LOGARITHMIC STABLE RECOVERY OF THE SOURCE AND THE INITIAL STATE OF TIME FRACTIONAL DIFFUSION EQUATIONS

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ABSTRACT. In this paper we study the inverse problem of identifying a source or an initial state in a time-fractional diffusion equation from the knowledge of a single boundary measurement. We derive logarithmic stability estimates for both inverse problems. These results show that the ill-posedness increases exponentially when the fractional derivative order tends to zero, while it exponentially decreases when the regularity of the source or the initial state becomes larger.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Settings.** Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be a bounded domain containing the origin, with C^2 boundary $\partial\Omega$. With reference to [34], the Riemann-Liouville integral operator of order β, denoted by I^β, is defined by

$$I^\beta h(t, \cdot) := \frac{1}{\Gamma(\beta)} \int_0^t \frac{h(\tau, \cdot)}{(t-\tau)^{1-\beta}} \, d\tau,$$

and the Riemann-Liouville fractional derivative of order β is $D_t^\beta := \partial_t \circ I^{1-\beta}$. Set

$$\partial_t^\beta h := D_t^\beta (h - h(0, \cdot)), \ \ h \in C([0, +\infty); L^2(\Omega)).$$

The operator ∂_t^β is called the Caputo fractional derivative of order β, as we have

$$\partial_t^\beta h = I^{1-\beta} \partial_t h, \ \ h \in W^{1,1}_{loc}(\mathbb{R}_+; L^2(\Omega)),$$

where $\mathbb{R}_+ := (0, +\infty)$.

For $\alpha \in (0, 1)$ fixed, we consider the following initial boundary value problem (IBVP)

$$\begin{cases} \partial_t^\alpha u(t,x) - \Delta u(t,x) = g(t) f(x), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$, $g \in L^\infty(\mathbb{R}_+)$ and $u_0 \in H_0^1(\Omega)$. In this article, assuming that the function g is known and satisfies an appropriate condition that we will make precise further, we aim to study the stability issue in the inverse problem of determining either the source term f or the initial state u_0, from a single boundary measurement $\partial_\nu u = \nabla u \cdot \nu$ on $\mathbb{R}_+ \times \partial\Omega$, of the solution u to (1). Here and in the remaining part of this text, we denote by ν the outward unit normal vector to $\partial\Omega$.

The time-fractional diffusion system (1) describes anomalous diffusion in homogeneous media. It has multiple engineering applications in geophysics, environmental science and biology, see e.g., [1, 10, 22]. From a mathematical viewpoint, the time-fractional diffusion equation of (1) can be seen as a corresponding macroscopic model to microscopic diffusion processes governed by a continuous-time random walk, see e.g., [8, 32].

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1.2. **A brief review of the existing literature.** Inverse source problems have received a lot of attention from the mathematical community over the last decade, owing it to the major impact they made in many areas, including medical diagnosis and industrial nondestructive testing. We refer the reader to [18, 6] for an overview of inverse source problems for partial differential equations, and to [22, 31] for the study of these problems in the framework of fractional diffusion equations.

While several authors have already addressed the inverse problem of retrieving the space-varying part of the source term in a fractional diffusion equation, see e.g., [14, 37], only uniqueness results are available in the mathematical literature, see e.g., [19, 22, 25, 28, 36] (see also [23] for some related inverse problem) with the exception of the recent stability result of [11] stated with specific norms. This is not surprising since the classical methods used to build a stability estimate for the source of parabolic ($\alpha = 1$) or hyperbolic ($\alpha = 2$) systems, see e.g., [12, 17, 27, 3], do not apply in a straightforward way to time-fractional diffusion equations. One of the reasons for this is that Carleman estimates, which play an important role in the analysis of the stability issue of parabolic or hyperbolic inverse problems, and whose derivation heavily relies on partial time integration, generally cannot be adapted to time-fractional equations. Moreover, we point out that the Carleman estimate designed in [16, 41] specifically for (1) with $\alpha = \frac{1}{2}$ (upon reduction to a multidimensional fourth-order differential system), boils down to internal measurement of the solution and is not suitable for solving inverse source problems with lateral data.

As far as we know, the only mathematical work dealing with the stability issue of the inverse source problem under consideration in the present article, can be found in [11, 20, 29]. While the authors of [20, 29] studied this problem in the peculiar framework of a cylindrical domain $\Omega' \times (-\ell, \ell)$, where Ω' is an open subset of $\mathbb{R}^{d-1}$ and $\ell \in \mathbb{R}_+$, the Lipschitz stability estimate presented in [11] is derived for $\alpha \in (1, 2)$ under a specifically designed topology induced by the adjoint system of the fractional wave equation.

However, we could not find such thing as a stability inequality with respect to the usual norm in $L^2(\Omega)$, of the space-dependent part of the source term of time-fractional diffusion equations posed in a general bounded domain, in the mathematical literature. Besides, the main achievement of this article is the logarithmic-stable determination through one Neumann data, of either the space-varying part of the source term or the initial state of the IBVP (1). The corresponding stability estimates are given in Theorems 1.1 and 1.2 below, but prior to stating these two inverse results, we turn our attention to the direct problem associated with (1).

1.3. **The direct problem.** With reference to [13, 25, 30], we define a weak solution to the IBVP as a function $u \in L^1_{loc}(\mathbb{R}_+; L^2(\Omega))$ satisfying the three following conditions simultaneously:

i) $D_t^\alpha[u - u_0](t, x) - \Delta u(t, x) = f(x)g(t)$ in the distributional sense in $\mathbb{R}_+ \times \Omega$;
ii) $I^{1-\alpha}u \in W^{1,1}_{loc}(\mathbb{R}_+; H^{-2}(\Omega))$ and $I^{1-\alpha}[u - u_0](0, x) = 0$ for a.e. $x \in \Omega$;
iii) $p_0 := \inf\{\tau > 0 : \ e^{-\tau t}u \in L^1(\mathbb{R}_+; L^2(\Omega))\} < \infty$ and

$$\exists p_1 \geq p_0, \ \forall p \in \mathbb{C}, \ \Re p > p_1 \Longrightarrow \widetilde{u}(p, \cdot) := \int_0^{+\infty} e^{-pt}u(t, \cdot) \ dt \in H^1_0(\Omega).$$

Here and the remaining part of this article, $\Re z$ (resp., $\Im z$) denotes the real part (resp., the imaginary part) of the complex number z. Notice from iii) that $\widetilde{u}$ is the Laplace transform in time of u with respect to the time-variable t.

As long as finite time evolution is concerned, we can rely on the results of [25, 30, 37]. Indeed, they ensure us for all $\alpha \in (0, 1)$, all $T > 0$ and all $(f, g, u_0) \in L^2(\Omega) \times L^\infty(\mathbb{R}_+) \times \left(H^1_0(\Omega) \cap H^2(\Omega)\right)$ that the IBVP (1) admits a unique weak solution $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^2(\Omega) \cap H^1_0(\Omega))$ satisfying

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^\alpha u\|_{L^2((0,T)\times\Omega)} \leq C \left(\|g\|_{L^\infty(0,T)}\|f\|_{L^2(\Omega)} + \|u_0\|_{H^k(\Omega)}\right),$$

for some constant $C > 0$ depending only on α, Ω and T. However, since C may blow up as T tends to infinity, it is not clear how the global in time properties of the solution on $\mathbb{R}_+$, needed by the analysis of the inverse problem under study in this article, can be inferred from the above estimate. For this reason we proceed by establishing the following global time existence and uniqueness result.

Proposition 1.1. *Let $\alpha \in (0,1)$ and let $(f, g, u_0) \in L^2(\Omega) \times \left(L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)\right) \times \left(H_0^1(\Omega) \cap H^2(\Omega)\right)$. Then, there exists a unique weak solution $u \in L^r(\mathbb{R}_+; H^{\frac{7}{4}}(\Omega))$, $r > \alpha^{-1}$, to the IBVP (1). Moreover, we have*

$$\|u\|_{L^r(\mathbb{R}_+;H^{\frac{7}{4}}(\Omega))} \leq C \left(\|g\|_{L^r(\mathbb{R}_+)} \|f\|_{L^2(\Omega)} + \|u_0\|_{H^2(\Omega)} \right), \tag{2}$$

for some positive constant C depending only on α and Ω.

Notice that similar results were obtained in [5, 7] upon applying the operator-valued Fourier multiplier theorem of [40] to fractional diffusion equations with a Riemann-Liouville time fractional derivative. In light of [21, Theorem 3], this method applies to the system (1) with initial state u_0 which is identically zero. In this peculiar framework, the solution u to (1) lies in $L^r(\mathbb{R}_+; H^2(\Omega))$ for all $r > 1$, and satisfies the energy estimate

$$\|u\|_{L^r(\mathbb{R}_+;H^2(\Omega))} \leq C \|g\|_{L^r(\mathbb{R}_+)} \|f\|_{L^2(\Omega)}.$$

Nevertheless, as it remains to be seen whether the approach of [5, 7] can be adapted to a non-trivial u_0, it is not clear how Proposition 1.1 could be derived from the analysis developed in [5, 7]. Therefore we provide an extensive proof of this result, which can be found in Appendix A.

Having expressed Proposition 1.1, we turn now to stating the two main results of this article. Each of them is concerned with one of the inverse problems described in the introduction.

1.4. **Inverse problems.** In this section we present two stability estimates. The first one is associated with the determination of the source term f of the IBVP (1), by one boundary measurement of the solution over the entire time-span $\mathbb{R}_+$. The second one is related to the same inverse problem where the unknown is replaced by the initial state u_0. Prior to stating these two inequalities, we introduce the following notations. Firstly, for all $\tau \in \mathbb{R}$, we note $\lfloor \tau \rfloor$ the integer part of τ. Secondly, for all $k \in \mathbb{N} := \{1, 2, \ldots\}$, we denote by $H_0^k(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the topology of the k-th order Sobolev space $H^k(\Omega)$.

We start with the determination of the source term f.

Theorem 1.1. *Let $\alpha \in (0,1)$, let $f \in H_0^k(\Omega)$ for some $k \in \mathbb{N}$ and let $g \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ satisfy the condition*

$$\exists c_0 > 0, \ \forall p \in \mathbb{R}_+, \ |\widetilde{g}(p)| \geq c_0. \tag{3}$$

Assume that $u_0 = 0$ and denote by u the $L^{\frac{2}{\alpha}}(\mathbb{R}_+; H^{\frac{7}{4}}(\Omega))$-solution to (1), and let $s = 1 + \lfloor \frac{2}{\alpha} \rfloor$.
Then, for all $\theta \in (0,1)$, there exists $\varepsilon_0 = \varepsilon_0 \left(\Omega, d, k, \theta, \frac{\|f\|_{H^k(\Omega)}}{\|f\|_{L^1(\Omega)}} \right) \in (0,1)$ such that we have

$$\|f\|_{L^2(\Omega)} \leq C \|f\|_{H^k(\Omega)} \begin{cases} \left| \ln \|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))} \right|^{-\frac{k}{1+\theta}} & \text{if } s \leq d + \frac{d^2}{2k} \\ \left| \ln \|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))} \right|^{-\frac{d^2}{2(s-d)(1+\theta)}} & \text{if } s > d + \frac{d^2}{2k}, \end{cases} \tag{4}$$

whenever $\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))} \in (0, \varepsilon_0)$. Here, C is a positive constant depending only on Ω, d, k, θ and c_0.

It is worth noticing that the condition (3) (which is especially true when g is a non-trivial function that does not change sign) is purely technical in the sense that it is needed by the proof of Theorem 1.1, displayed in Section 2 below, to establish the key estimate (10) with the aid of (9). This being said, we may now state the following corresponding result to Theorem 1.1 but for the determination of the initial state u_0 instead of the space-moving part of the source term f.

Theorem 1.2. *Let $\alpha \in (0,1)$ and let $u_0 \in H_0^k(\Omega)$, where $k \in \mathbb{N}$. Assume that $f = 0$, denote by u the $L^{\frac{2}{\alpha}}(\mathbb{R}_+; H^{\frac{7}{4}}(\Omega))$-solution to the IBVP (1), and let $s = 1 + \lfloor \frac{2}{\alpha} \rfloor$.*
Then, for all $\theta \in (0,1)$, there exists $\varepsilon_0 = \varepsilon_0 \left(\Omega, d, k, \theta, \frac{\|u_0\|_{H^k(\Omega)}}{\|u_0\|_{L^1(\Omega)}} \right) \in (0,1)$ such that we have

$$\|u_0\|_{L^2(\Omega)} \leq C \|u_0\|_{H^k(\Omega)} \begin{cases} \left| \ln \|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))} \right|^{-\frac{k}{1+\theta}} & \text{if } s \leq d + \frac{d^2}{2k} \\ \left| \ln \|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))} \right|^{-\frac{d^2}{2(s-d)(1+\theta)}} & \text{if } s > d + \frac{d^2}{2k}, \end{cases} \tag{5}$$

provided $\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))} \in (0,\varepsilon_0)$*. Here,* C *is a positive constant depending only on* Ω*,* d*,* k *and* θ*.*

The result of Theorem 1.2 may also be seen as a weak observability inequality that could be helpful in control theory, see e.g. [42, 4, 3], for proving approximate controllability of the time-fractional diffusion equation appearing in (1). This open problem is of great importance in photoacoustic imaging in the sub-diffusion regime [2, 38, 35].

To our knowledge, Theorem 1.1 (resp., Theorem 1.2) is the only existing mathematical result on the stable recovery of the space-varying part of the source term (resp., the initial state) of a fractional diffusion equation posed in a general spatial domain. As already mentioned in Section 1.2, there are two comparable results available in [20, 29] but they only apply to cylindrical shaped domains.

Notice that the stability of the recovery of either f or u_0, expressed in (4) and (5), respectively, degenerates exponentially fast as the fractional order α tends to zero. In contrast, this stability exponentially improves as their Sobolev regularity order k increases. Moreover, we point out that it is for simplicity's sake that $\|\cdot\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+,L^2(\partial\Omega))}$ was used in (4) and (5), as it can be seen from Proposition 1.1 below that any norm $\|\cdot\|_{L^r(\mathbb{R}_+,L^2(\partial\Omega))}$ with $r > \alpha^{-1}$, is suitable for such inequalities.

Remark 1.1. *It is not clear whether the approach developed in this article applies to fractional wave equations (that is to say to the system* (1) *with* $\alpha \in (1,2)$*). Indeed, most of the intermediate technical results used by the analysis carried out in this article, such as the unique continuation presented in Section 2.4, are specifically written for the sub-diffusive case* $\alpha < 1$*, and it remains to be seen whether they can be adapted to the super-diffusive case* $\alpha \in (1,2)$*.*

The data used in Theorem 1.1 and 1.2 involve measuring the Neumann trace of the solution u *to* (1) *over the infinite time interval* $\mathbb{R}_+$*. This is because the analysis carried out in the present article is based on the use of the Laplace transform with respect to the time variable. The question to know whether these data could be replaced by a finite time observation of lateral measurements of* $\partial_\nu u$ *still remains unanswered, but several lines of research can be pursued. One of them is to estimate* $\partial_\nu u|_{\mathbb{R}_+\times\partial\Omega}$ *in terms of* $\partial_\nu u|_{(0,T)\times\partial\Omega}$ *for some* $T > 0$*. Another one is to try to adapt the method developed in this manuscript to the case of Laplace transform of periodised functions.*

1.5. **Outline.** This article is organized as follows. The proof of the two stability estimates (4)-(5) is presented in Section 2. Their derivation boils down to a unique continuation result, that is established in Section 2.4. The analysis of the direct problem associated with the IBVP (1) can be found in Appendix A, which contains the proof of Proposition 1.1.

2. Proof of Theorems 1.1 and 1.2

In this section we derive the two stability inequalities (4) and (5). Our strategy is to study the IBVP (1) in the so-called frequency domain, that is to say that we examine the elliptic system derived from (1) upon applying the Laplace transform with respect to the time-variable t. This is made precise in the coming section, which is a preamble to the proof of Theorems 1.1 and 1.2.

2.1. **Preliminaries.** Let u be the weak solution to (1) given by Propisition 1.1. In light of iii) in Section 1.3, the Laplace transform in time $\widetilde{u}(p,\cdot)$ of u is well-defined provided the real part of $p \in \mathbb{C}$ is sufficiently large. As a matter of fact, it can be checked from [24, Theorem 1.3] or [25, Theorem 4.1] that $\widetilde{u}(p,\cdot)$ is well-defined for all $p \in \mathbb{C}_+ := \{z \in \mathbb{C},\ \mathfrak{R}(z) > 0\}$ and solves the following boundary value problem (BVP)

$$\left\{ \begin{array}{ll} (-\Delta + p^\alpha)\widetilde{u}(p,\cdot) = \widetilde{g}(p) f + p^{\alpha-1} u_0 & \text{in } \Omega, \\ \widetilde{u}(p,\cdot) = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{6}$$

Notice that since $g \in L^\infty(\mathbb{R}_+)$, $\widetilde{g}(p,\cdot)$ is well-defined for $p \in \mathbb{C}_+$ as well.

Next, for all $z \in \mathbb{C}^* \setminus i\mathbb{R}_+ := \{\tau \in \mathbb{C},\ -i\tau \notin [0,+\infty)\}$ and for all $q \in \mathbb{R}$, we set $z^q := e^{q \log z}$, where log denotes the complex logarithm function defined and holomorphic on $\mathbb{C}^* \setminus i\mathbb{R}_+$. Thus, for all $\omega \in \mathbb{R}_+$,

$U(\omega,\cdot):=\widetilde{u}(\omega^{\frac{2}{\alpha}},\cdot)$ is a solution to

$$\left\{ \begin{array}{ll} (-\Delta+\omega^2)U=\widetilde{g}(\omega^{\frac{2}{\alpha}})f+\omega^{2-\frac{2}{\alpha}}u_0 & \text{in } \Omega, \\ U=0 & \text{on } \partial\Omega. \end{array} \right. \tag{7}$$

Let us denote by $(\lambda_k)_{k\geq 1}\in\mathbb{R}_+^{\mathbb{N}}$ the eigenvalues of the self-adjoint operator A in $L^2(\Omega)$, acting as $-\Delta$ on its domain $H_0^1(\Omega)\cap H^2(\Omega)$, which is positive and has a compact resolvent. Since $F:=\widetilde{g}(\omega^{\frac{2}{\alpha}})f+\omega^{2-\frac{2}{\alpha}}u_0\in L^2(\Omega)$, the BVP (7) admits a unique solution $U=(A+\omega^2)^{-1}F$ within the space $H_0^1(\Omega)\cap H^2(\Omega)$, provided we have $\omega\in\mathbb{C}\setminus\{\pm i\lambda_k,\ k\geq 1\}$. In particular, this entails that $U(\omega,\cdot)\in H^2(\Omega)$ for all $\omega\in\mathbb{C}_+$.

Further, recalling that the Fourier transform $\widehat{\phi}$ of a function $\phi\in L^1(\Omega)$ reads

$$\widehat{\phi}(\xi)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int_\Omega e^{-ix\cdot\xi}\phi(x)dx,\ \xi\in\mathbb{R}^d, \tag{8}$$

we pick $\xi\in\mathbb{S}^{d-1}$, multiplying the first equation of (7) by $e^{\omega x\cdot\xi}$, where $\omega\in\mathbb{R}_+$ is fixed, and we integrate by parts over Ω. We obtain that

$$\widetilde{g}(\omega^{\frac{2}{\alpha}})\widehat{f}(i\omega\xi)+\omega^{2-\frac{2}{\alpha}}\widehat{u_0}(i\omega\xi)=(2\pi)^{-\frac{d}{2}}\int_{\partial\Omega}\partial_\nu U(\omega,x)e^{\omega x\cdot\xi}d\sigma(x),\ \xi\in\mathbb{S}^{d-1},\ \omega\in\mathbb{R}_+. \tag{9}$$

The above identity is a stepping stone to the proof of Theorems 1.1 and 1.2.

2.2. **Proof of Theorem 1.1.** Since $u_0=0$ by assumption and $|\widetilde{g}(p)|\geq c_0>0$ for all $p\in\mathbb{R}_+$, according to (3), it follows from (9) that

$$\left|\widehat{f}(i\omega\xi)\right|\leq c_0^{-1}(2\pi)^{-\frac{d}{2}}\left|\partial\Omega\right|^{\frac{1}{2}}e^{\kappa_\Omega\omega}\|\partial_\nu U(\omega,\cdot)\|_{L^2(\partial\Omega)},\ \xi\in\mathbb{S}^{d-1},\ \omega\in\mathbb{R}_+, \tag{10}$$

where $\kappa_\Omega:=\sup_{x\in\partial\Omega}|x|$.
Next, we have $u\in L^{\frac{2}{\alpha}}(\mathbb{R}_+;H^{\frac{7}{4}}(\Omega))$ from Proposition 1.1, and hence $\partial_\nu u\in L^{\frac{2}{\alpha}}(\mathbb{R}_+;H^{\frac{1}{4}}(\partial\Omega))\subset L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))$, and

$$\partial_\nu U(\omega,x)=\widetilde{\partial_\nu u}(\omega^{\frac{2}{\alpha}},x),\ x\in\partial\Omega,\ \omega\in\mathbb{R}_+,$$

from [26, Step 2 in the proof of Theorem 2.2]. Thus, applying Hölder's inequality, we get that

$$\begin{aligned} \|\partial_\nu U(\omega,\cdot)\|_{L^2(\partial\Omega)} &\leq \int_0^{+\infty}e^{-\omega^{\frac{2}{\alpha}}t}\|\partial_\nu u(t,\cdot)\|_{L^2(\partial\Omega)}dt \\ &\leq \left(\int_0^{+\infty}e^{-\frac{2}{2-\alpha}\omega^{\frac{2}{\alpha}}t}dt\right)^{\frac{2-\alpha}{2}}\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))} \\ &\leq \omega^{\frac{\alpha-2}{\alpha}}\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))}. \end{aligned} \tag{11}$$

Now, putting (10) together with (11), we find for all $\xi\in\mathbb{S}^{d-1}$ that

$$\left|\widehat{f}(i\omega\xi)\right|\leq c_0^{-1}(2\pi)^{-\frac{d}{2}}\left|\partial\Omega\right|^{\frac{1}{2}}\omega^{\frac{\alpha-2}{\alpha}}e^{\kappa_\Omega\omega}\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))},\ \omega\in\mathbb{R}_+.$$

Since $\omega^{\frac{\alpha-2}{\alpha}}\leq\left(\frac{2+\omega}{\omega}\right)^s$ for all $\omega\in\mathbb{R}_+$, where $s:=1+\lfloor\frac{2}{\alpha}\rfloor$, we deduce from the above line that

$$\left(\frac{\omega}{2+\omega}\right)^s e^{-\kappa_\Omega\omega}\left|\widehat{f}(i\omega\xi)\right|\leq c_0^{-1}(2\pi)^{-\frac{d}{2}}\left|\partial\Omega\right|^{\frac{1}{2}}\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))},\ \omega\in[0,+\infty). \tag{12}$$

Further, the right-hand-side on (12) being independent of ξ, we substitute $-\xi$ for ξ in (12) and obtain that

$$\left(\frac{-\omega}{2-\omega}\right)^s e^{\kappa_\Omega\omega}\left|\widehat{f}(i\omega\xi)\right|\leq c_0^{-1}(2\pi)^{-\frac{d}{2}}\left|\partial\Omega\right|^{\frac{1}{2}}\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))},\ \omega\in(-\infty,0].$$

Then we use the fact that $e^{\kappa_\Omega\omega}\geq e^{-2\kappa_\Omega}e^{-\kappa_\Omega\omega}$ and $(2-\omega)^{-1}\geq 3^{-1}(2+\omega)^{-1}$ whenever $\omega\in[-1,0)$, to get that $\left(\frac{-\omega}{2+\omega}\right)^s e^{-\kappa_\Omega\omega}\left|\widehat{f}(i\omega\xi)\right|\leq 3^s c_0^{-1}(2\pi)^{-\frac{d}{2}}e^{2\kappa_\Omega}\left|\partial\Omega\right|^{\frac{1}{2}}\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))}$ for all $\omega\in[-1,0)$. From this and

(12) it then follows for all $\xi \in \mathbb{S}^{d-1}$ that

$$\left| \left(\frac{\omega}{2+\omega} \right)^s e^{-\kappa_\Omega \omega} \widehat{f}(i\omega\xi) \right| \leq \epsilon,\ \omega \in [-1, +\infty), \tag{13}$$

where

$$\epsilon := 3^s c_0^{-1} (2\pi)^{-\frac{d}{2}} e^{2\kappa_\Omega} \left| \partial\Omega \right|^{\frac{1}{2}} \left\| \partial_\nu u \right\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+; L^2(\partial\Omega))}. \tag{14}$$

The remaining part of our analysis boils down to a quantitative unique continuation result which is expressed in Lemma 2.1 below. For this purpose, we consider $\phi \in L^1(\Omega)$, put $\mathcal{H} := \{z \in \mathbb{C},\ \Re z > -1\}$ and with reference to (8), we set

$$F(z) := \left(\frac{z}{2+z} \right)^s e^{-\kappa_\Omega z} \widehat{\phi}(iz\xi),\ z \in \overline{\mathcal{H}} = \{z \in \mathbb{C},\ \Re z \geq -1\}, \tag{15}$$

where we recall that $\widehat{\phi}$ is the Fourier transform of ϕ and $\kappa_\Omega = \sup_{x\in\Omega} |x|$ is the Euclidean diameter of Ω. Evidently, the function F is holomorphic in the half-plane $\mathcal{H}$ and continuous on $\partial\mathcal{H} = \{z \in \mathbb{C},\ \Re z = -1\}$. Moreover, we have

$$|F(z)| \leq (2\pi)^{-d/2} e^{2\kappa_\Omega} \|\phi\|_{L^1(\Omega)},\ z \in \overline{\mathcal{H}}, \tag{16}$$

directly from (8). With reference to (16) we set for further use

$$M := 1 + (2\pi)^{-d/2} e^{2\kappa_\Omega} \|\phi\|_{L^1(\Omega)} \in [1, +\infty). \tag{17}$$

Next, as we aim to compute a suitable upper bound of F in the quadrant $Q := \{z \in \mathbb{C},\ \Re z > -1 \text{ and } \Im z < 0\}$, we introduce

$$w(z) := \frac{2}{\pi} \left(\frac{\pi}{2} + \arg(z+1) \right),\ z \in \overline{Q} \setminus \{-1\} = \{z \in \mathbb{C},\ \Re z \geq -1 \text{ and } \Im z \leq 0\} \setminus \{-1\}. \tag{18}$$

Here and below, $\arg z$, for all $z \in \mathbb{C}$ satisfying $\Re z > 0$, denotes the angle $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $z = |z|\, e^{i\theta}$.

The unique continuation result that we have in mind can now be stated as follows.

Lemma 2.1. *Let F be defined by (15), let M be given by (17) and let w be the same as in (18). Then, we have*

$$|F(z)| \leq M m^{w(z)},\ z \in \overline{Q} \setminus \{-1\}, \tag{19}$$

where $m := \sup_{t\in[-1,+\infty)} |F(t)|$.

The proof of Lemma 2.1 being independent of the one of Theorem 1.1, we postpone it to Section 2.4.

Now, putting $F(z) := \left(\frac{z}{2+z}\right)^s e^{-\kappa_\Omega z} \widehat{f}(iz\xi)$ for all $z \in \mathbb{C}$ such that $\Re z \geq -1$, we get upon substituting f for ϕ in Lemma 2.1, that

$$|F(z)| \leq M m^{w(z)},\ z \in \overline{Q} \setminus \{-1\}, \tag{20}$$

where $M := 1 + (2\pi)^{\frac{d}{2}} e^{2\kappa_\Omega} \|f\|_{L^1(\Omega)}$. Further, since $m \leq \epsilon$ from (13)-(14), and since $w(z) \geq 0$ for all $z \in \overline{Q} \setminus \{-1\}$, (20) yields that

$$\left| \left(\frac{z}{2+z} \right)^s e^{-\kappa_\Omega z} \widehat{f}(iz\xi) \right| \leq M \epsilon^{w(z)},\ z \in \overline{Q} \setminus \{-1\}.$$

In the particular case where $z = -it$, $t \in \mathbb{R}_+$, this leads to $\frac{t^s}{(4+t^2)^{\frac{s}{2}}} \left| \widehat{f}(t\xi) \right| \leq M \epsilon^{w(-it)}$ and $w(-it) = \frac{2}{\pi}\left(\frac{\pi}{2} + \arg(1-it)\right) = \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan t\right) = \frac{2}{\pi} \arctan t^{-1}$. As a consequence we have

$$\left| \widehat{f}(t\xi) \right| \leq M \frac{(4+t^2)^{\frac{s}{2}}}{t^s} \epsilon^{\frac{2}{\pi} \arctan t^{-1}},\ t \in \mathbb{R}_+,\ \xi \in \mathbb{S}^{d-1}. \tag{21}$$

Therefore, for all $\delta \in (0,1)$ and all $R \in (1, +\infty)$, we get

$$\|\widehat{f}\|_{L^2(B_{R,\delta}(0))} \leq h_s(\delta, R), \tag{22}$$

through standard computations, where

$$h_s(\delta, R) := 5^{\frac{s}{2}} M C_d \epsilon^{\frac{2}{\pi}\arctan R^{-1}} k_s(\delta, R) \text{ and } k_s(\delta, R) := \begin{cases} R^d & \text{if } s < d, \\ -\ln\delta + R^d & \text{if } s = d, \\ \delta^{-(s-d)} + R^d & \text{if } s > d, \end{cases} \tag{23}$$

and C_d is a positive constant depending only on d. Here and in the remaining part of this article, $B_\rho(0)$ (resp., $\overline{B_\rho(0)}$), $\rho \in \mathbb{R}_+$, denotes the open (resp., closed) ball of $\mathbb{R}^d$ centered at 0 and with radius ρ, i.e., $B_\rho(0) := \{x \in \mathbb{R}^d,\ |x| < \rho\}$ (resp., $\overline{B_\rho(0)} := \{x \in \mathbb{R}^d,\ |x| \leq \rho\}$), and $B_{R,\delta}(0) := B_R(0) \setminus \overline{B_\delta(0)}$. Notice that the upper-bound on $\|\widehat{f}\|_{L^2(B_{R,\delta}(0))}$ given by (23) was obtained upon decomposing the annulus $B_{R,\delta}(0)$ into $B_{R,1}(0) \cup B_{1,\delta}(0)$ and majorizing $\left|\widehat{f}(t\xi)\right|$ by $5^{\frac{s}{2}} M \epsilon^{\frac{2}{\pi}\arctan R^{-1}}$ for all $t \in B_{R,1}(0)$, and by $5^{\frac{s}{2}} M t^{-s} \epsilon^{\frac{2}{\pi}\arctan R^{-1}}$ for all $t \in B_{1,\delta}(0)$.

Further, since $f \in L^1(\Omega)$, we have $\left|\widehat{f}(\xi)\right| \leq (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\Omega)}$ for all $\xi \in \mathbb{R}^d$ and hence $\left|\widehat{f}(\xi)\right| \leq (2\pi)^{-\frac{d}{2}} |\Omega|^{\frac{1}{2}} \|f\|_{H^k(\Omega)}$ by the Cauchy-Schwarz inequality. This entails that

$$\|\widehat{f}\|_{L^2(B_\delta(0))} \leq C\delta^{\frac{d}{2}} \|f\|_{H^k(\Omega)}, \tag{24}$$

where, here and below, C denotes a generic positive constant depending only on d and Ω, which may change from line to line. Next, since the function $f \in H_0^k(\Omega)$ extended by zero in $\mathbb{R}^d \setminus \Omega$, lies in $H^k(\mathbb{R}^d)$, we infer from the Fourier-Plancherel theorem that

$$\begin{aligned} \|\widehat{f}\|^2_{L^2(\mathbb{R}^d \setminus B_R(0))} &= \int_{\mathbb{R}^d \setminus B_R(0)} \left|\widehat{f}(\xi)\right|^2 d\xi \\ &\leq R^{-2k} \int_{\mathbb{R}^d \setminus B_R(0)} (1+|\xi|^2)^k \left|\widehat{f}(\xi)\right|^2 d\xi \\ &\leq R^{-2k} \int_{\mathbb{R}^d} (1+|\xi|^2)^k \left|\widehat{f}(\xi)\right|^2 d\xi \\ &\leq R^{-2k} \|f\|^2_{H^k(\Omega)}. \end{aligned}$$

From this and (24) it then follows that $\|\widehat{f}\|_{L^2(\mathbb{R}^d \setminus B_{R,\delta}(0))} \leq C(\delta^{\frac{d}{2}} + R^{-k})\|f\|_{H^k(\Omega)}$, which together with (22) yields $\|\widehat{f}\|_{L^2(\mathbb{R}^d)} \leq h_s(\delta, R) + C(\delta^{\frac{d}{2}} + R^{-k})\|f\|_{H^k(\Omega)}$. And since $\|f\|_{L^2(\Omega)} = \|\widehat{f}\|_{L^2(\mathbb{R}^d)}$ from the Fourier-Plancherel theorem, we have

$$\|f\|_{L^2(\Omega)} \leq h_s(\delta, R) + C(\delta^{\frac{d}{2}} + R^{-k})\|f\|_{H^k(\Omega)}. \tag{25}$$

The rest of the proof is to estimate $h_s(\delta, R)$ with the aid of (23). This leads us to examine the three cases $s < d$, $s = d$ and $s > d$ separately.

First case: $s < d$. Taking $\delta = R^{-\frac{2k}{d}}$ in (25), we find that

$$\|\widehat{f}\|_{L^2(\mathbb{R}^d)} \leq 5^{\frac{s}{2}} C_d M R^d \epsilon^{\frac{2}{\pi}\arctan R^{-1}} + CR^{-k}\|f\|_{H^k(\Omega)}. \tag{26}$$

Next, assuming without loss of generality that $\epsilon \in (0,1)$, we pick $\theta \in (0,1)$, choose $R = (-\ln\epsilon)^{\frac{1}{1+\theta}}$ in (26) and get that

$$\|f\|_{L^2(\Omega)} \leq 5^{\frac{s}{2}} C_d M (-\ln\epsilon)^{\frac{d}{1+\theta}} e^{-\frac{2}{\pi}(-\ln\epsilon)\arctan(-\ln\epsilon)^{-\frac{1}{1+\theta}}} + C(-\ln\epsilon)^{-\frac{k}{1+\theta}} \|f\|_{H^k(\Omega)}.$$

Since $\arctan u = \int_0^u \frac{1}{1+v^2} dv$ and since the function $v \mapsto \frac{1}{1+v^2}$ is decreasing on $[0,+\infty)$, we obtain $\arctan u \geq \frac{u}{1+u^2}$ for all $u \geq 0$, and hence

$$\begin{aligned}\|f\|_{L^2(\Omega)} &\leq 5^{\frac{s}{2}} C_d M(-\ln\epsilon)^{\frac{d}{1+\theta}} e^{-\frac{2}{\pi}\frac{(-\ln\epsilon)^{\frac{\theta}{1+\theta}}}{1+(-\ln\epsilon)^{-\frac{2}{1+\theta}}}} + C(-\ln\epsilon)^{-\frac{k}{1+\theta}}\|f\|_{H^k(\Omega)} \\ &\leq (-\ln\epsilon)^{-\frac{k}{1+\theta}}\left(5^{\frac{s}{2}} C_d M(-\ln\epsilon)^{\frac{d+k}{1+\theta}} e^{-\frac{2}{\pi}\frac{(-\ln\epsilon)^{\frac{2+\theta}{1+\theta}}}{1+(-\ln\epsilon)^{\frac{2}{1+\theta}}}} + C\|f\|_{H^k(\Omega)}\right).\end{aligned}$$

This entails that

$$\|f\|_{L^2(\Omega)} \leq (-\ln\epsilon)^{-\frac{k}{1+\theta}}\left(5^{\frac{s}{2}} C_d M(-\ln\epsilon)^{\frac{d+k}{1+\theta}} e^{-\frac{1}{\pi}(-\ln\epsilon)^{\frac{2+\theta}{1+\theta}}} + C\|f\|_{H^k(\Omega)}\right),\ \epsilon\in(0,e^{-1}). \tag{27}$$

Now, since $\lim_{\epsilon\downarrow 0}(-\ln\epsilon)^{\frac{d+k}{1+\theta}} e^{-\frac{1}{\pi}(-\ln\epsilon)^{\frac{2+\theta}{1+\theta}}} = 0$, there exists $\epsilon_0 > 0$, depending only on Ω, d, k, s, θ and $\frac{\|f\|_{L^1(\Omega)}}{\|f\|_{H^k(\Omega)}}$, such that we have

$$\|f\|_{L^2(\Omega)} \leq C\|f\|_{H^k(\Omega)}(-\ln\epsilon)^{-\frac{k}{1+\theta}},\ \epsilon\in(0,\epsilon_0), \tag{28}$$

which immediately yields (4).

Second case: $d = s$. Choosing $\delta = e^{-R^d}$ in (25), we get (26) where the constant C_d is substituted for $2C_d$. This leads to (4) upon arguing as in the *First case*.

Third case: $s > d$. Taking $\delta = R^{-\frac{d}{s-d}}$ in (25), we obtain that

$$\|f\|_{L^2(\Omega)} \leq 5^{\frac{s}{2}} C_d M R^d \epsilon^{\frac{2}{\pi}\arctan R^{-1}} + CR^{-k}\left(1 + R^{\frac{k}{s-d}\left(s-d\left(1+\frac{d}{2k}\right)\right)}\right)\|f\|_{H^k(\Omega)}, \tag{29}$$

where $2C_d$ is replaced by C_d.

i) If $s \leq d + \frac{d^2}{2k}$ then it is apparent that (29) yields (26), from where we get (4) by following the exact same path as in the *First case*.

ii) If $s > d + \frac{d^2}{2k}$ then (29) may be equivalently rewritten as

$$\|f\|_{L^2(\Omega)} \leq 5^{\frac{s}{2}} C_d M R^d \epsilon^{\frac{2}{\pi}\arctan R^{-1}} + CR^{-\frac{d^2}{2(s-d)}}\|f\|_{H^k(\Omega)},$$

which is the same estimate as (26) where the power k of the second occurence of R is replaced by $\frac{d^2}{2(s-d)}$. Thus, by arguing in the same way as in the derivation of (28) from (26), we obtain that

$$\|f\|_{L^2(\Omega)} \leq C\|f\|_{H^k(\Omega)}(-\ln\epsilon)^{-\frac{d^2}{(1+\theta)(s-d)}},\ \epsilon\in(0,\epsilon_0),$$

for some positive constant ϵ_0 depending only on Ω, d, k, s, θ and $\frac{\|f\|_{L^1(\Omega)}}{\|f\|_{H^k(\Omega)}}$, giving (4).

2.3. **Proof of Theorem 1.2.** We keep the notations of Section 2.2. Assuming that $f = 0$, we infer from (9) that

$$|\widehat{u_0}(i\omega\xi)| \leq (2\pi)^{-\frac{d}{2}}|\partial\Omega|^{\frac{1}{2}} e^{\kappa_\Omega\omega}\omega^{\frac{2-2\alpha}{\alpha}}\|\partial_\nu U(\omega,\cdot)\|_{L^2(\partial\Omega)},\ \omega\in\mathbb{R}_+. \tag{30}$$

Plugging (11) into (30) then yields

$$|\widehat{u_0}(i\omega\xi)| \leq (2\pi)^{-\frac{d}{2}}|\partial\Omega|^{\frac{1}{2}}\omega^{-1}e^{\kappa_\Omega\omega}\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))},\ \omega\in\mathbb{R}_+,$$

and since $\omega^{-1} \leq \left(\frac{2+\omega}{\omega}\right)^s$ for all $\omega\in\mathbb{R}_+$, where we recall that $s := 1 + \lfloor\frac{2}{\alpha}\rfloor \in [3,+\infty)$, we end up getting that

$$\left(\frac{\omega}{2+\omega}\right)^s e^{-\kappa_\Omega\omega}|\widehat{u_0}(i\omega\xi)| \leq (2\pi)^{-\frac{d}{2}}|\partial\Omega|^{\frac{1}{2}}\|\partial_\nu u\|_{L^{\frac{2}{\alpha}}(\mathbb{R}_+;L^2(\partial\Omega))},\ \omega\in[0,+\infty). \tag{31}$$

The above estimate being similar to (12) where f replaced by u_0 and c_0 is equal to one, the desired result then follows from (31) by arguing in the same way as in the derivation of Theorem 1.1 from (12).

2.4. **Unique continuation.** In this section we prove the quantitative unique continuation result stated in Lemma 2.1. We start by recalling the following technical result, which is borrowed from [15].

Lemma 2.2. *The function w defined in (18) is a harmonic measure and is the unique solution to the system*

$$\left\{ \begin{array}{ll} \Delta w(z) = 0, & z \in Q, \\ w(t) = 1, & t \in (-1, +\infty), \\ w(-1-it) = 0, & t \in (0, +\infty). \end{array} \right. \tag{32}$$

Proof. To make the present paper self-contained and for the convenience of the reader we include the proof of this result. First, we notice that $w(t) = \frac{2}{\pi}\left(\frac{\pi}{2} + \arg(t+1)\right) = \frac{2}{\pi}\left(\frac{\pi}{2} + 0\right) = 1$ for all $t \in (-1, +\infty)$ and that $w(-1-it) = \frac{2}{\pi}\left(\frac{\pi}{2} + \arg(-it)\right) = \frac{2}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = 0$ for $t \in (0, +\infty)$. Next, in order to show that $\Delta w = 0$ in Q, it is enough to we rewrite w as

$$w(z) = \frac{2}{\pi}\left(\frac{\pi}{2} + \Im \log(z+1)\right), \ z \in Q,$$

where log denotes the complex logarithmic function log, defined in $\{z \in \mathbb{C}, \ -iz \notin [0, +\infty)\}$. The desired result then follows from the holomorphicity of $z \mapsto \log(z+1)$ in the quadrant Q. □

Armed with Lemma 2.2 we can now complete the proof of Lemma 2.1. For this purpose we combine the estimate $|F(t)| \leq m$ for all $t \in (-1, +\infty)$, with the basic identity $m = M^{1-1}m^1$ and the second line of (32), and get that

$$|F(t)| \leq M^{1-w(t)} m^{w(t)}, \ t \in (-1, +\infty). \tag{33}$$

Further we have

$$|F(z)| \leq M, \ z \in \overline{Q}, \tag{34}$$

from (16)-(17), hence $|F(-1-it)| \leq M$ for all $t \in (0, +\infty)$. From this, the identity $M = M^{1-0}m^0$ and the third line of (32), it then follows that

$$|F(-1-it)| \leq M^{1-w(-1-it)} m^{w(-1-it)}, \ t \in (0, +\infty), \tag{35}$$

Summing up (33) and (35), we have

$$|F(z)| \leq M^{1-w(z)} m^{w(z)}, \ z \in \partial Q \setminus \{-1\}. \tag{36}$$

Since F is holomorphic in Q and w is a harmonic measure of Q by Lemma 2.2, (34), (36) and the Two-constants theorem (see, e.g., [33, Chap. III, Section 2.1] or [39]) yield

$$|F(z)| \leq M^{1-w(z)} m^{w(z)}, \ z \in Q.$$

Thus, by continuity of F on ∂Q and w on $\partial Q \setminus \{-1\}$, we obtain that

$$|F(z)| \leq M^{1-w(z)} m^{w(z)}, \ z \in \overline{Q} \setminus \{-1\}.$$

Finally, (19) follows readily from this and the inequality $M \geq 1$ arising from (17). □

Appendix A. The direct problem

This section is devoted to the proof of Proposition 1.1. For this purpose we introduce the self-adjoint operator A in $L^2(\Omega)$, acting as $-\Delta$ on his domain $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$, that is to say the operator generated in $L^2(\Omega)$ by the closed quadratic form $a(u) := \int_\Omega |\nabla u|^2 \, dx$, $u \in D(a) := H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the operator A has a compact resolvent and consequently a discrete spectrum. We denote by $(\lambda_k)_{k \in \mathbb{N}}$ the non-decreasing sequence of the eigenvalues of A, and by $\{\phi_k, \ k \in \mathbb{N}\}$ a set of eigenfunctions such that $A\phi_k = \lambda_k \phi_k$, which form an orthonormal basis in $L^2(\Omega)$.

With reference to [30, 37], the $L^2_{\rm loc}(\mathbb{R}_+;H^2(\Omega)\cap H^1_0(\Omega))$-solution u to (1) reads

$$u(t)=S_0(t)u_0+\int_0^t S_1(t-s)fg(s)ds,\ t\in\mathbb{R}_+, \tag{37}$$

where

$$S_0(t)h:=\sum_{k=1}^{+\infty}E_{\alpha,1}(-\lambda_k t^\alpha)\langle h,\phi_k\rangle_{L^2(\Omega)},\ t\in\mathbb{R}_+,\ h\in L^2(\Omega), \tag{38}$$

$$S_1(t)h:=\sum_{k=1}^{+\infty}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha)\langle h,\phi_k\rangle_{L^2(\Omega)},\ t\in\mathbb{R}_+,\ h\in L^2(\Omega), \tag{39}$$

and E_{β_1,β_2}, for $(\beta_1,\beta_2)\in\mathbb{R}_+^2$, is the the Mittag-Leffler function defined by

$$E_{\beta_1,\beta_2}(z):=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\beta_1 k+\beta_2)},\ z\in\mathbb{C}. \tag{40}$$

The rest of the proof is to show that the function u defined by (37)-(40), lies in $L^r(\mathbb{R}_+;H^{\frac{7}{4}}(\Omega))$ for all $r>\alpha^{-1}$, and that u satisfies the energy estimate (2). As will appear further, this essentially boils down to the following properties of the Mittag-Leffler functions defined by (40), which can be found in, e.g., [34, Section 1.2.7 (pp. 34–35)].

Lemma A.1. *Let $\beta_1\in(0,2)$ and all $\beta_2\in\mathbb{R}$. Then, there exists a constant $c=c(\beta_1,\beta_2)>0$ such that*

$$|E_{\beta_1,\beta_2}(\tau)|\leq\frac{c}{1+|\tau|},\ \tau\in(-\infty,0], \tag{41}$$

and it holds true for all $N\in\mathbb{N}$ that

$$E_{\beta_1,\beta_2}(\tau)=-\sum_{k=1}^{N}\frac{\tau^{-k}}{\Gamma(\beta_2-\beta_1 k)}+\underset{\tau\to-\infty}{\mathcal{O}}(|\tau|^{-N-1}). \tag{42}$$

Here and in the remaining part of this appendix, we follow the convention used in [34] by setting

$$\frac{1}{\Gamma(m)}:=0,\ m\in\mathbb{Z}\setminus\mathbb{N}:=\{\dots,-2,-1,0\}. \tag{43}$$

We recall that for all $\gamma>0$, $D(A^\gamma)=\{v\in L^2(\Omega),\ \sum_{k=1}^{+\infty}\lambda_k^{2\gamma}\left|\langle v,\varphi_k\rangle_{L^2(\Omega)}\right|^2<\infty\}$ is a Hilbert space with the norm

$$\|v\|_{D(A^\gamma)}:=\left(\sum_{k=1}^{+\infty}\lambda_k^{2\gamma}\left|\langle v,\varphi_k\rangle_{L^2(\Omega)}\right|^2\right)^{\frac{1}{2}},\ v\in D(A^\gamma),$$

and that $D(A^\gamma)\subset H^{2\gamma}(\Omega)$, the injection being continuous. In light of this we write $\|S_0(t)u_0\|_{H^2(\Omega)}\leq C\|S_0(t)u_0\|_{D(A)}$, where, from now on, C denotes a generic positive constant depending only on Ω and α, which may change from line to line, and then deduce from (41) that

$$\begin{aligned}\|S_0(t)u_0\|^2_{H^2(\Omega)}&\leq C\sum_{k=1}^{+\infty}\lambda_k^2\left|E_{\alpha,1}(-\lambda_k t^\alpha)\right|^2\left|\langle u_0,\phi_k\rangle_{L^2(\Omega)}\right|^2\\&\leq C\sum_{k=1}^{+\infty}\frac{\lambda_k^2}{(1+\lambda_k t^\alpha)^2}\left|\langle u_0,\phi_k\rangle_{L^2(\Omega)}\right|^2\\&\leq\frac{C}{(1+\lambda_1 t^\alpha)^2}\sum_{k=1}^{+\infty}\lambda_k^2\left|\langle u_0,\phi_k\rangle_{L^2(\Omega)}\right|^2.\end{aligned}$$

Thus, we have $\|S_0(t)u_0\|_{H^2(\Omega)} \leq \frac{C}{1+\lambda_1 t^\alpha}\|u_0\|_{D(A)}$, and consequently

$$\|S_0(t)u_0\|_{H^2(\Omega)} \leq \frac{C\|u_0\|_{D(A)}}{1+t^\alpha},\ t \in \mathbb{R}_+.$$

For $r > \alpha^{-1}$, this entails that $t \mapsto S_0(t)u_0 \in L^r\left(\mathbb{R}_+; H^{\frac{7}{4}}(\Omega)\right)$, with

$$\|S_0(\cdot)u_0\|_{L^r\left(\mathbb{R}_+;H^{\frac{7}{4}}(\Omega)\right)} \leq C\|u_0\|_{H^2(\Omega)}. \tag{44}$$

Similarly, since $\|S_1(t)f\|_{H^{\frac{7}{4}}(\Omega)} \leq C\|S_1(t)f\|_{D(A^{\frac{7}{8}})}$ for all $t \in \mathbb{R}_+$, we have

$$\begin{aligned}\|S_1(t)f\|^2_{H^{\frac{7}{4}}(\Omega)} &\leq Ct^{2(\alpha-1)}\sum_{k=1}^{+\infty}\lambda_k^{\frac{7}{4}}\left|E_{\alpha,\alpha}(-\lambda_k t^\alpha)\right|^2\left|\langle f,\phi_k\rangle_{L^2(\Omega)}\right|^2 \\ &\leq Ct^{2(\alpha-1)}\sum_{k=1}^{+\infty}\frac{\lambda_k^{\frac{7}{4}}}{(1+\lambda_k t^\alpha)^2}\left|\langle f,\phi_k\rangle_{L^2(\Omega)}\right|^2.\end{aligned}$$

Taking into account that $\frac{\lambda_k^{\frac{7}{4}}}{(1+\lambda_k t^\alpha)^2} = t^{-\frac{7\alpha}{4}}\left(\frac{\lambda_k t^\alpha}{1+\lambda_k t^\alpha}\right)^{\frac{7}{4}}\frac{1}{(1+\lambda_k t^\alpha)^{\frac{1}{4}}} \leq t^{-\frac{7\alpha}{4}}$ for all $k \in \mathbb{N}$, we get from the above estimate that $\|S_1(t)f\|^2_{H^{\frac{7}{4}}(\Omega)} \leq Ct^{2(\alpha-1)}t^{-\frac{7}{4}\alpha}\sum_{k=1}^{+\infty}\left|\langle f,\phi_k\rangle_{L^2(\Omega)}\right|^2$, which entails that

$$\|S_1(t)f\|_{H^{\frac{7}{4}}(\Omega)} \leq Ct^{2\left(\frac{\alpha}{8}-1\right)}\|f\|_{L^2(\Omega)},\ t \in \mathbb{R}_+. \tag{45}$$

Further, applying (42) with $\beta_1 = \beta_2 = \alpha$ and $N = 2$, we have $E_{\alpha,\alpha}(\tau) = \frac{\tau^{-2}}{\Gamma(-\alpha)} + \underset{\tau\to-\infty}{\mathcal{O}}(|\tau|^{-3})$, by virtue of (43). Thus, there exists a positive constant C, depending only on α, such that the following estimate

$$\left|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha)\right| \leq Ct^{\alpha-1}(\lambda_k t^\alpha)^{-2} \leq C\lambda_k^{-2}t^{-(\alpha+1)},\ t \in [1,+\infty),$$

holds uniformly in $k \in \mathbb{N}$. As a consequence we have

$$\begin{aligned}\|S_1(t)f\|^2_{H^2(\Omega)} &\leq C\|S_1(t)f\|^2_{D(A)} \\ &\leq C\sum_{k=1}^{+\infty}\lambda_k^2\left|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha)\right|^2\left|\langle f,\phi_k\rangle_{L^2(\Omega)}\right|^2 \\ &\leq C\sum_{k=1}^{+\infty}\lambda_k^{-2}t^{-2(\alpha+1)}\left|\langle f,\phi_k\rangle_{L^2(\Omega)}\right|^2,\end{aligned}$$

and hence $\|S_1(t)f\|_{H^2(\Omega)} \leq Ct^{-(\alpha+1)}\|f\|_{L^2(\Omega)}$ for all $t \in [1,+\infty)$, where we used that $\lambda_k^{-2} \leq \lambda_1^{-2}$ for all $k \in \mathbb{N}$. From this and (45) it then follows that

$$\|S_1(t)f\|_{H^{\frac{7}{4}}(\Omega)} \leq C\left(t^{\frac{\alpha}{8}-1}\mathbb{1}_{(0,1)}(t) + t^{-(\alpha+1)}\mathbb{1}_{[1,+\infty)}(t)\right),\ t \in \mathbb{R}_+, \tag{46}$$

where the notation $\mathbb{1}_I$ stands for the characteristic function of any subinterval $I \subset \mathbb{R}$. Putting (45) and (46) together, we obtain that $S_1(t)f \in L^1\left(\mathbb{R}_+; H^{\frac{7}{4}}(\Omega)\right)$, with

$$\|S_1(\cdot)f\|_{L^1\left(\mathbb{R}_+;H^{\frac{7}{4}}(\Omega)\right)} \leq C\|f\|_{L^2(\Omega)}. \tag{47}$$

Moreover, we have

$$\begin{aligned}\|u(t)\|_{H^{\frac{7}{4}}(\Omega)} &\leq \|S_0(t)u_0\|_{H^{\frac{7}{4}}(\Omega)} + \int_0^t\|S_1(t-s)f\|_{H^{\frac{7}{4}}(\Omega)}|g(s)|\,ds \\ &\leq \|S_0(t)u_0\|_{H^{\frac{7}{4}}(\Omega)} + \left(\|S_1(\cdot)f\|_{H^{\frac{7}{4}}(\Omega)}\mathbb{1}_{\mathbb{R}_+}\right)\star\left(|g|\,\mathbb{1}_{\mathbb{R}_+}\right)(t),\ t \in \mathbb{R}_+,\end{aligned}$$

from (37), and hence

$$\begin{aligned} \|u\|_{L^r(\mathbb{R}_+;H^{\frac{7}{4}}(\Omega))} &\leq \|S_0(\cdot)u_0\|_{L^r(\mathbb{R}_+;H^{\frac{7}{4}}(\Omega))} + \|S_1(\cdot)f\|_{L^1(\mathbb{R}_+;H^{\frac{7}{4}}(\Omega))}\|g\|_{L^r(\mathbb{R}_+)} \\ &\leq C\left(\|u_0\|_{H^2(\Omega)} + \|g\|_{L^r(\mathbb{R}_+)}\|f\|_{L^2(\Omega)}\right), \end{aligned}$$

upon combining Young's convolution inequality with (44) and (47). This completes the proof of Proposition 1.1.

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