On Time-Fractional Diffusion Equations with Space-Dependent Variable Order

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Abstract. This paper deals with mathematical problems related to space-dependent anomalous diffusion processes. Namely, we investigate diffusion equations with time-fractional derivatives of space-dependent variable order. We establish that variable order time-fractional Cauchy problems admit a unique weak solution and prove that the space-dependent variable order coefficient is uniquely determined by the knowledge of a suitable time-sequence of partial Dirichlet-to-Neumann maps.

1. Introduction

1.1. Statement of the problem

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geqslant 2$, with Lipschitz continuous boundary $\partial\Omega$, and let $(a_{i,j})_{1 \leqslant i,j \leqslant d} \in L^\infty(\Omega; \mathbb{R}^{d^2})$ be symmetric, i.e., fulfill $a_{i,j} = a_{j,i}$ a.e. in Ω, for $i, j = 1, \ldots, d$, and satisfy the ellipticity condition

$$\exists c > 0,\ \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \geqslant c|\xi|^2,\ x \in \Omega,\ \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d. \tag{1.1}$$

For $\kappa \in (d, +\infty]$ and $q \in L^\kappa(\Omega)$, such that

$$q(x) \geq 0,\ x \in \Omega, \tag{1.2}$$

we introduce the formal differential operators

$$\mathcal{A}_0 u(x) = -\sum_{i,j=1}^{d} \partial_{x_i}\left(a_{i,j}(x)\partial_{x_j} u(x)\right) \text{ and } \mathcal{A}_q u(x) := \mathcal{A}_0 u(x) + q(x)u(x),\ x \in \Omega,$$

where we set $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, d$.

Given $T \in (0, +\infty]$ and two functions $\alpha \in L^\infty(\Omega)$ and $\rho \in L^\infty(\Omega)$ satisyfing

$$0 < \alpha_0 \leqslant \alpha(x) \leqslant \alpha_M < 1 \text{ and } 0 < \rho_0 \leqslant \rho(x) \leqslant \rho_M < +\infty,\ x \in \Omega, \tag{1.3}$$

we consider the initial-boundary value problem for a space-dependent variable order (VO) fractional diffusion equation

$$\begin{cases} (\rho(x)\partial_t^{\alpha(x)} + \mathcal{A}_q)u(t,x) &= f(t,x), \quad (t,x) \in Q := (0,T) \times \Omega, \\ u(t,x) &= 0, \qquad (t,x) \in \Sigma := (0,T) \times \partial\Omega, \\ u(0,x) &= u_0(x), \quad x \in \Omega. \end{cases} \tag{1.4}$$

Here and below, $\partial_t^{\alpha(x)}$ denotes the Caputo fractional derivative of order $\alpha(x)$ with respect to t, defined by

$$\partial_t^{\alpha(x)} u(t,x) := \frac{1}{\Gamma(1-\alpha(x))} \int_0^t (t-s)^{-\alpha(x)} \partial_s u(s,x) ds, \ (t,x) \in Q,$$

where Γ is the Gamma function.

In this paper, we pursue two goals. The first one is to establish the well-posedness of the initial-boundary value problem (1.4) for a suitable source term f and initial value u_0 (actually, in what follows we rather focus on the existence and uniqueness issues of a weak solution to (1.4), than on its stability properties with respect to the data, but for the sake of shortness, we may refer by a slight abuse of language to Theorem 1.1 as a well-posedness result). The second one is to analyse the uniqueness in an inverse problem of determining simultaneously the fractional order α and two coefficients ρ and q of the diffusion equation in (1.4) by partial Cauchy data.

1.2. Physical motivations

Anomalous diffusion in complex media is a rapidly growing field of academic research with multiple engineering applications in geophysics, environmental science and biology. The diffusion properties of homogeneous media are currently modeled, see e.g., [1, 6], by constant order (CO) time-fractional diffusion processes where in (1.4) the mapping $x \mapsto \alpha(x)$ is maintained constant over Ω. However, in complex media, the presence of heterogeneous regions displays space inhomogeneous variations and the CO fractional dynamic models are not robust for long times, see [11]. In this background the VO time-fractional model is more relevant for describing the space-dependent anomalous diffusion process, see e.g., [47]. As a matter of fact, several VO diffusion models have been successfully applied to numerous areas of applied sciences and engineering, such as chemistry [10], rheology [45], biology [13], hydrogeology [4] and physics [46, 51].

Notice that the VO time-fractional kinetic equation is usually a corresponding macroscopic model to continuous time random walk (CTRW) driven stochastic diffusion processes with space-dependent diffusion coefficient. We refer to [37, 40] for a rigorous derivation of the fractional heat equation from a CTRW scheme with space-dependent diffusion coefficient.

1.3. A short review of the mathematical literature of time-fractional diffusion equations

The mathematical study of ordinary and partial differential equations with fractional derivatives has attracted a lot of attention during the last decades.

We refer to [25, 36, 38, 44] for a general introduction to fractional calculus and to, e.g., [2, 15, 33] for a more specific focus on partial differential equations with time-fractional derivatives.

Autonomous CO fractional Cauchy problems, i.e. CO time-fractional diffusion equations associated with a time-independent elliptic part, have been extensively studied in the mathematical literature, see e.g., [14, 24, 43]. In this framework, the solution is unique and analytic with respect to the time variable, as it is expressed in terms of the Mittag-Leffler functions. This result is extended to the solution of Cauchy problems with almost sectorial operators in [48]. Existence and uniqueness results for autonomous CO fractional Cauchy systems, i.e. CO time-fractional diffusion equations associated with a time-dependent coefficients, are derived in [27, 50], by mean of the variational theory. In the particular case where the spatial domain $\Omega = \mathbb{R}^d$, an integro-differential approach of this problem can be found in [3].

As for distributed order (DO) time-fractional Cauchy problems, we refer to [26, 29, 34] for the analysis of the well-posedness issue, and to [29, 31] for the study of the asymptotic behavior of the solution. However, in contrast with CO or DO time-fractional equations, there is, to the best of our knowledge, no result available in the mathematical literature for VO time-fractional diffusion equations.

Quite similarly, there is only a small number of mathematical papers dealing with inverse problems associated with time-fractional diffusion processes, which are listed below. In the one-dimensional case, [9] proves simultaneous determination of the constant fractional differential order and the time-independent diffusion coefficient by Dirichlet boundary measurements of the solution. In dimension 2 or greater, [17] determines the constant fractional order from measurements at one point of the solution over the entire time span. In [12, 43], the time-varying factor in the source term or in the zeroth order coefficient is stably determined by pointwise observation of the solution. For half-order fractional diffusion equations, the zeroth order coefficient is stably reconstructed in [8, 49], with the aid of a Carleman estimate. In [42] the authors determine an unknown boundary condition in a time-fractional Cauchy problem and in [35] they recover an unknown semilinear term in a time-fractional reaction-diffusion equation. We also mention that a unique continuation result for CO time-fractional diffusion equations can be found in [7, 32].

As for multiple CO time-fractional systems, we refer to [28], claiming unique determination of the number of time-fractional derivatives with the corresponding differential orders and several spatially varying coefficients. Finally, in [23], general space-dependent coefficients defined on a Riemanian manifold, along with the Riemanian metric, are simultaneously recovered by the partial Dirichlet-to-Neumann map taken at one arbitrarily fixed time.

1.4. Main results

The first result of this paper is given for a Lipschitz continuous bounded domain Ω. It establishes the existence, the uniqueness and the regularity properties of the weak solution to the initial-boundary value problem (1.4) in the sense of Definition 2.2 below.

As a preliminary we introduce the following notations, used throughout the entire text. The interval $(0,T]$ (resp., $[0,T]$) should be understood as $(0,+\infty)$ (resp., $[0,+\infty)$) for the case of $T=+\infty$. Next we define the contour in $\mathbb{C}$,

$$\gamma(\varepsilon,\theta) := \gamma_-(\varepsilon,\theta) \cup \gamma_0(\varepsilon,\theta) \cup \gamma_+(\varepsilon,\theta), \tag{1.5}$$

oriented in the counterclockwise direction, where for every $(\varepsilon,\theta) \in (0,1) \times \left(\frac{\pi}{2},\pi\right)$,

$$\gamma_0(\varepsilon,\theta) := \{\varepsilon e^{i\beta};\ \beta \in [-\theta,\theta]\} \text{ and } \gamma_\pm(\varepsilon,\theta) := \{s e^{\pm i\theta};\ s \in [\varepsilon,+\infty)\}, \tag{1.6}$$

and the two copies of the $\pm$ sign in the above identity must both be replaced in the same way. Furthermore, we denote by A_q the self-adjoint realization in $L^2(\Omega)$ of the operator $\mathcal{A}_q$ with the homogeneous Dirichlet boundary condition and for $p \in \mathbb{C} \setminus \mathbb{R}_-$, $(A_q + \rho(x)p^{\alpha(x)})^{-1}$ is the resolvent operator of $A_q + \rho(x)p^{\alpha(x)}$. Henceforth, the notation $\langle t \rangle$ stands for $(1+t^2)^{\frac{1}{2}}$.

Then the existence and uniqueness result of a weak solution to (1.4) is as follows.

Theorem 1.1. *Suppose that* (1.1) *and* (1.2) *are fulfilled. Let* $u_0 \in L^2(\Omega)$. *We assume that* $f \in L^\infty(0,T;L^2(\Omega)) \cap \mathcal{C}((0,T],L^2(\Omega))$ *in the case of* $T < +\infty$, *and* $f \in \mathcal{C}((0,+\infty),L^2(\Omega))$ *satisfies* $\langle t \rangle^{-\zeta} f \in L^\infty(\mathbb{R}_+;L^2(\Omega))$ *with some* $\zeta \in \mathbb{R}_+$ *in the case of* $T=+\infty$. *Then there exists a unique weak solution* $u \in \mathcal{C}((0,T];L^2(\Omega))$ *to* (1.4), *which is expressed by*

$$u(t) = u(t,\cdot) = S_0(t)u_0 + \int_0^t S_1(t-\tau)f(\tau)d\tau + S_2 f(t),\ t \in (0,T], \tag{1.7}$$

where we set

$$S_0(t)\psi := \frac{1}{2i\pi}\int_{\gamma(\varepsilon,\theta)} e^{tp}(A_q + \rho(x)p^{\alpha(x)})^{-1}\rho(x)p^{\alpha(x)-1}\psi dp,$$

$$S_1(t)\psi := \frac{1}{2i\pi}\int_{\gamma(\varepsilon,\theta)} e^{tp}(A_q + \rho(x)p^{\alpha(x)})^{-1}\psi dp$$

and

$$S_2\psi := \frac{1}{2i\pi}\int_{\gamma(\varepsilon,\theta)} p^{-1}(A_q + \rho(x)p^{\alpha(x)})^{-1}\psi dp$$

for all $\psi \in L^2(\Omega)$, *the three above integrals being independent of the choice of* $\varepsilon \in (0,1)$ *and* $\theta \in \left(\frac{\pi}{2},\pi\right)$.

Moreover, if $f=0$, *then the mapping* $u : (0,T) \longrightarrow L^2(\Omega)$ *is analytic in* $(0,T)$.

Remark 1. *If* $x \mapsto \alpha(x)$ *is constant in* Ω *then it is not hard to see that the operator* S_2 *is identically zero. Therefore,* (1.7) *reduces to the classical Duhamel formula*

$$u(t) = S_0(t)u_0 + \int_0^t S_1(t-\tau)f(\tau)d\tau,\ t \in (0,T], \tag{1.8}$$

which is consistent with the representation formula of the solution presented in [43, Theorems 2.1-2.2]. *Moreover, it can be checked that* (1.8) *remains valid for all* $\alpha_0 \in \left(0, \frac{1}{2}\right)$, *provided we have* $\alpha_M \in (\alpha_0, 2\alpha_0)$.

The second result deals with the inverse problem of determining the unknown functions α, ρ, q of the time fractional diffusion equation in (1.4) by partial boundary data of the solution. More precisely, we assume that $\partial\Omega$ is $\mathcal{C}^{1,1}$ and

$$a_{i,j}(x) = \delta_{i,j},\ x \in \Omega,\ i,j = 1,\dots,d, \tag{1.9}$$

where $\delta_{i,j}$ is equal to 1 whenever $i = j$, and to 0 otherwise. Then we fix $k \in \mathbb{N} \setminus \{1\}$, where $\mathbb{N} := \{1,2,\dots\}$, and consider the following system

$$\begin{cases} (\rho(x)\partial_t^{\alpha(x)} + \mathcal{A}_q)u(t,x) &= 0, & (t,x) \in (0,+\infty) \times \Omega, \\ u(t,x) &= t^k g(x), & (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(0,x) &= 0, & x \in \Omega, \end{cases} \tag{1.10}$$

with suitable g. Given two non empty subsets S_{in} and S_{out} of $\partial\Omega$, we introduce the following boundary operator

$$\mathcal{N}_{\alpha,\rho,q}(t) : \mathcal{H}_{\text{in}} \ni g \mapsto \partial_\nu u_g(t,\cdot)_{|S_{\text{out}}}, \quad t \in (0,+\infty), \tag{1.11}$$

where $\mathcal{H}_{\text{in}} := \{g \in H^{3/2}(\partial\Omega);\ \text{supp}\, g \subset \overline{S_{\text{in}}}\}$. Here by u_g we denote a unique solution in $\mathcal{C}([0,+\infty); H^2(\Omega))$ to (1.10), whose existence is guaranteed by Proposition 3.1 stated below, ν is the outward normal unit vector to $\partial\Omega$, and $\partial_\nu u_g(t,x) := \nabla u_g(t,x) \cdot \nu(x)$ for $(t,x) \in (0,+\infty) \times \partial\Omega$.

We discuss the uniqueness in the inverse problem of determining the coefficients (α,ρ,q) from the knowledge of the boundary operators $\{\mathcal{N}_{\alpha,\rho,q}(t_n);\ n \in \mathbb{N}\}$ associated with a time sequence t_n, $n \in \mathbb{N}$ fulfilling

$$\text{the set } \{t_n;\ n \in \mathbb{N}\} \text{ has an accumulation point in } (0,+\infty). \tag{1.12}$$

Moreover we assume that Ω, S_{in} and S_{out} satisfy the following conditions.

(i) **Case:** $d = 2$.
It is required that $\partial\Omega$ is composed of a finite number of smooth closed contours. In this case, we choose $S_{\text{in}} = S_{\text{out}} := \gamma$, where γ is any arbitrary non-empty relatively open subset of $\partial\Omega$, and the set of admissible unknown functions reads

$$\mathcal{E}_2 := \big\{(\alpha,\rho,q);\ \alpha \in W^{1,r}(\Omega) \text{ and } \rho \in W^{1,r}(\Omega) \text{ fulfill (1.3) and } \\ q \in W^{1,r}(\Omega;\mathbb{R}_+) \text{ with } r \in (2,+\infty)\big\}.$$

(ii) **Case:** $d \geqslant 3$.
We choose $x_0 \in \mathbb{R}^d$ outside the convex hull of $\overline{\Omega}$. Then we assume that

$$\{x \in \partial\Omega;\ (x - x_0)\cdot\nu \geqslant 0\} \subset S_{\text{in}} \text{ and } \{x \in \partial\Omega;\ (x - x_0)\cdot\nu \leqslant 0\} \subset S_{\text{out}}.$$

Furthermore we define the set of admissible unknown functions by

$$\mathcal{E}_d := \{(\alpha,\rho,q);\ \alpha \in L^\infty(\Omega) \text{ and } \rho \in L^\infty(\Omega) \text{ fulfill (1.3) and } q \in L^\infty(\Omega;\mathbb{R}_+)\}.$$

The uniqueness result for our inverse coefficients problem is as follows.

Theorem 1.2. *Let t_n, $n \in \mathbb{N}$ fulfill* (1.12) *and assume that either* (*i*) *or* (*ii*) *is satisfied. If*

$$\mathcal{N}_{\alpha_1,\rho_1,q_1}(t_n) = \mathcal{N}_{\alpha_2,\rho_2,q_2}(t_n),\ n \in \mathbb{N}, \quad (\alpha_j, \rho_j, q_j) \in \mathcal{E}_d,\ j = 1, 2, \tag{1.13}$$

then we have $(\alpha_1, \rho_1, q_1) = (\alpha_2, \rho_2, q_2)$.

We stress out that the statement of Theorem 1.2 remains valid upon replacing the polynomial time-varying part $t \mapsto t^k$ of the boundary condition in (1.10) by any function $t \mapsto h(t)$ that is analytic in $(0, +\infty)$, and whose Laplace transform, defined for all $p \in \mathbb{C}_+ := \{z \in \mathbb{C};\ \Re z > 0\}$ by $H(p) := \int_0^{+\infty} e^{-pt} h(t)\, dt$, admits a holomorphic extension to $\mathbb{C} \setminus \mathbb{R}_-$, still denoted by H, fulfilling the following condition:

$$\exists C > 0,\ \exists \varepsilon > 0,\ |H(z)| \leq C\, |z|^{-(2+\varepsilon)},\ z \in \mathbb{C} \setminus \mathbb{R}_-.$$

This can be checked from the proof of Theorem 1.2, displayed in Section 3.2.

1.5. Comments and outline

As was already mentioned above, the present paper is, to our best knowledge, the first mathematical work dealing with existence and uniqueness results of a weak solution to time-fractional Cauchy systems of space-dependent order. Such a weak solution is defined by Definition 2.2, which was inspired by [24, Definition 1.1]. In the particular case where the mapping $x \mapsto \alpha(x)$ is constant in Ω, it turns out that these two definitions are equivalent (see also Remark 1). Moreover, it can be checked that Definition 2.2 is equivalent to [48, Definition 4.1]. Nevertheless, we believe that the formulation of Definition 2.2, arising from the Laplace decomposition of (1.4), is more natural than the one of [48, Definition 4.1], which is based on an abstract integral equation. Finally, we point out that Definition 2.2 applies without change to DO time-fractional diffusion equations (see [29, Definition 1.1]).

In Theorem 1.1 we prove existence and uniqueness of the weak solution to (1.4), by applying the Brownwich-Mellin formula to the Laplace transform of the expected solution. This step requires that the non self-adjoint operators $\rho^{-1} A_q + p^\alpha$, for $p \in \mathbb{C} \setminus \mathbb{R}_-$, be boundedly invertible in $L^2(\Omega; \rho dx)$ and that their resolvent operator be appropriately estimated as in Proposition 2.1. This technical estimation is the main difference with the analysis carried out for CO or DO time-fractional Cauchy systems in e.g., [29, 43]. Indeed, CO (resp., DO) time-fractional diffusion equations with time-independent elliptic operator $\rho^{-1} A_q$ decompose into a family of CO (resp. DO) time-fractional ordinary differential equations, obtained by projection onto the eigenspaces of $\rho^{-1} A_q$, which are explicitly solvable in terms of the Mittag-Leffler functions (resp., suitable special functions associated with the density function appearing in the DO), see e.g., [43, Theorems 2.1 and 2.2] (resp., [29, Proposition 2.1]). Obviously, this is no longer true when $x \mapsto \alpha(x)$ is non-constant, which makes the analysis of the well-posedness of VO time-fractional Cauchy problems technically more challenging than the one of CO or DO time-fractional Cauchy systems.

The method carried out in this paper for the analysis of the forward problem under investigation, is powerful enough to show that the weak solution is analytic with respect to the time variable, which is a cornerstone in the derivation of unique identification result stated in Theorem 1.2. This is all the more remarkable given that the strategy developed in [50] for building variational solutions to CO time-fractional diffusion equations does not seem to work for VO time-fractional Cauchy problems, and that the integro-differential approach implemented in [3] is not applicable to time-fractional Cauchy problems with a general elliptic operator defined on a bounded domain, such as $\rho^{-1} A_q$.

Notice from (1.7) that the weak solution to (1.4) is not represented by the classical Duhamel formula, as the right hand side of (1.7) is the superposition of three terms. The two first terms are the usual expressions arising from the initial state u_0 and the forcing term f, and they are similar the ones appearing in the representation of the weak solution to CO or DO time-fractional Cauchy systems, see [43, Theorems 2.1 and 2.2] and [29, Proposition 2.1]. In the case of VO time-fractional Cauchy problems, there is an additional term (denoted by $S_2 f$ in (1.7)) appearing in the representation formula of the weak solution. As already noted in Remark 1, this additional term is uniformly zero when $x \mapsto \alpha(x)$ is constant in Ω. Moreover, we can also prove that the operator S_2 vanishes for all fractional orders $\alpha : \Omega \to [\alpha_0, \alpha_M] \subset (0,1)$ such that $\alpha_0 \in \left(0, \frac{1}{2}\right)$ and $\alpha_M \in (\alpha_0, 2\alpha_0)$. This suggests that the occurence of a non-vanishing operator S_2 in (1.7) is tied to a sufficiently large magnitude of variation of α, indicating that the non-standard Duhamel formula (1.7) is an effect of the x-dependency of the fractional order.

The paper is organized as follows. In Section 2, we discuss the well-posedness of the initial-boundary value problem (1.4). More precisely, the weak solution to the VO time-fractional diffusion equation appearing in (1.4), is defined in Section 2.2, and Section 2.3 proves Theorem 1.1, which is by means of a technical resolvent estimate of the elliptic part of the diffusion equation given in Section 2.1. The proof of the statement of Remark 1 can be found in Section 2.4. The analysis of the inverse problem of identifying the three unknown functions α, ρ and q in the first line of (1.4) by partial boundary data is carried out in Section 3. That is, the partial boundary operators (1.11) are rigorously defined in Section 3.1, and Section 3.2 provides the proof of Theorem 1.2.

2. Analysis of the forward problem

2.1. Elliptic operator: self-adjointness and resolvent estimate

Let A_0 be the operator generated by the quadratic form

$$a_0(u) := \sum_{i,j=1}^{d} \int_\Omega a_{i,j}(x)\partial_{x_i}u(x)\partial_{x_j}u(x)dx,\ u \in H_0^1(\Omega).$$

Since there exists a constant $\tilde{c}_0 > 0$ such that

$$a_0(u) \geqslant c_0 \|\nabla u\|_{L^2(\Omega)^d}^2 \geqslant \tilde{c}_0 \|u\|_{H^1(\Omega)}^2,\ u \in H_0^1(\Omega), \tag{2.1}$$

by (1.1) and the Poincaré inequality, the operator A_0 is self-adjoint in $L^2(\Omega)$ and acts as $\mathcal{A}_0$ on its dense domain $D(A_0)$ in $L^2(\Omega)$, that is, $A_0 u = \mathcal{A}_0 u$ for all $u \in D(A_0)$.

Put $r := 2\kappa/(\kappa - 2)$ and notice from the Hölder inequality that

$$\|qu\|_{L^2(\Omega)} \leqslant \|q\|_{L^\kappa(\Omega)} \|u\|_{L^r(\Omega)},\ u \in L^r(\Omega). \tag{2.2}$$

Furthermore we have $H^1(\Omega) = W^{1,2}(\Omega) \subset W^{r_0,r}(\Omega)$ with $r_0 := 1 - d/\kappa \in (0,1)$ by the Sobolev embedding theorem (e.g., [16, Theorem 1.4.4.1]), and the embedding is continuous:

$$\exists c > 0,\ \|u\|_{W^{r_0,r}(\Omega)} \leqslant c\|u\|_{H^1(\Omega)},\ u \in H^1(\Omega). \tag{2.3}$$

Therefore, by (2.2)-(2.3), we have $\|qu\|_{L^2(\Omega)} \leqslant c\|q\|_{L^\kappa(\Omega)}\|u\|_{H^1(\Omega)}$ for every $u \in H^1(\Omega)$, and so it follows from (2.1) that

$$\begin{aligned}\|qu\|_{L^2(\Omega)}^2 &\leqslant \frac{c^2\|q\|_{L^\kappa(\Omega)}^2}{\tilde{c}_0^2} \langle A_0 u, u\rangle_{L^2(\Omega)} \\ &\leqslant \frac{c^2\|q\|_{L^\kappa(\Omega)}^2}{2\tilde{c}_0^2} \left(\varepsilon \|A_0 u\|_{L^2(\Omega)}^2 + \varepsilon^{-1}\|u\|_{L^2(\Omega)}^2\right),\ u \in D(A_0),\ \varepsilon \in (0,+\infty).\end{aligned}$$

Thus, taking $\varepsilon > 0$ so small that $\varepsilon c^2 \|q\|_{L^\kappa(\Omega)}^2 < 2\tilde{c}_0^2$, we see that the multiplier by q in $L^2(\Omega)$ is A_0-bounded with relative bound zero. As a consequence, $A_q := A_0 + q$ is self-adjoint in $L^2(\Omega)$ with domain $D(A_q) = D(A_0)$ by the Kato-Rellich theorem (see e.g., [21, Theorem V.4.3], [39, Theorem X.12]). Moreover A_q acts as $\mathcal{A}_q$ on $D(A_q) = D(A_0)$.

In this article, we suppose (1.2) in such a way that $A_q \geqslant \tilde{c}_0$ in the operator sense, where $\tilde{c}_0$ is the constant appearing in (2.1). This hypothesis is quite convenient for proving Proposition 2.1 below stated, which is essential for the proof of Theorem 1.1 and Proposition 3.1, but it could be removed at the price of greater unessential technical difficulties. Nevertheless, for simplicity, we shall not go further into this direction.

In what follows $\mathcal{B}(X,Y)$ denotes the Banach space of all the bounded linear operators from a Banach space X to another Banach space Y, and we set $\mathcal{B}(X) := \mathcal{B}(X,X)$.

Proposition 2.1. *For all $p \in \mathbb{C} \setminus \mathbb{R}_-$, the operator $A_q + \rho(x)p^{\alpha(x)}$ is boundedly invertible in $L^2(\Omega)$ and $(A_q + \rho(x)p^{\alpha(x)})^{-1}$ maps $L^2(\Omega)$ into $D(A_0)$.*

Moreover, we have

$$\left\| (A_q + \rho(x) r^{\alpha(x)} e^{i\beta\alpha(x)})^{-1} \right\|_{\mathcal{B}(L^2(\Omega))} \leqslant C(r,\beta) \max_{j=0,M} r^{-\alpha_j},\ r \in (0,+\infty),\ \beta \in (-\pi,\pi) \quad (2.4)$$

with

$$C(r,\beta) := \begin{cases} 2\rho_0^{-1}, & \text{if } |\beta| \leqslant \theta_*(r), \\ \rho_0^{-1} c_*(\beta), & \text{otherwise}, \end{cases} \quad (2.5)$$

and

$$\theta_*(r) := \alpha_M^{-1} \min_{\sigma=\pm 1} \arctan\left(\frac{\rho_0}{3\rho_M} r^{\sigma(\alpha_M - \alpha_0)} \right),\ c_*(\beta) := \max_{j=0,M} |\sin(\alpha_j \beta)|^{-1}. \quad (2.6)$$

Furthermore the mapping $p \mapsto (A_q + \rho(x) p^{\alpha(x)})^{-1}$ *is bounded holomorphic in* $\mathbb{C} \setminus \mathbb{R}_-$ *as operator with values in* $\mathcal{B}(L^2(\Omega))$.

Proof. 1) Let us first establish (2.4). We shall do it only for $r \in [1,+\infty)$, because the corresponding estimate for $r \in (0,1)$ can be derived in the same way. We examine the two cases $|\beta| \in (\theta_*(r),\pi)$ and $|\beta| \in (0,\theta_*(r)]$ separately.

a) We start by assuming that $\beta \in (0,\beta_*(r))$, the case of $\beta \in (-\pi,-\beta_*(r))$ being similarly treated. For $p = re^{i\beta}$, let $B_{q,p}$ be the self-adjoint part of the operator $A_q + \rho(x) p^{\alpha(x)}$, i.e.,

$$B_{q,p} := A_q + \rho(x) r^{\alpha(x)} \cos(\beta\alpha(x)). \quad (2.7)$$

Since the multiplication operator by $\rho(x) r^{\alpha(x)} \cos(\beta\alpha(x))$ is bounded (by $\rho_M r^{\alpha_M}$) in $L^2(\Omega)$, the Kato-Rellich theorem ensures us that $B_{q,p}$ is self-adjoint in $L^2(\Omega)$, with domain $D(B_{q,p}) = D(A_q) = D(A_0)$.

Next we define the multiplication operator U_β in $L^2(\Omega)$ by $U_\beta f(x) = u_\beta(x) f(x)$ for all $f \in L^2(\Omega)$, where

$$u_\beta(x) := \Big(\rho(x) r^{\alpha(x)} \sin(\beta\alpha(x)) \Big)^{1/2},\ x \in \Omega. \quad (2.8)$$

Evidently iU_β^2 is the skew-adjoint part of the operator $A_q + \rho(x) p^{\alpha(x)}$, so we have

$$A_q + \rho(x) p^{\alpha(x)} = B_{q,p} + iU_\beta^2, \quad (2.9)$$

in virtue of (2.7). Further, putting $m_\beta := \min_{j=0,M} \sin(\alpha_j \beta)$, we get that

$$0 < \rho_0^{1/2} m_\beta^{1/2} r^{\alpha_0/2} \leqslant u_\beta(x) \leqslant \rho_M^{1/2} r^{\alpha_M/2},\ x \in \Omega.$$

Therefore the self-adjoint operator U_β is bounded and boundedly invertible in $L^2(\Omega)$, with

$$\|U_\beta^{-1}\|_{\mathcal{B}(L^2(\Omega))} \leqslant \rho_0^{-1/2} m_\beta^{-1/2} r^{-\alpha_0/2}. \quad (2.10)$$

Moreover, since U_β^{-1}, self-adjoint and bounded in $L^2(\Omega)$, has a closed image and finite dimensional (actually, trivial) kernel, then [18, Corollary 2] entails that the linear operator $U_\beta^{-1} B_{q,p} U_\beta^{-1}$, endowed with domain $D(U_\beta^{-1} B_{q,p} U_\beta^{-1})$

$:= U_\beta D(A_0)$, is self-adjoint[1] in $L^2(\Omega)$. Its spectrum $\sigma(U_\beta^{-1} B_{q,p} U_\beta^{-1})$ is thus real, i.e. $\sigma(U_\beta^{-1} B_{q,p} U_\beta^{-1}) \subset \mathbb{R}$, so the operator $U_\beta^{-1} B_{q,p} U_\beta^{-1} + i$ is invertible in $L^2(\Omega)$ and we have

$$\|(U_\beta^{-1} B_{q,p} U_\beta^{-1} + i)^{-1}\|_{\mathcal{B}(L^2(\Omega))} \leqslant \operatorname{dist}\left(-i, \sigma(U_\beta^{-1} B_{q,p} U_\beta^{-1})\right)^{-1} \leqslant 1, \quad (2.11)$$

by [21, Section V.3.5, Eq. (3.16)]. Now, recalling (2.9), we get that $A_q + \rho(x)p^{\alpha(x)} = U_\beta \left(U_\beta^{-1} B_{q,p} U_\beta^{-1} + i\right) U_\beta$, hence $A_q + \rho(x)p^{\alpha(x)}$ is invertible in $L^2(\Omega)$, with

$$(A_q + \rho(x)p^{\alpha(x)})^{-1} = U_\beta^{-1}(U_\beta^{-1} B_{q,p} U_\beta^{-1} + i)^{-1} U_\beta^{-1}.$$

As a consequence, $(A_q+\rho(x)p^{\alpha(x)})^{-1}$ maps $L^2(\Omega)$ into $U_\beta^{-1} D(U_\beta^{-1} B_{q,p} U_\beta^{-1}) = D(A_0)$ and we infer from (2.10)-(2.11) that

$$\begin{aligned} &\|(A_q + \rho(x)p^{\alpha(x)})^{-1}\|_{\mathcal{B}(L^2(\Omega))} \\ &\qquad \leqslant \|(U_\beta^{-1} B_{q,p} U_\beta^{-1} + i)^{-1}\|_{\mathcal{B}(L^2(\Omega))} \|U_\beta^{-1}\|^2_{\mathcal{B}(L^2(\Omega))} \leqslant \rho_0^{-1} m_\beta^{-1} r^{-\alpha_0}, \end{aligned}$$

which is the claim of (2.4) for $|\beta| \in (\theta_*(r), \pi)$.

b) Let us now address the case of $|\beta| \leq \theta_*(r)$. Since $r \in [1, +\infty)$ by assumption, it holds true that $\alpha_M \theta_*(r) = \arctan(\rho_0 \rho_M^{-1} r^{-(\alpha_M - \alpha_0)}/3) \in (0, \pi/6)$. Therefore we have simultaneously $\cos(\alpha_M \theta_*(r))/3 = \rho_0^{-1} \rho_M r^{\alpha_M - \alpha_0} \times \sin(\alpha_M \theta_*(r))$ and $2\cos(\alpha_M \theta_*(r))/3 > 1/\sqrt{3} > 1/2$, whence

$$\cos(\alpha_M \theta_*(r)) > \frac{1}{2} + \rho_0^{-1} \rho_M r^{\alpha_M - \alpha_0} \sin(\alpha_M \theta_*(r)). \quad (2.12)$$

Next, as $\alpha(x)|\beta| \leqslant \alpha_M \theta_*(r) < \pi/2$, we have $\cos(\alpha(x)\beta) \geqslant \cos(\alpha_M \theta_*(r)) > 0$. From this, (1.1)-(1.2) and (2.10), we get that $B_{q,p} \geqslant \rho_0 r^{\alpha_0} \cos(\alpha_M \theta_*(r))$ in the operator sense. Therefore $B_{q,p}$ is boundedly invertible in $L^2(\Omega)$ and

$$\|B_{q,p}^{-1}\|_{\mathcal{B}(L^2(\Omega))} \leqslant \frac{\rho_0^{-1} r^{-\alpha_0}}{\cos(\alpha_M \theta_*(r))}. \quad (2.13)$$

Similarly, since $|\sin(\alpha(x)\beta)| \leqslant \sin(\alpha_M \theta_*(r))$, we infer from (2.8) that

$$\|U_\beta\|_{\mathcal{B}(L^2(\Omega))} \leqslant \rho_M^{1/2} r^{\alpha_M/2} \sin(\alpha_M \theta_*(r))^{1/2},$$

where we recall that U_β is the multiplication operator in $L^2(\Omega)$ by the function u_β defined in (2.8). This and (2.13) yield

$$\begin{aligned} \|B_{q,p}^{-1} U_\beta^2\|_{\mathcal{B}(L^2(\Omega))} &\leqslant \|B_{q,p}^{-1}\|_{\mathcal{B}(L^2(\Omega))} \|U_\beta\|^2_{\mathcal{B}(L^2(\Omega))} \\ &\leqslant \rho_0^{-1} \rho_M r^{\alpha_M - \alpha_0} \tan(\alpha_M \theta_*(r)) < 1. \end{aligned}$$

[1] This can be directly deduced from the self-adjointness of the operators $U_\beta^{\pm 1}$ and $B_{q,p}$, and from the boundedness of $U_\beta^{\pm 1}$, in $L^2(\Omega)$. But, in order to avoid the inadequate expense of the size of this article, we rather establish this claim by invoking a stronger result established by S. S. Holland in [18].

Furthermore, bearing in mind that $A_q+\rho(x)p^{\alpha(x)}=B_{q,p}(I+iB_{q,p}^{-1}U_\beta^2)$, where I denotes the identity operator in $L^2(\Omega)$, we find that $A_q+\rho(x)p^{\alpha(x)}$ is invertible in $L^2(\Omega)$, with

$$\begin{aligned}\|(A_q+\rho(x)p^{\alpha(x)})^{-1}\|_{\mathcal{B}(L^2(\Omega))} &\leqslant \frac{\|B_{q,p}^{-1}\|_{\mathcal{B}(L^2(\Omega))}}{1-\|B_{q,p}^{-1}U_\beta^2\|_{\mathcal{B}(L^2(\Omega))}}\\ &\leqslant \frac{\rho_0^{-1}r^{-\alpha_0}}{\cos(\alpha_M\theta_*(r))-\rho_0^{-1}\rho_M r^{\alpha_M-\alpha_0}\sin(\alpha_M\theta_*(r))}.\end{aligned}$$

From this and (2.12) it then follows that $\|(A_q+\rho(x)p^{\alpha(x)})^{-1}\|_{\mathcal{B}(L^2(\Omega))}<2\rho_0^{-1}r^{-\alpha_0}$, which entails (2.4) when $|\beta|\le\theta_*(r)$.

2) We turn now to proving that $p\mapsto(A_q+\rho(x)p^{\alpha(x)})^{-1}$ is bounded holomorphic in $\mathbb{C}\setminus\mathbb{R}_-$. To this purpose, we introduce the closed sequilinear form

$$a_{q,p}(u):=a_0(u)+\int_\Omega(q(x)+\rho(x)p^{\alpha(x)})|u(x)|^2dx,\ u\in H_0^1(\Omega),$$

which is associated with the operator $A_q+\rho(x)p^{\alpha(x)}$ in $L^2(\Omega)$. In light of (1.2) and (2.1), we have

$$\Re a_{q,p}(u)\geqslant\left(\tilde{c}_0-\rho_M\max_{j=0,M}|p|^{\alpha_j}\right)\|u\|_{L^2(\Omega)}^2$$

and

$$\Im a_{q,p}(u)\leqslant\left(\rho_M\max_{j=0,M}|p|^{\alpha_j}\right)\|u\|_{L^2(\Omega)}^2$$

for all $u\in H_0^1(\Omega)$, involving that $a_{q,p}$ is sectorial for every $p\in\mathbb{C}\setminus\mathbb{R}_-$. Here and henceforth $\Re$ and $\Im$ mean the real part and the imaginary part of a complex number under consideration, respectively.

Moreover, since $p\mapsto a_{q,p}(u)$ is holomorphic in $\mathbb{C}\setminus\mathbb{R}_-$, $\{a_{q,p};\ p\in\mathbb{C}\setminus\mathbb{R}_-\}$ is an analytic family of sesquilinear forms of type (a) in the sense of Kato (see [21, Section VII.4.2]). Thus [21, Theorem VII.4.2] yields that $\{A_q+\rho(x)p^{\alpha(x)};\ p\in\mathbb{C}\setminus\mathbb{R}_-\}$ is an analytic family of operators. Therefore $p\mapsto(A_q+\rho(x)p^{\alpha(x)})^{-1}$ is holomorphic in $\mathbb{C}\setminus\mathbb{R}_-$, by [21, Theorem VII.1.3]. And the proof of Proposition 2.1 is complete. □

We point out that $\theta_*(r)$ behaves likes $\min_{\sigma=\pm1}r^{\sigma(\alpha_M-\alpha_0)}$ as r becomes either sufficiently small or sufficiently large (that is, like $r^{-(\alpha_M-\alpha_0)}$ as $r\to0$, and like $r^{(\alpha_M-\alpha_0)}$ as $r\to+\infty$). Indeed, bearing in mind that $\arctan u=\int_0^u\frac{dv}{1+v^2}$ for all $u\in[0,+\infty)$, we see that $\arctan u\in\left[\frac{u}{1+u^2},u\right]$, and so we infer from (2.6) that

$$\frac{\frac{\rho_0}{3\rho_M}\min_{\sigma=\pm1}r^{\sigma(\alpha_M-\alpha_0)}}{1+\frac{\rho_0^2}{9\rho_M^2}\min_{\sigma=\pm1}r^{2\sigma(\alpha_M-\alpha_0)}}\leqslant\alpha_M\theta_*(r)\leqslant\frac{\rho_0}{3\rho_M}\min_{\sigma=\pm1}r^{\sigma(\alpha_M-\alpha_0)},\ r\in(0,+\infty).$$

Since $\min_{\sigma=\pm 1} r^{\sigma(\alpha_M-\alpha_0)} \in (0,1]$, the denominator of the left-hand side of the above inequality is majorized by $1+\frac{\rho_0^2}{9\rho_M^2} \leqslant \frac{10}{9}$, so that we have

$$\frac{3\rho_0}{10\rho_M} \min_{\sigma=\pm 1} r^{\sigma(\alpha_M-\alpha_0)} \leqslant \alpha_M \theta_*(r) \leqslant \frac{\rho_0}{3\rho_M} \min_{\sigma=\pm 1} r^{\sigma(\alpha_M-\alpha_0)},\ r \in (0,+\infty).$$

Therefore it follows readily from (2.5)-(2.6) and the inequality $\sin u \geqslant \frac{u}{2}$ for all $u \in [0,1]$ that

$$C(r,\beta) \leqslant \rho_0^{-1} \left(\sin \left(\frac{3\alpha_0\rho_0}{10\alpha_M\rho_M} \min_{\sigma=\pm 1} r^{\sigma(\alpha_M-\alpha_0)} \right) \right)^{-1} \leqslant \frac{20\alpha_M\rho_M}{3\alpha_0\rho_0^2} \max_{\sigma=\pm 1} r^{\sigma(\alpha_M-\alpha_0)},$$

when r is sufficiently close to either 0 or $+\infty$. As a consequence, there exists a constant $C>0$, which is independent of r and β such that we have

$$C(r,\beta) \leqslant C \max_{\sigma=\pm 1} r^{\sigma(\alpha_M-\alpha_0)},\ r \in (0,+\infty),\ \beta \in (-\pi,\pi). \tag{2.14}$$

2.2. Weak solution

Let $\mathcal{S}'(\mathbb{R};L^2(\Omega))$ be the space dual to $\mathcal{S}(\mathbb{R};L^2(\Omega))$. We denote by $\mathcal{S}'(\mathbb{R}_+;L^2(\Omega)) := \{v \in \mathcal{S}'(\mathbb{R};L^2(\Omega));\ \mathrm{supp}\, v \subset [0,+\infty) \times \overline{\Omega}\}$ the set of distributions in $\mathcal{S}'(\mathbb{R};L^2(\Omega))$ supported in $[0,+\infty) \times \overline{\Omega}$. Otherwise stated, $v \in \mathcal{S}'(\mathbb{R};L^2(\Omega))$ lies in $\mathcal{S}'(\mathbb{R}_+;L^2(\Omega))$ if and only if $\langle v,\varphi\rangle_{\mathcal{S}'(\mathbb{R};L^2(\Omega)),\mathcal{S}(\mathbb{R};L^2(\Omega))} = 0$ whenever $\varphi \in \mathcal{S}(\mathbb{R};L^2(\Omega))$ vanishes in $\mathbb{R}_+ \times \overline{\Omega}$. As a consequence, for a.e. $x \in \Omega$, we have

$$\langle v(\cdot,x),\varphi\rangle_{\mathcal{S}'(\mathbb{R}),\mathcal{S}(\mathbb{R})} = \langle v(\cdot,x),\psi\rangle_{\mathcal{S}'(\mathbb{R}),\mathcal{S}(\mathbb{R})},\quad \varphi,\psi \in \mathcal{S}(\mathbb{R}), \tag{2.15}$$

provided $\varphi=\psi$ in $\mathbb{R}_+$. Furthermore we say that $\varphi \in \mathcal{S}(\mathbb{R}_+)$ if φ is the restriction to $\mathbb{R}_+$ of a function $\tilde{\varphi} \in \mathcal{S}(\mathbb{R})$. Then we set

$$x \mapsto \langle v(\cdot,x),\varphi\rangle_{\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+)} := x \mapsto \langle v(\cdot,x),\tilde{\varphi}\rangle_{\mathcal{S}'(\mathbb{R}),\mathcal{S}(\mathbb{R})},\ v \in \mathcal{S}'(\mathbb{R}_+;L^2(\Omega)). \tag{2.16}$$

Notice from (2.15) that $\tilde{\varphi}$ may be any function in $\mathcal{S}(\mathbb{R})$ such that $\tilde{\varphi}(t)=\varphi(t)$ for all $t \in \mathbb{R}_+$.

For $p \in \mathbb{C}_+$, we put

$$e_p(t) := \exp(-pt),\ t \in \mathbb{R}_+.$$

Evidently, $e_p \in \mathcal{S}(\mathbb{R}_+)$. For $v \in \mathcal{S}'(\mathbb{R}_+;L^2(\Omega))$, we define the Laplace transform $\mathcal{L}[v]$ in t of v by

$$\mathcal{L}[v](p) := x \mapsto \langle v(\cdot,x),e_p\rangle_{\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+)},\ p \in \mathbb{C}_+,$$

and notice that $p \mapsto \mathcal{L}[v](p) \in \mathcal{C}^\infty(\mathbb{C}_+;L^2(\Omega))$. Having seen this, we define the weak solution to (1.4) as follows.

Definition 2.2. *Let $u_0 \in L^2(\Omega)$. For $T<+\infty$, we assume that $f \in L^1(0,T;L^2(\Omega))$ and, for $T=+\infty$, we assume that there exists $m \in \mathbb{N}$ such that $(1+t)^{-m} f \in L^1(\mathbb{R}_+;L^2(\Omega))$. We say that u is a weak solution to (1.4) if u is the restriction to Q of a distribution $v \in \mathcal{S}'(\mathbb{R}_+;L^2(\Omega))$ and the Laplace transform $V := \mathcal{L}[v]$ verifies*

$$V(p) = \left(A_q + \rho(x)p^{\alpha(x)}\right)^{-1} \left(F(p) + \rho(x)p^{\alpha(x)-1}u_0\right),\ p \in (0,+\infty). \tag{2.17}$$

Here $F(p) := \mathcal{L}[f(t,\cdot)1_{(0,T)}(t)](p) = \int_0^T e^{-pt} f(t,\cdot)dt$, *where* 1_I *denotes the characteristic function of a set* $I \subset \mathbb{R}$.

Remark 2. *Notice from* (2.17) *and Proposition* 2.1 *that* $V(p) \in D(A_q) = D(A_0) \subset H_0^1(\Omega)$ *for all* $p \in (0,+\infty)$, *which entails that* $V(p) = 0$ *on* $\partial\Omega$. *Actually it is clear that* (2.17) *can be equivalently replaced by the condition*

$$\begin{cases} (\mathcal{A}_q + \rho(x)p^{\alpha(x)})V(p) & = & F(p) + \rho(x)p^{\alpha(x)-1}u_0, & \textit{in } \Omega, \\ V(p) & = & 0, & \textit{on } \partial\Omega \end{cases}$$

for all $p \in (0,+\infty)$.

Remark 3. *For all* $h \in \mathcal{C}^1(\mathbb{R}^+)$ *such that*

$$\varepsilon_0 := \inf\left\{\varepsilon \in (0,+\infty);\ e^{-\varepsilon t}\frac{d^k h(t)}{dt^k} \in L^1(\mathbb{R}^+),\ k = 0,1\right\} \in \mathbb{R}_+,$$

we know from [38, Eq. (2.140)] *that*

$$\mathcal{L}[\partial^{\alpha(x)}h](p) = p^{\alpha(x)}H(p) - p^{\alpha(x)-1}h(0),\ p \in (\varepsilon_0,+\infty),$$

where $H(p) := \mathcal{L}[h](p) = \int_0^{+\infty} e^{-pt}h(t)dt$. *Therefore, in the particular case where the mapping* $x \mapsto \alpha(x)$ *is constant, we infer from* [43, Theorems 2.1 and 2.2] *that the initial-boundary value problem* (1.4) *admits a unique weak solution to* (1.4) *in the sense of Definition* 2.2, *provided* u_0 *and* f *are sufficiently smooth.*

2.3. Proof of Theorem 1.1

Since the first line of (1.4) is a linear PDE, we may invoke the superposition principle and divide the proof into two independent parts: In the first one, which is concerned with S_0, we assume that $f = 0$, while in the second one, dealing with the operators S_1 and S_2, we take $u_0 = 0$ in (1.4).

2.3.1. Case $f = 0$.

The construction of S_0 is rather lengthy and consists of a succession of three lemmas. Prior to starting the proof we put

$$W(p) := p^{-2}(A_q + \rho(x)p^{\alpha(x)})^{-1}\rho(x)p^{\alpha(x)-1}u_0,\ p \in \mathbb{C} \setminus \mathbb{R}_-,$$

and then state the first lemma as follows.

Lemma 2.3. *The function*

$$w(t) := \frac{1}{2i\pi}\int_{1-i\infty}^{1+i\infty} e^{tp}W(p)dp,$$

is well defined for all $t \in \mathbb{R}$. *Moerover it satisfies:*

$$\mathcal{L}[w](p) = W(p),\ p \in \{z \in \mathbb{C};\ \Re z \in (1,+\infty)\}. \tag{2.18}$$

Proof. For $\mu \in [1,+\infty)$, we infer from (2.4) that

$$\|W(\mu + i\eta)\|_{L^2(\Omega)} \leqslant C(r,\beta)\rho_M|\mu + i\eta|^{-3+\alpha_M-\alpha_0},\ \eta \in \mathbb{R}, \tag{2.19}$$

where $C(r,\beta)$ is given by (2.5)-(2.6) with $r = |\mu + i\eta| \in [1,+\infty)$ and $\beta = \arg(\mu + i\eta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. According to (2.14), there exists a constant $C = C(\alpha_0, \alpha_M, \rho_0, \rho_M)$ such that

$$C(r,\beta) \leqslant C|\mu + i\eta|^{\alpha_M - \alpha_0}, \quad \mu \in [1,+\infty),\ \eta \in \mathbb{R}.$$

Thus (2.19) yields

$$\|W(\mu + i\eta)\|_{L^2(\Omega)} \leqslant C\langle\eta\rangle^{-3+2(\alpha_M-\alpha_0)},\ \mu \in [1,+\infty),\ \eta \in \mathbb{R}, \tag{2.20}$$

upon substituting C for $\rho_M C$. As a consequence, we have for each $k = 1, 2$,

$$\begin{aligned} C_k &:= \sup_{\mu\in[1,+\infty)} \|W(\mu + i\cdot)\|_{L^k(\mathbb{R};L^2(\Omega))} \\ &= \sup_{\mu\in[1,+\infty)} \left(\int_{\mathbb{R}} \|W(\mu + i\eta)\|_{L^2(\Omega)}^k\, d\eta\right)^{\frac{1}{k}} < \infty, \end{aligned} \tag{2.21}$$

and hence

$$\omega(t) := \frac{1}{2i\pi}\int_{-i\infty}^{i\infty} e^{tp}W(p+1)dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it\eta}W(1+i\eta)d\eta \tag{2.22}$$

is well defined for all $t \in \mathbb{R}$. Moreover the mapping $p \mapsto e^{tp}W(p+1)$ is holomorphic in $\mathbb{C} \setminus (-\infty,-1]$ by Proposition 2.1, and so we infer from the Cauchy formula that

$$\omega(t) = \frac{1}{2i\pi}\int_{s-i\infty}^{s+i\infty} e^{tp}W(p+1)dp,\ s \in (0,+\infty). \tag{2.23}$$

Indeed, for all $R \in (1,+\infty)$ and $s \in (0,+\infty)$, we have

$$\begin{aligned} &\int_{s-iR}^{s+iR} e^{tp}W(p+1)dp - \int_{-iR}^{iR} e^{tp}W(p+1)dp \\ &\qquad = \sum_{\sigma=\pm1} \sigma \int_0^s e^{t(\mu+i\sigma R)}W(\mu+1+i\sigma R)d\mu \end{aligned} \tag{2.24}$$

from the Cauchy formula, and

$$\begin{aligned} &\left\|\int_0^s e^{t(\mu+i\sigma R)}W(\mu+1+i\sigma R)d\mu\right\|_{L^2(\Omega)} \\ &\qquad \leqslant Cs\max(1,e^{st})\langle R\rangle^{-3+2(\alpha_M-\alpha_0)},\ \sigma = \pm1, \end{aligned}$$

by (2.20). Hence (2.23) follows by letting R to $+\infty$ in (2.24). Next, in view of (2.23), we obtain that

$$\begin{aligned} \|\omega(t)\|_{L^2(\Omega)} &= \frac{1}{2\pi}\left\|\int_{\mathbb{R}} e^{t(s+i\eta)}W(s+1+i\eta)d\eta\right\|_{L^2(\Omega)} \\ &\leqslant \frac{e^{st}}{2\pi}\sup_{\mu\in[1,+\infty)} \|W(\mu+i\cdot)\|_{L^1(\mathbb{R};L^2(\Omega))} \end{aligned}$$

for all $t \in \mathbb{R}$ and $s \in (0, +\infty)$, and consequently that

$$\|\omega(t)\|_{L^2(\Omega)} \leqslant \frac{C_1}{2\pi} e^{ts} \tag{2.25}$$

according to (2.21). Now, letting s to $+\infty$ on the right-hand side of (2.25), we have $\omega(t) = 0$, $t \in (-\infty, 0)$. Similarly, by letting s to 0 in (2.25), we find that $\|\omega(t)\|_{L^2(\Omega)} \leqslant \frac{C_1}{2\pi}$ for all $t \in [0, +\infty)$. Therefore, we have $\omega \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap \mathcal{S}'(\mathbb{R}_+; L^2(\Omega))$, and since $p \mapsto W(p+1)$ is holomorphic in $\mathbb{C}_+$, we infer from (2.21) with $k = 2$, (2.22), Theorem 19.2 and the following remark in [41] that $\mathcal{L}[\omega](p) = W(p+1)$ for all $p \in \mathbb{C}_+$. As a consequence, the function

$$w(t) = e^t \omega(t) = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} e^{t(p+1)} W(p+1) dp = \frac{1}{2i\pi} \int_{1-i\infty}^{1+i\infty} e^{tp} W(p) dp,\ t \in \mathbb{R}$$

verifies $\mathcal{L}[w](p) = \mathcal{L}[\omega](p-1) = W(p)$ for all $p \in \{z \in \mathbb{C};\ \Re z \in (1, +\infty)\}$. This establishes (2.18) and terminates the proof of Lemma 2.3. □

Further we deduce from (2.4)–(2.6) that

$$\|W(1+i\eta)\|_{L^2(\Omega)} \leqslant \rho_0^{-1} \max\left(2, c_*\left(\frac{\pi}{4}\right)\right) \langle \eta \rangle^{-3+\alpha_M-\alpha_0},\ \eta \in \mathbb{R} \setminus (-1, 1),$$

and from (2.21) with $k = 1$ that the mapping $\eta \mapsto (1+i\eta)W(1+i\eta) \in L^1(\mathbb{R}; L^2(\Omega))$. Therefore we have

$$y(t) := \partial_t w(t) = \frac{1}{2i\pi} \int_{1-i\infty}^{1+i\infty} e^{tp} p W(p) dp,\ t \in \mathbb{R} \tag{2.26}$$

by the very definition of w in Lemma 2.3, and

$$\mathcal{L}[y](p) = p\mathcal{L}[w](p) = pW(p),\ p \in \{z \in \mathbb{C};\ \Re z \in (1, +\infty)\} \tag{2.27}$$

from (2.18). Let us now collect several useful properties of y in the following:

Lemma 2.4. *For any $\varepsilon \in (0, 1)$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$, it holds true that*

$$y(t) = \frac{1}{2i\pi} \int_{\gamma(\varepsilon,\theta)} e^{tp} p W(p) dp,\ t \in \mathbb{R}_+, \tag{2.28}$$

where $\gamma(\varepsilon, \theta)$ is defined by (1.5)-(1.6). As a consequence we have $y \in \mathcal{S}'(\mathbb{R}_+; L^2(\Omega))$.

Proof. The proof of (2.28) boils down to (2.26) and the analyticity of the mapping $p \mapsto e^{tp} p W(p)$ in $\mathbb{C} \setminus \mathbb{R}_-$, arising from Proposition 2.1. This can be seen by applying the Cauchy formula upon taking advantage of the fact that

$$\lim_{\eta \to +\infty} \int_{\eta((\tan\theta)^{-1} \pm i)}^{1 \pm i\eta} e^{tp} p W(p) dp = 0,\ t \in \mathbb{R}_+$$

in $L^2(\Omega)$. Indeed, for any sufficiently large $\eta \in (1,+\infty)$ and all $t \in \mathbb{R}_+$, (2.4)–(2.6) yield

$$\begin{aligned}&\left\| \int_{\eta((\tan\theta)^{-1}\pm i)}^{1\pm i\eta} e^{tp} pW(p)dp \right\|_{L^2(\Omega)} \\ &= \left\| \int_{\eta(\tan\theta)^{-1}}^{1} e^{t(\mu\pm i\eta)}(\mu\pm i\eta)W(\mu\pm i\eta)d\mu \right\|_{L^2(\Omega)} \\ &\leqslant Ce^t(1-\eta(\tan\theta)^{-1})\eta^{-2+\alpha_M-\alpha_0}\|u_0\|_{L^2(\Omega)}\end{aligned}$$

for some positive constant C depending only on θ, α_0, α_M, ρ_0 and ρ_M.

We turn now to showing that $y \in \mathcal{S}'(\mathbb{R}_+; L^2(\Omega))$. We proceed by estimating the right-hand side of (2.28). First, by performing the change of variable $p = \varepsilon e^{i\beta}$ with $\beta \in (-\theta,\theta)$ in the integral $\int_{\gamma_0(\varepsilon,\theta)} e^{tp}pW(p)dp$, we derive from (2.4) and (2.14) that

$$\begin{aligned}&\left\| \int_{\gamma_0(\varepsilon,\theta)} e^{tp} pW(p)dp \right\|_{L^2(\Omega)} \\ &\leqslant \rho_M \left(\int_{-\theta}^{\theta} C(\varepsilon,\beta) e^{t\varepsilon\cos\beta} d\beta \right) \varepsilon^{-(1+\alpha_M-\alpha_0)} \|u_0\|_{L^2(\Omega)} \\ &\leqslant Ce^{t\varepsilon}\varepsilon^{-(1+2(\alpha_M-\alpha_0))}\|u_0\|_{L^2(\Omega)},\ t\in\mathbb{R}_+. \end{aligned} \tag{2.29}$$

Next, (2.5)-(2.6) yield the existence of a positive constant C_θ depending only on α_0, α_M, ρ_0, ρ_M, and θ, such that the estimate $C(r,\theta) \leqslant C_\theta$ holds uniformly in $r \in (0,+\infty)$. Then it follows from (2.4) that

$$\begin{aligned}&\left\| \int_{\gamma_\pm(\varepsilon,\theta)} e^{tp} pW(p)dp \right\|_{L^2(\Omega)} \\ &\leqslant \rho_M C_\theta \left(\int_\varepsilon^1 r^{-(2+\alpha_M-\alpha_0)}dr + \int_1^{+\infty} r^{-(2-(\alpha_M-\alpha_0))}dr \right) \|u_0\|_{L^2(\Omega)} \\ &\leqslant \frac{\rho_M C_\theta}{1+\alpha_M-\alpha_0} \left(\varepsilon^{-(1+\alpha_M-\alpha_0)} + \frac{2}{1-(\alpha_M-\alpha_0)} \right) \|u_0\|_{L^2(\Omega)}. \end{aligned} \tag{2.30}$$

Now, taking $\varepsilon = t^{-1} \in (0,1)$ in (2.29)-(2.30), we deduce from (1.5) and (2.28) that

$$\|y(t)\|_{L^2(\Omega)} \leqslant Ct^{1+2(\alpha_M-\alpha_0)}\|u_0\|_{L^2(\Omega)},\ t \in (1,+\infty) \tag{2.31}$$

for some positive constant C depending only on θ, α_j and ρ_j for $j = 0, M$. Similarly, by choosing $\varepsilon = 1/2$ in (2.29)-(2.30), we find that $\|y(t)\|_{L^2(\Omega)} \leqslant C\|u_0\|_{L^2(\Omega)}$ for all $t \in [0,1]$, where $C \in (0,+\infty)$ is independent of t. Therefore we have $t \mapsto \langle t\rangle^{-3}y(t) \in L^\infty(\mathbb{R}; L^2(\Omega))$, and consequently $y \in \mathcal{S}'(\mathbb{R}_+; L^2(\Omega))$. □

Both functions $p \mapsto \mathcal{L}[y](p)$ and $p \mapsto pW(p)$ are holomorphic in $\mathbb{C}_+$, and (2.18) entails that $\mathcal{L}[y](p) = pW(p)$ for all $p \in \mathbb{C}_+$, by the unique continuation. As a consequence, $v := \partial_t y \in \mathcal{S}'(\mathbb{R}_+; L^2(\Omega))$ satisfies $\mathcal{L}[v](p) =$

$p^2W(p) = (A_q + \rho(x)p^{\alpha(x)})^{-1}\rho(x)p^{\alpha(x)-1}u_0$ for every $p \in \mathbb{C}_+$, which shows that $u := v_{|Q}$ is a weak solution to (1.4) associated with $f = 0$. Moreover, since u is unique, as can be seen from Definition 2.2, we are left with the task of establishing (1.7) in the case where $f = 0$:

Lemma 2.5. *For all $t \in (0,T]$, we have:*

$$u(t) = \frac{1}{2i\pi}\int_{\gamma(\varepsilon,\theta)} e^{tp}(A_q + \rho(x)p^{\alpha(x)})^{-1}\rho(x)p^{\alpha(x)-1}u_0 dp.$$

Proof. This equality follows from Proposition 2.1 and the identity $u = \partial_t y$ in $(C_0^\infty)'(0,T;L^2(\Omega))$. Indeed, for all $p \in \gamma(\varepsilon,\theta)$, the mapping $t \mapsto e^{tp}pW(p)$ is continuously differentiable in $(0,T)$, and (2.4)–(2.6) yield the existence of a constant $C = C(\alpha_0,\alpha_M,\rho_0,\rho_M,\theta) \in (0,+\infty)$ such that we have

$$\begin{aligned}&\|e^{tp}p^2W(p)\|_{L^2(\Omega)}\\ &\qquad\leqslant Ce^{tr\cos\theta}\max_{\sigma=\pm 1} r^{-1+\sigma(\alpha_M-\alpha_0)}\|u_0\|_{L^2(\Omega)},\ p = re^{\pm i\theta},\ r \in (\varepsilon,+\infty).\end{aligned}$$

Moreover, by $\cos\theta \in (-1,0)$, we see that $r \mapsto e^{tr\cos\theta}\max_{\sigma=\pm1} r^{-1+\sigma(\alpha_M-\alpha_0)} \in L^1(\varepsilon,+\infty)$ for each $t \in (0,T]$, and so the integral $\int_{\gamma(\varepsilon,\theta)} e^{tp}p^2W(p)dp$ is well-defined. Therefore we obtain the claim of Lemma 2.5 by this and $u \in C((0,T];L^2(\Omega))$. $\square$

Finally, combining Lemma 2.5 with arguments similar to Lemma 3.2 below (see Subsection 3.2), we deduce that $u : (0,T) \to L^2(\Omega)$ is analytic.

2.3.2. Case $u_0 = 0$. As we aim for building S_1 and S_2, we turn now to establishing (1.7) in the case where $u_0 = 0$. To this purpose, we introduce the following family of operators acting in $L^2(\Omega)$,

$$\widetilde{W}(p) := p^{-2}(A_q + \rho(x)p^{\alpha(x)})^{-1},\ p \in \mathbb{C}\setminus\mathbb{R}_-.$$

For any $\mu \in [1,+\infty)$ and $\eta \in \mathbb{R}$, it follows from (2.4) and (2.14) that $\left\|\widetilde{W}(\mu+i\eta)\right\|_{\mathcal{B}(L^2(\Omega))}$ is majorized by $\langle\eta\rangle^{-2+\alpha_M-2\alpha_0}$ up to some multiplicative constant that is independent of η and μ. Therefore we have

$$\sup_{\mu\in[1,+\infty)}\int_{\mathbb{R}}\left\|\widetilde{W}(\mu+i\eta)\right\|_{\mathcal{B}(L^2(\Omega))}^k d\eta < \infty,\ k = 1,2.$$

Thus, by arguing exactly in the same way as in the first part of the proof, we see that

$$S_1(t) := \frac{1}{2i\pi}\int_{1-i\infty}^{1+i\infty} e^{tp}p\widetilde{W}(p)dp,\ t \in \mathbb{R} \tag{2.32}$$

lies in $\mathcal{S}'(\mathbb{R}_+;\mathcal{B}(L^2(\Omega)))$,

$$t \mapsto \langle t\rangle^{-\alpha_M}S_1(t) \in L^\infty(\mathbb{R};\mathcal{B}(L^2(\Omega))) \tag{2.33}$$

and

$$\mathcal{L}[S_1\psi](p) = p\widetilde{W}(p)\psi,\ p \in \mathbb{C}_+,\ \psi \in L^2(\Omega). \tag{2.34}$$

By $\tilde{f}$ we denote the extension of a function f by 0 on $(\mathbb{R}\times\Omega)\setminus((0,T)\times\Omega)$. We recall that there exists $\zeta\in\mathbb{R}_+$ such that

$$\langle t\rangle^{-\zeta}\tilde{f}\in L^\infty(\mathbb{R};L^2(\Omega)), \tag{2.35}$$

and consider the convolution of S_1 with $\tilde{f}$, that is,

$$(S_1*\tilde{f})(t,x)=\int_0^t S_1(t-s)f(s,x)1_{(0,T)}(s)ds,\ (t,x)\in\mathbb{R}\times\Omega.$$

Evidently, $(S_1*\tilde{f})(t)=0$ for all $t\in\mathbb{R}_-$, and we infer from (2.33) and (2.35) that

$$\begin{aligned}&\left\|(S_1*\tilde{f})(t)\right\|_{L^2(\Omega)}\\&\leqslant\left\|\langle t\rangle^{-\alpha_M}S_1\right\|_{L^\infty(\mathbb{R}_+;\mathcal{B}(L^2(\Omega)))}\left\|\langle t\rangle^{-\zeta}f\right\|_{L^\infty(\mathbb{R}_+;L^2(\Omega))}\langle t\rangle^{1+\alpha_M+\zeta},\ t\in\mathbb{R}_+.\end{aligned} \tag{2.36}$$

Therefore $t\mapsto\langle t\rangle^{-(1+\alpha_M+\zeta)}(S_1*\tilde{f})(t)\in L^\infty(\mathbb{R};L^2(\Omega))$, and consequently $S_1*\tilde{f}\in\mathcal{S}'(\mathbb{R}_+;L^2(\Omega))$. Moreover, again by (2.33) and (2.35), we see that

$$\begin{aligned}&\inf\{\varepsilon\in\mathbb{R}_+;\ e^{-\varepsilon t}S_1\in L^1(\mathbb{R};\mathcal{B}(L^2(\Omega)))\}\\&\qquad\qquad=\inf\{\varepsilon\in\mathbb{R}_+;\ e^{-\varepsilon t}\tilde{f}\in L^1(\mathbb{R};L^2(\Omega))\}=0,\end{aligned}$$

which entails

$$\mathcal{L}[S_1*\tilde{f}](p)=\mathcal{L}[S_1](p)\mathcal{L}[\tilde{f}](p)=\mathcal{L}[S_1](p)F(p),\ p\in\mathbb{C}_+,$$

with $\mathcal{L}[S_1](p)=\int_0^{+\infty}S_1(t)e^{-pt}dt$ and $\mathcal{L}[\tilde{f}](p)=\int_0^{+\infty}\tilde{f}(t)e^{-pt}dt$. Thus, setting $\tilde{v}:=\partial_t(S_1*\tilde{f})\in\mathcal{S}'(\mathbb{R}_+;L^2(\Omega))$, we derive from (2.34) that

$$\mathcal{L}[\tilde{v}](p)=p\mathcal{L}[S_1*\tilde{f}](p)=p\mathcal{L}[S_1](p)F(p)=(A_q+\rho(x)p^{\alpha(x)})^{-1}F(p),\ p\in\mathbb{C}_+.$$

The last step of the proof is to establish the following:

Lemma 2.6. *We have:*

$$\tilde{v}(t)=\int_0^t S_1(t-\tau)\tilde{f}(\tau)d\tau+S_2\tilde{f}(t),\ t\in[0,T]. \tag{2.37}$$

Proof. The proof of (2.37) can be done with the aid of (2.32), yielding

$$(S_1*\tilde{f})(t)=\frac{1}{2i\pi}\int_0^t\int_{1-i\infty}^{1+i\infty}e^{(t-s)p}p^{-1}(A_q+\rho(x)p^{\alpha(x)})^{-1}\tilde{f}(s)dpds,\ t\in\mathbb{R}_+.$$

Indeed, we notice with a slight adaptation of the reasoning used in the derivation of (2.28) that the integral $\int_{1-i\infty}^{1+i\infty}e^{(t-s)p}p^{-1}(A_q+\rho(x)p^{\alpha(x)})^{-1}f(s)dp$ can be replaced on the right-hand side of the above identity by $\int_{\gamma(\varepsilon,\theta)}e^{(t-s)p}$ $p^{-1}(A_q+\rho(x)p^{\alpha(x)})^{-1}\tilde{f}(s)dp$ associated with any $\varepsilon\in(0,1)$ and $\theta\in\left(\frac{\pi}{2},\pi\right)$. Therefore we have

$$(S_1*\tilde{f})(t)=\frac{1}{2i\pi}\int_0^t\int_{\gamma(\varepsilon,\theta)}e^{(t-s)p}p^{-1}(A_q+\rho(x)p^{\alpha(x)})^{-1}\tilde{f}(s)dp\ ds.$$

Hence, by (2.4), (2.14) and (2.35), we infer from the Fubini theorem that

$$(S_1 * \tilde{f})(t) = \frac{1}{2i\pi} \int_{\gamma(\varepsilon,\theta)} g_q(t,p)dp,\ t \in \mathbb{R}_+$$

with

$$g_q(t,p) := \int_0^t e^{(t-s)p} p^{-1} (A_q + \rho(x) p^{\alpha(x)})^{-1} \tilde{f}(s) ds,\ p \in \gamma(\varepsilon,\theta). \tag{2.38}$$

Therefore, for all $t \in \mathbb{R}_+$ and all $p \in \gamma(\varepsilon,\theta)$, we have

$$\partial_t g_q(t,p) = \int_0^t e^{(t-s)p} (A_q + \rho(x)p^{\alpha(x)})^{-1} \tilde{f}(s)ds + p^{-1}(A_q + \rho(x)p^{\alpha(x)})^{-1}\tilde{f}(t),$$

and consequently

$$\begin{aligned} &\|\partial_t g_q(t,p)\|_{L^2(\Omega)} \\ &\leqslant \left\| (A_q + \rho(x)p^{\alpha(x)})^{-1} \right\|_{\mathcal{B}(L^2(\Omega))} \left(\int_0^t e^{s\Re p} ds + |p|^{-1} \right) \left\| \tilde{f} \right\|_{L^\infty(0,t+1;L^2(\Omega))}. \end{aligned}$$

From this and (2.4)–(2.6), it follows that

$$\begin{aligned} &\|\partial_t g_q(t,p)\|_{L^2(\Omega)} \\ &\qquad \leqslant \rho_0^{-1} \max(2, c_*(\theta)) (1 + |\cos\theta|^{-1}) |p|^{-(1+\alpha_0)} \left\| \tilde{f} \right\|_{L^\infty(0,t+1;L^2(\Omega))}. \end{aligned}$$

As a consequence, the mapping $p \mapsto \partial_t g_q(t,p) \in L^1(\gamma(\varepsilon,\theta); L^2(\Omega))$ for any fixed $t \in \mathbb{R}_+$ and we have $\tilde{v}(t) = \partial_t[S_1 * \tilde{f}](t) = \frac{1}{2i\pi} \int_{\gamma(\varepsilon,\theta)} \partial_t g_q(t,p)dp$, or equivalently $2i\pi\tilde{v}(t)$ equals

$$\int_{\gamma(\varepsilon,\theta)} \left(\int_0^t e^{(t-s)p} (A_q + \rho(x)p^{\alpha(x)})^{-1} \tilde{f}(s)ds + p^{-1}(A_q + \rho(x)p^{\alpha(x)})^{-1}\tilde{f}(t) \right) dp$$

in virtue of (2.38). Now, applying the Fubini theorem to the right-hand side of the above identity, we obtain (2.37). $\square$

In view of Lemma 2.6, the restriction to Q of the function expressed by the right-hand side of (2.37), is a weak solution to (1.4) associated with $u_0 = 0$. Evidently such a function lies in $\mathcal{C}([0,T]; L^2(\Omega))$. Moreover it is unique from Definition 2.2.

2.4. Proof of Remark 1

We use the notations of Section 2.3. For $\varepsilon \in (0,1)$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $R \in [1, +\infty)$, we introduce $\gamma_R(\varepsilon,\theta) := \{z \in \gamma(\varepsilon,\theta);\ |z| \in [0,R]\}$ and put $\mathcal{C}_R(\theta) := \{z \in \mathbb{C};\ z = Re^{i\beta},\ \beta \in [-\theta,\theta]\}$. In light of Proposition 2.1, the Cauchy formula yields

$$\int_{\gamma_R(\varepsilon,\theta) \cup \mathcal{C}_R(\theta)^-} p^{-1}(A_q + \rho(x)p^{\alpha(x)})^{-1}\psi\ dp = 0,\ \psi \in L^2(\Omega),$$

where the notation $\mathcal{C}_R(\theta)^-$ stands for the counterclockwise oriented path $\mathcal{C}_R(\theta)$. Thus, by letting R to $+\infty$ in the above identity, we obtain

$$S_2\psi = \lim_{R\to+\infty} \int_{\mathcal{C}_R(\theta)} p^{-1}(A_q + \rho(x)p^{\alpha(x)})^{-1}\psi\, dp,\ \psi \in L^2(\Omega) \tag{2.39}$$

from the definition of S_2. Furthermore, for any $R \in [1, +\infty)$, from (2.4) and (2.14), we have

$$\left\| \int_{\mathcal{C}_R(\theta)} p^{-1}(A_q + \rho(x)p^{\alpha(x)})^{-1}\, dp \right\|_{\mathcal{B}(L^2(\Omega))} \leqslant CR^{\alpha_M - 2\alpha_0}, \tag{2.40}$$

where C is a positive constant which is independent of R. Since $\alpha_M - 2\alpha_0$ is negative by the assumption, we have $S_2\psi = 0$ for any $\psi \in L^2(\Omega)$ directly from (2.39)-(2.40). Finally (1.8) follows readily from this and (1.7).

3. Analysis of the inverse problem

In this section, we suppose that $\partial\Omega$ is $\mathcal{C}^{1,1}$ and (1.9) holds, that is, $A_0 = -\Delta$ and

$$D(A_0) = H_0^1(\Omega) \cap H^2(\Omega).$$

We recall for further use that the norm in $H^2(\Omega)$ is equivalent to the norm in $D(A_0)$ or in $D(A_q)$.

First we prove that the boundary operator $\mathcal{N}_{\alpha,\rho,q}(t)$ expressed by (1.11), is well-defined for all $t \in (0, T]$.

3.1. Definition of the boundary operator

By (1.11) and the continuity of the trace operator $\varphi \mapsto \partial_\nu\varphi$ from $H^2(\Omega)$ into $L^2(\partial\Omega)$, it suffices to prove the following well-posedness for the initial-boundary value problem (1.10).

Proposition 3.1. *Let α, ρ and q be the same as in Theorem* 1.1. *Then, for all $g \in H^{3/2}(\partial\Omega)$, there exists a unique weak solution in $\mathcal{C}([0, +\infty); H^2(\Omega))$ to* (1.10).

Proof. Let $G \in H^2(\Omega)$ satisfy $G = g$ on $\partial\Omega$. Then we notice that $u = u_g$ is a solution to (1.10) if and only if the function $v(t,x) := u(t,x) - t^kG(x)$ is a solution to the system

$$\left\{ \begin{array}{rcll} (\rho(x)\partial_t^{\alpha(x)} + \mathcal{A}_q)v(t,x) & = & f(t,x), & (t,x) \in (0,+\infty) \times \Omega, \\ v(t,x) & = & 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ v(0,x) & = & 0, & x \in \Omega, \end{array} \right. \tag{3.1}$$

where $f(t,x) := -\left(\rho(x)\partial_t^{\alpha(x)} t^k + t^k\mathcal{A}_q\right) G(x)$.

Furthermore, since $f \in \mathcal{C}((0,+\infty); L^2(\Omega))$ and $(1+t)^{-k-1}f \in L^\infty(0,+\infty; L^2(\Omega))$, the initial-boundary value problem (3.1) admits a unique weak solution $v \in \mathcal{C}((0,+\infty); L^2(\Omega))$ according to Theorem 1.1. Let us now prove that $v \in \mathcal{C}([0,+\infty);H^2(\Omega))$. For this purpose, we infer from the basic identity $\mathcal{L}[t^k](p) := \int_0^{+\infty} t^k e^{-pt} dt = \frac{k!}{p^{k+1}}$ that

$$F(p,x) := \mathcal{L}[f(\cdot,x)](p) = -\frac{k!}{p^{k+1}} \left(\mathcal{A}_q + \rho(x) p^{\alpha(x)} \right) G(x),\ (p,x) \in \mathbb{C}_+ \times \Omega.$$

Next, upon extending the expression of the right-hand side of the above equality to all $p \in \mathbb{C} \setminus \mathbb{R}_-$, we obtain from the first equation of (3.1) that $V := \mathcal{L}[v]$ reads

$$V(p,x) = -\frac{k!}{p^{k+1}} (A_q + \rho(x) p^{\alpha(x)})^{-1} \left(\mathcal{A}_q + \rho(x) p^{\alpha(x)} \right) G(x),\ (p,x) \in (\mathbb{C} \setminus \mathbb{R}_-) \times \Omega. \tag{3.2}$$

Therefore, arguing in the same way as in the proof of Theorem 1.1, we obtain for any fixed $\varepsilon \in (0,1)$ and $\theta \in (\pi/2, \pi)$ that

$$v(t,x) = \frac{1}{2i\pi} \int_{\gamma(\varepsilon,\theta)} e^{tp} V(p,x) dp,\ (t,x) \in [0,+\infty) \times \Omega. \tag{3.3}$$

On the other hand, (3.2) and Proposition 2.1 yield that $V(p,\cdot) \in D(A_q)$ for all $p \in \mathbb{C} \setminus \mathbb{R}_-$ with

$$\begin{aligned} A_q V(p,x) = \frac{k!}{p^{k+1}} & \left(\rho(x) p^{\alpha}(x) (A_q + \rho(x) p^{\alpha(x)})^{-1} - I \right) \\ & \times \left(\rho(x) p^{\alpha(x)} + \mathcal{A}_q \right) G(x),\ x \in \Omega. \end{aligned} \tag{3.4}$$

Here the symbol I stands for the identity operator in $L^2(\Omega)$. Applying (2.4)-(2.5), we deduce from (3.4) that

$$\begin{aligned} \left\| A_q V(re^{\pm i\theta}) \right\|_{L^2(\Omega)} & \leqslant C r^{-(k+1)} \max(r^{2\alpha_M - \alpha_0}, r^{2\alpha_0 - \alpha_M}) \\ & \leqslant C \varepsilon^{-3(\alpha_M - \alpha_0)} r^{2\alpha_M - \alpha_0 - k - 1} \\ & \leqslant C \varepsilon^{-3(\alpha_M - \alpha_0)} r^{1 - k - \alpha_0},\ r \in [\varepsilon, +\infty) \end{aligned} \tag{3.5}$$

with some positive constant $C = C(\theta, M, \|g\|_{H^{3/2}(\partial\Omega)}, \|q\|_{L^\infty(\Omega)}, \alpha_0, \alpha_M, \rho_0, \rho_M)$ which is independent of ε. Therefore we have $r \mapsto A_q V(re^{\pm i\theta}, \cdot) \in L^1(\varepsilon, +\infty; L^2(\Omega))$ and hence $r \mapsto V(re^{\pm i\theta}, \cdot) \in L^1(\varepsilon, +\infty; D(A_q))$. From this and (3.3), it follows that $v(t,\cdot) \in D(A_q)$ for all $t \in [0,+\infty)$ with

$$A_q v(t,\cdot) = \frac{1}{2i\pi} \int_{\gamma(\varepsilon,\theta)} e^{tp} A_q V(p) dp, \tag{3.6}$$

proving that $A_q v \in \mathcal{C}([0,+\infty); L^2(\Omega))$. As a consequence, we have $v \in \mathcal{C}([0,+\infty); D(A_q))$ and the desired result follows immediately from this and the identity $D(A_q) = H_0^1(\Omega) \cap H^2(\Omega)$. $\square$

3.2. Proof of Theorem 1.2

The proof of Theorem 1.2 is by means of the analytic properties of the mapping $t \mapsto \mathcal{N}_{\alpha,\rho,q}(t)$, defined by (1.11), that are preliminarily established in the coming subsection.

3.2.1. On the analyticity of the boundary operator. We first introduce the following notations. Let X be a Hilbert space, and let $\mathcal{O}$ be either a subinterval of $\mathbb{R}$ or an open subset of $\mathbb{C}$. We denote by $\mathcal{A}(\mathcal{O};X)$ the space of X-valued functions that are analytic in $\mathcal{O}$.

Lemma 3.2. *Let $g \in H^{3/2}(\partial\Omega)$ and let u be the solution in $\mathcal{C}([0,+\infty);H^2(\Omega))$ to* (1.10) *associated with g, whose existence is guaranteed by Proposition* 3.1. *Then the mapping $t \mapsto \partial_\nu u(t,\cdot)_{|\partial\Omega}$ lies in $\mathcal{A}((0,+\infty);L^2(\partial\Omega))$.*

Proof. By the definitions and the notations used in the proof of Proposition 3.1, the solution u to (1.10) reads $u(t,x) = t^k G(x) + v(t,x)$ for a.e. $(t,x) \in (0,+\infty)\times\Omega$, where $v \in \mathcal{C}([0,+\infty);H^2(\Omega))$ is a solution to (3.1). Since $G \in H^2(\Omega)$, it is apparent that $t \mapsto t^k\partial_\nu G_{|\partial\Omega} \in \mathcal{A}((0,+\infty);L^2(\partial\Omega))$. Therefore we are left with the task of showing that $t \mapsto \partial_\nu v(t,\cdot)_{|\partial\Omega} \in \mathcal{A}((0,+\infty);L^2(\partial\Omega))$. Since $D(A_q) = H_0^1(\Omega)\cap H^2(\Omega)$ and the trace map $w \mapsto \partial_\nu w_{|\partial\Omega}$ is continuous from $H^2(\Omega)$ into $L^2(\partial\Omega)$, it is sufficient to prove that $t \mapsto v(t,\cdot) \in \mathcal{A}((0,+\infty);D(A_q))$.

For this purpose, we fix $\theta_1 \in (0,\theta-\pi/2)\cap(0,\pi-\theta)$, put $\mathcal{O} := \{\tau e^{i\psi};\ \tau \in (0,+\infty),\ \psi \in (-\theta_1,\theta_1)\}$, and we extend v into a function of $\mathcal{A}(\mathcal{O};D(A_q))$. This can be done with the help of (3.5)-(3.6) by noticing

$$|e^{zp}| = |e^{\tau r e^{i(\pm\theta+\psi)}}| = e^{\tau r\cos(\pm\theta+\psi)} \qquad \text{for all } z=\tau e^{i\psi}\in\mathcal{O} \text{ and } p = re^{\pm i\theta} \text{ with } r\in[\varepsilon,+\infty).$$

Indeed, since we have $\theta+\psi \in (\theta-\theta_1,\theta+\theta_1)\subset(\pi/2,\pi)$ and $-\theta+\psi \in (-\theta-\theta_1,-\theta+\theta_1)\subset(-\pi,-\pi/2)$, it holds true that $\cos(\pm\theta+\psi)\leqslant \cos(\theta-\theta_1)$ and

$$|e^{zp}| \leqslant e^{|z|r\cos(\theta-\theta_1)},\ z\in\mathcal{O},\ p=re^{\pm i\theta},\ r\in[\varepsilon,+\infty). \tag{3.7}$$

Furthermore, since $\cos(\theta-\theta_1)\in(-1,0)$, it follows from (3.5) and (3.7) that

$$\mathcal{W}: z \mapsto \frac{1}{2i\pi}\int_{\gamma(\varepsilon,\theta)} e^{zp}A_qV(p)dp$$

is well defined in $\mathcal{O}$. Moreover, for any compact subset $K\subset\mathcal{O}$ in $\mathbb{C}$, we infer from (3.5) that

$$\left\| e^{z(re^{\pm i\theta})}A_qV(re^{\pm i\theta})\right\|_{L^2(\Omega)} \leqslant C\varepsilon^{-3(\alpha_M-\alpha_0)}e^{\delta r\cos(\theta-\theta_1)}r^{1-k-\alpha_0},\ z\in K,\ r\in[\varepsilon,+\infty),$$

where $\delta := \inf\{|z|;\ z\in K\}>0$ and C is the constant in (3.5). Next, as $z\mapsto e^{zp}A_qV(p)\in\mathcal{A}(\mathcal{O};L^2(\Omega))$ for all $p\in\{re^{\pm i\theta};\ r\in[\varepsilon,+\infty)\}$, this implies that $\mathcal{W}\in\mathcal{A}(\mathcal{O};L^2(\Omega))$. Furthermore, since $\mathcal{W}(t) = A_qv(t,\cdot)$ for all $t\in(0,+\infty)$, we obtain by (3.5) that

$$t\mapsto A_qv(t,\cdot)\in\mathcal{A}((0,+\infty);L^2(\Omega)). \tag{3.8}$$

Finally, arguing in the same way as above, we deduce from (3.3) that $t\mapsto v(t,\cdot)\in\mathcal{A}((0,+\infty);L^2(\Omega))$. This and (3.8) yield that $t\mapsto v(t,\cdot)\in\mathcal{A}((0,+\infty);D(A_q))$, which proves the result. $\square$

In terms of Lemma 3.2, we can complete the proof of Theorem 1.2.

3.2.2. Completion of the proof. For $j = 1, 2$, we denote by u_j the weak solution to the initial-boundary value problem (1.10) associated with $g \in \mathcal{H}_{\mathrm{in}}$, $(\alpha, \rho, q) = (\alpha_j, \rho_j, q_j)$, and $T = +\infty$. The proof is divided into three steps. The first one is to establish that

$$\partial_\nu u_1(t, \cdot)_{|S_{\mathrm{out}}} = \partial_\nu u_2(t, \cdot)_{|S_{\mathrm{out}}},\ t \in (0, +\infty), \tag{3.9}$$

and the second one is to derive from (3.9) that the functions $U_j := \mathcal{L}[u_j]$, $j = 1, 2$, verify

$$\partial_\nu U_1(p, \cdot)_{|S_{\mathrm{out}}} = \partial_\nu U_2(p, \cdot)_{|S_{\mathrm{out}}}, \quad p \in (0, +\infty). \tag{3.10}$$

The third step corresponds to the end of the proof, which is by means of the existing results for the Calderón problem with partial Cauchy data.
Step 1. Put $h(t, x) := \partial_\nu u_1(t, x) - \partial_\nu u_2(t, x)$ for $(t, x) \in (0, +\infty) \times S_{\mathrm{out}}$. We recall from Lemma 3.2 that $h \in \mathcal{A}((0, +\infty); L^2(S_{\mathrm{out}}))$, and from (1.13) that

$$h(t_n) = 0,\ n \in \mathbb{N}.$$

Therefore, by (1.12), the set of the zeros of the analytic function h has accumulation point in $(0, +\infty)$, so that h identically vanishes, and (3.9) follows.
Step 2. For $j = 1, 2$, let v_j denote the solution to (3.1) where (α_j, ρ_j, q_j) is substituted into (α, ρ, q) such that

$$u_j(t, x) = t^k G(x) + v_j(t, x),\ (t, x) \in Q. \tag{3.11}$$

Furthermore, putting $V_j := \mathcal{L} v_j$, we deduce from (3.2) and (3.5) that

$$\begin{aligned} \left\| \int_{\gamma_\pm(\varepsilon,\theta)} e^{tp} A_{q_j} V_j(p) dp \right\|_{L^2(\Omega)} & \leqslant C\varepsilon^{-3(\alpha_M - \alpha_0)} \int_\varepsilon^{+\infty} r^{1-k-\alpha_0} dr \\ & \leqslant \frac{C}{k + \alpha_0 - 2} \varepsilon^{2-k-\alpha_0-3(\alpha_M-\alpha_0)}, \end{aligned} \tag{3.12}$$

where the constant C is the same as in (3.5). Similarly, by Lemma 2.1, we infer from (2.14) and (3.4) that

$$\begin{aligned} \left\| \int_{\gamma_0(\varepsilon,\theta)} e^{tp} A_{q_j} V_j(p) dp \right\|_{L^2(\Omega)} & \leqslant C\varepsilon^{-(k+\alpha_M)} \left(\int_{-\theta}^{\theta} e^{t\varepsilon\cos\beta} C_\beta d\beta \right) \\ & \leqslant C e^{t\varepsilon} \varepsilon^{-(k+2\alpha_M-\alpha_0)}, \end{aligned} \tag{3.13}$$

where another constant $C > 0$ is independent of ε. Thus, for all $t \in (1, +\infty)$, by taking $\varepsilon = t^{-1}$ in (3.12)-(3.13) we see that $\left\| \int_{\gamma(\varepsilon,\theta)} e^{tp} A_{q_j} V_j(p) dp \right\|_{L^2(\Omega)}$ is upper bounded by $t^{k+2\alpha_0-\alpha_M}$ up to some positive constant C_j, which is independent of t. In light of (3.6), this entails that $\|v_j(t, \cdot)\|_{H^2(\Omega)} \leqslant C_j t^{k+2\alpha_0-\alpha_M}$ for every $t \in (1, +\infty)$. Therefore, by (3.11) we have

$$\|u_j(t, \cdot)\|_{H^2(\Omega)} \leqslant C_j t^{k+2\alpha_0},\ t \in (1, +\infty).$$

Moreover, since $v_j \in L^\infty(0, 1; H^2(\Omega))$ in virtue of Lemma 3.2, and hence $u_j \in L^\infty(0, 1; H^2(\Omega))$ by (3.11), we obtain that $t \mapsto e^{-pt} u_j(t, \cdot) \in L^1(0, +\infty;$

$H^2(\Omega))$ for all $p \in \mathbb{C}_+$. This and the continuity of the trace map $v \mapsto \partial_\nu v_{|\partial\Omega}$ from $H^2(\Omega)$ into $L^2(\partial\Omega)$, yield that

$$\mathcal{L}[\partial_\nu u_j](p) = \partial_\nu U_j(p),\ j = 1, 2,\ p \in \mathbb{C}_+.$$

Now (3.10) follows from this and (3.9).
Step 3. We can complete the proof by [20, Theorem 7] (see also [19]) when $d = 2$ and [22, Theorem 1.2] when $d \geqslant 3$.

Theorem 3.3. *Assume that $\partial\Omega$ is smooth and that Ω is connected. For*

$$V \in \mathcal{V} := \{q \in L^\infty(\Omega);\ 0 \textit{ lies in the resolvent set of } A_q\},$$

let Λ_V be the partial Dirichlet-to-Neumann map $\mathcal{H}_{\rm in} \ni \varphi \mapsto \partial_\nu w_{|S_{\rm out}}$, where w is the solution to

$$\begin{cases} -\Delta w + V(x)w &= 0, & x \in \Omega, \\ w(x) &= \varphi(x), & x \in \partial\Omega. \end{cases} \tag{3.14}$$

For $j = 1, 2$, pick V_j in $\mathcal{V} \cap W^{1,r}(\Omega)$ with $r \in (2, +\infty)$, if $d = 2$, and in $\mathcal{V}$ if $d \geqslant 3$. Then

$$\Lambda_{V_1} = \Lambda_{V_2} \textit{ yields } V_1 = V_2. \tag{3.15}$$

It is clear for all $p \in (0, +\infty)$ that $\tilde{U}_j(p) := \frac{p^{k+1}}{k!} U_j(p)$, $j = 1, 2$, is a solution to (3.14) associated with $V = q_j + \rho_j p^{\alpha_j}$ and $\varphi = g$. As a consequence, we have

$$\Lambda_{q_1+\rho_1 p^{\alpha_1}} g = \Lambda_{q_2+\rho_2 p^{\alpha_2}} g, \quad p \in (0, +\infty)$$

by (3.10), and since g is arbitrary in $\mathcal{H}_{\rm in}$, this immediately entails that

$$\Lambda_{q_1+\rho_1 p^{\alpha_1}} = \Lambda_{q_2+\rho_2 p^{\alpha_2}}, \quad p \in (0, +\infty). \tag{3.16}$$

Moreover, from the definition of $\mathcal{E}_d$, for every $p \in (0, +\infty)$, we have $q_j + \rho_j p^{\alpha_j} \in W^{1,r}(\Omega)$ with $r \in (2, +\infty)$ if $d = 2$, and $q_j + \rho_j p^{\alpha_j} \in L^\infty(\Omega)$ if $d \geqslant 3$. Therefore, applying (3.15) with $V_j = q_j + \rho_j p^{\alpha_j}$, we infer from (3.16) that

$$q_1 + \rho_1 p^{\alpha_1} = q_2 + \rho_2 p^{\alpha_2},\ p \in (0, +\infty). \tag{3.17}$$

Letting p to zero in (3.17), we see that $q_1 = q_2$. Thus, taking $p = 1$ in (3.17), we obtain that $\rho_1 = \rho_2$. Finally, applying (3.17) with $p = e$, we find that $e^{\alpha_1} = e^{\alpha_2}$, which yields that $\alpha_1 = \alpha_2$.

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