mathcal{S}'_+(\mathbb{R}; D'(\Omega))$. Moreover, since $\widehat{D_{t,K}u}(p,\cdot) = p\widehat{I_K u}(p,\cdot) = p\widehat{K}(p,\cdot)\widehat{u}(p,\cdot)$ and $\widehat{\mathcal{A}u}(p,\cdot) = \mathcal{A}\widehat{u}(p,\cdot)$ for all $p \in \mathbb{C}_+$, we get upon applying the Laplace transform to both sides of (1.1), that

$$p\widehat{K}(p,x)\widehat{u}(p,x) + \mathcal{A}\widehat{u}(p,x),\ p \in \mathbb{C}_+,\ x \in \Omega.$$

From this and the condition $\widehat{u}(p,\cdot) \in H_0^1(\Omega)$ for all $p \in \mathbb{C}_+$, imposed by Definition 1.1(ii), it then follows that $\widehat{u}(p,\cdot) = 0$ in Ω, according to [18, Proposition 2.1] and [13, Lemma 4.1]. Thus, we have $u = 0$ in $\mathbb{R}_+ \times \Omega$ by injectivity of the Laplace transform. This proves that a weak solution to (1.1), if any, is unique.

Existence. Let us establish that the distribution $u \in D'_+(\mathbb{R}, L^2(\Omega))$, which is expressed by (2.3), is a weak solution to the IBVP (1.1). For this purpose we apply [7, Theorem 8.3.1] to $F \in \mathcal{S}'(\mathbb{R}, L^2(\Omega))$ and get $F_1 \in L^1_{\mathrm{loc}}(\mathbb{R}, L^2(\Omega))$ such that $F = \partial_t^{N_1} F_1$ for some $N_1 \in \mathbb{N}$ and $(1+t^2)^{-N_2}F_1 \in L^\infty(\mathbb{R}, L^2(\Omega))$ for some $N_2 \in \mathbb{N}$. Thus, applying (2.3) with $u_0 = 0$, we obtain that

$$u = S_{1,K} * F = S_{1,K} * (\partial_t^{N_1} F_1) = \partial_t^{N_1}(S_{1,K} * F_1), \tag{3.3}$$

where for all $t \in \mathbb{R}$, we have $S_{1,K} * F_1(t) = \int_0^t S_{1,K}(t-s)F_1(s)ds$ because $F_1 \in L^1_{\mathrm{loc}}(\mathbb{R}, L^2(\Omega))$. Next, since K is defined by either (1.3), (1.4) or (1.5), we know from [13, Lemmas 2.1, 3.1 and 4.2] that

$$\|S_{1,K}(t)\|_{\mathcal{B}(L^2(\Omega))} \leqslant C \max(1, t^{-r_1}),\ t \in \mathbb{R}_+,$$

where $r_1 \in (0,1)$ and C is a positive constant C. It follows readily from this and $(1+t^2)^{-N_2}F_1 \in L^\infty(\mathbb{R}, L^2(\Omega))$ that $S_{1,K} * F_1 \in \mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$. Therefore, we have $u \in \mathcal{S}'_+(\mathbb{R}, L^2(\Omega))$ directly from (3.3).

The next step is to infer from $(1+t^2)^{-N_2}F_1 \in L^\infty(\mathbb{R}, L^2(\Omega))$ that $(1+t)^{-2N_2-2}F_1 \in L^1(\mathbb{R}_+, L^2(\Omega))$, and then to apply [13, Propositions 2.2, 3.2 & 4.5]: We get that

$$\mathcal{L}(S_{1,K} * F_1)(p,\cdot) = \left(A + p\widehat{K}(p,\cdot)\right)^{-1} \widehat{F_1}(p,\cdot),\ p \in \mathbb{C}_+, \tag{3.4}$$

where $\mathcal{L}$ stands for the Laplace transformation with respect to t. Further, in view of (3.3) we have

$$\widehat{u}(p,\cdot)=\mathcal{L}\left(\partial_t^{N_1}S_{1,K}*F_1\right)(p,\cdot)=p^{N_1}\mathcal{L}\left(S_{1,K}*F_1\right)(p,\cdot),\ p\in\mathbb{C}_+,$$

so we deduce from (3.4) that

$$\begin{aligned}\widehat{u}(p,\cdot)=\left(A+p\widehat{K}(p,\cdot)\right)^{-1}p^{N_1}\widehat{F_1}(p,\cdot) &= \left(A+p\widehat{K}(p,\cdot)\right)^{-1}\mathcal{L}\left(\partial_t^{N_1}F_1\right)(p,\cdot)\\ &= \left(A+p\widehat{K}(p,\cdot)\right)^{-1}\widehat{F}(p,\cdot).\end{aligned} \tag{3.5}$$

Since $\widehat{F}(p,\cdot)\in L^2(\Omega)$ for all $p\in\mathbb{C}_+$, (3.5) yields that $\widehat{u}(p,\cdot)\in D(A)$. Thus, bearing in mind that $D(A)\subset H_0^1(\Omega)$, this shows that u fulfills Definition 1.1(ii). Finally, in light of (3.5) and Lemma 3.1, we see that $v:=D_t^K u+\mathcal{A}u\in\mathcal{S}'_+(\mathbb{R},D'(\Omega))$ satisfies

$$\widehat{v}(p,\cdot)=(\mathcal{A}+p\widehat{K}(p,\cdot))\widehat{u}(p,\cdot)=(A+p\widehat{K}(p,\cdot))\left(A+p\widehat{K}(p,\cdot)\right)^{-1}\widehat{F}(p,\cdot)=\widehat{F}(p,\cdot),\ p\in\mathbb{C}_+.$$

This leads to $D_{t,K}u+\mathcal{A}u=F$ in $\mathcal{S}'_+(\mathbb{R},D'(\Omega))$ by injectivity of the Laplace transform $\mathcal{L}$, which establishes that u fulfills Definition 1.1(i) as well. As a consequence u is a weak solution to (1.1).

4. Proof of Theorem 2.2

One of the two main ingredients of the proof of Theorem 2.2 is the analyticity of the solution to (1.1) with respect to the time variable t. We start by establishing this technical property.

4.1. **Time-analyticity of the solution.** We aim to establish the following byproduct of Theorem 2.1.

Lemma 4.1. *Under the conditions of Theorem 2.2, the $L^2(\Omega)$-valued weak solution u to* (1.1) *with initial state $u_0\equiv 0$, is real-analytic with respect to $t\in(T_1,\infty)$.*

Proof. Pick $\theta_1\in(0,\min(\pi-\theta,\theta-\pi/2))$, $\theta\in(\pi/2,\pi)$ being the same as in Section 2.2, in such a way that $\Re(zp)<0$ whenever $z\in\mathcal{C}_{\theta_1}=\{\tau e^{i\psi}:\ \tau\in(0,\infty),\ \psi\in(-\theta_1,\theta_1)\}$ and $p\in\gamma_\pm(\delta,\theta)$, where $\gamma_\pm(\delta,\theta)$ is defined in (2.1). Then, with reference to (2.2), we get by arguing as in the proof of [14, Theorem 3.3.] or [19, Proposition 2.1], that the map $t\mapsto S_{1,K}(t)$ extends to a $\mathcal{B}(L^2(\Omega))$-valued holomorphic function $z\mapsto S_{1,K}(z)$ in $\mathcal{C}_{\theta_1}$. Therefore, for all $s\in(0,T_1)$, $z\mapsto S_{1,K}(z-s)$ is holomorphic in $\mathcal{O}_{\theta_1}=\{T_1+\tau e^{i\psi}:\ \tau\in(0,\infty),\ \psi\in(-\theta_1,\theta_1)\}$.

Next, bearing in mind that $\operatorname{supp}(\sigma) \subset (0, T_1)$, we introduce for all $z \in \mathcal{O}_{\theta_1}$, the linear bounded operator $U_\sigma(z)$ in $L^2(\Omega)$, as

$$U_\sigma(z)h = \langle \sigma, S_{1,K}(z - \cdot)h \rangle_{\mathcal{E}'(0,T_1), C^\infty(0,T_1)}, \ h \in L^2(\Omega), \tag{4.1}$$

where $\mathcal{E}'(0, T_1)$ denotes the space of compactly supported distributions in $D'(\mathbb{R})$, with support in $(0, T_1)$. Evidently, U_σ lies in $\mathscr{H}(\mathcal{O}_{\theta_1}, \mathcal{B}(L^2(\Omega)))$, the space of $\mathcal{B}(L^2(\Omega))$-valued holomorphic functions in $\mathcal{O}_{\theta_1}$, and the expected result follows from this since the weak solution u to (1.1) with $u_0 \equiv 0$, satisfies

$$u(t) = U_\sigma(t)f, \ t > T_1,$$

according to (2.3). □

Armed with this lemma, we may now complete the proof of Theorem 2.2.

4.2. **Completion of the proof.** First, we deduce from Lemma 4.1 and the assumptions $u_{|(0,T)\times\omega} \equiv 0$ and $T > T_1$, that

$$u(t, x) = 0, \ t \in \mathbb{R}_+, \ x \in \omega. \tag{4.2}$$

In order to show that either σ or f are uniformly zero, it is enough to prove that $f \equiv 0$ whenever $\sigma \not\equiv 0$. To this purpose, we suppose that $\sigma \not\equiv 0$. Next, with reference to Definition 1.1 and the proof of Theorem 2.1, we notice that the Laplace transform $\hat{u}(p, \cdot)$ of the solution u to (1.1), computed at $p \in \mathbb{R}_+$, solves

$$\left\{ \begin{array}{rcll} (p^\alpha + \mathcal{A})\hat{u}(p, x) & = & \hat{\sigma}(p)f(x), & x \in \Omega \\ \hat{u}(p, x) & = & 0, & x \in \mathbb{R}_+ \times \partial\Omega, \end{array} \right. \tag{4.3}$$

where $\hat{\sigma}$ is the Laplace transform of σ. Similarly, it follows readily from (4.2) that for all $p \in \mathbb{R}_+$,

$$\hat{u}(p, x) = 0, \ x \in \omega. \tag{4.4}$$

Since $p \mapsto \hat{\sigma}(p)$ is holomorphic in $\mathbb{C}$, by Schwartz's Paley-Wiener theorem, and since $\sigma \not\equiv 0$ by assumption, there exist two real numbers $0 < r_1 < r_2$ such that $\hat{\sigma}(p) \neq 0$ for all $p \in [r_1, r_2]$. Thus, putting

$$v(\tau, x) = \frac{\hat{u}(\tau^{\frac{1}{\alpha}}, x)}{\hat{\sigma}(\tau^{\frac{1}{\alpha}})}, \ \tau \in [r_1^\alpha, r_2^\alpha],$$

we infer from (4.3)-(4.4) ,

$$\begin{cases} (\tau+\mathcal{A})v(\tau,x) &=& f(x), & (\tau,x)\in[r_1^\alpha,r_2^\alpha]\times\Omega \\ v(\tau,x) &=& 0, & (\tau,x)\in[r_1^\alpha,r_2^\alpha]\times\partial\Omega \\ v(\tau,x) &=& 0, & (\tau,x)\in[r_1^\alpha,r_2^\alpha]\times\omega. \end{cases} \tag{4.5}$$

Now, let us introduce the $C([0,+\infty),H_0^1(\Omega))\cap C^1([0,+\infty),L^2(\Omega))$-solution w to

$$\begin{cases} (\partial_t+\mathcal{A})w(t,x) &=& 0, & (t,x)\in(0,\infty)\times\Omega, \\ w(t,x) &=& 0, & (t,x)\in(0,\infty)\times\partial\Omega, \\ w(0,x) &=& f(x), & x\in\Omega. \end{cases} \tag{4.6}$$

Then, $\hat{w}$, the Laplace transform (with respect to t) of w, is well-defined on $\mathbb{R}_+$, and we have

$$\hat{w}(\tau,\cdot)=v(\tau,\cdot),\ \tau\in[r_1^\alpha,r_2^\alpha],$$

by uniqueness of the solution to the two first lines of (4.5). Therefore, we have

$$w(t,x)=0,\ (t,x)\in(0,\infty)\times\omega,$$

from the third line of (4.5) and the injectivity of the Laplace transform. From this and the first line of (4.6), we get upon applying the unique continuation principle for parabolic equations, that $w\equiv 0$. As a consequence, we have $f\equiv 0$, and the proof is complete. □

Acknowledgments

This work was supported by the Agence Nationale de la Recherche (ANR) under grant ANR-17-CE40-0029 (projet MultiOnde).

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