ON TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS OF TIME-DEPENDENT PIECEWISE CONSTANT ORDER

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Abstract. This contribution considers the time-fractional subdiffusion with a time-dependent variable-order fractional operator of order $\beta(t)$. It is assumed that $\beta(t)$ is a piecewise constant function with a finite number of jumps. A proof technique based on the Fourier method and results from constant-order fractional subdiffusion equations has been designed. This novel approach results in the well-posedness of the problem.

1. Introduction

The idea of fractional calculus (FC) dates back to the 17th century when mathematicians like Leibniz and L'Hôpital pondered the meaning of differentiation and integration of noninteger orders. Significant progress was made in the 19th century thanks to the work of mathematicians like Liouville and Riemann.

Fractional calculus has applications in various scientific and engineering fields, including physics, engineering, signal processing, finance, and more. It has proven to be a powerful tool for describing systems with long-range memory, fractal phenomena, and non-local behaviour. The fractional calculus framework provides a deeper understanding of complex phenomena that integer-order calculus cannot adequately describe.

Machado et al. [22] summarised the historical perspective on the major developments in fractional calculus since the 1970s. Please note that most papers studied constant-order (CO) fractional operators. There exist various types of CO fractional derivatives (Caputo, Riemann-Liouville, Grünwald-Letnikov, etc.), see, e.g. [25]. The Caputo and the Riemann-Liouville variations can be expressed in terms of the Riemann-Liouville kernel $g_{1-\beta}$:

$$g_{1-\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \qquad t > 0,\ 0 < \beta < 1,$$

where Γ represents the Gamma function. This kernel obeys

$$(-1)^j g_{1-\beta}^{(j)}(t) \geq 0, \quad \forall t \geq 0, j = 0, 1, 2; \quad g'_{1-\beta} \not\equiv 0,$$

and, therefore, it is strongly positive definite by [23, Corollary 2.2]. This creates a powerful position when establishing energy estimates for a solution.

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In the literature, several definitions of fractional derivatives and integrals in variable-order (VO) FC can be found as generalisations of their CO counterparts, see e.g. [2, 6, 8, 10, 14, 15, 16, 20, 34, 35, 41, 33]. One of the possibilities uses the following time-dependent Riemann-Liouville kernel

$$\left(g_{1-\beta(t)}\right)(t) = \frac{t^{-\beta(t)}}{\Gamma(1-\beta(t))}, \qquad t > 0,\ 0 < \beta(t) < 1.$$

Then, the Caputo VO fractional derivative reads as follows

$$\left(\partial_t^{\beta(t)} u\right)(t) = \left(g_{1-\beta(t)} * \partial_t u\right)(t) = \int_0^t \frac{(t-s)^{-\beta(t-s)}}{\Gamma(1-\beta(t-s))} \partial_t u(s)\ \mathrm{d}s, \qquad t > 0,\ 0 < \beta(t) < 1.$$

Another possibility for a slightly different VO derivative is [6, 34]

$$\left(\partial_t^{\beta(t)} u\right)(t) = \frac{1}{\Gamma(1-\beta(t))} \int_0^t (t-s)^{-\beta(t)} \partial_t u(s)\ \mathrm{d}s, \qquad t > 0,\ 0 < \beta(t) < 1.$$

This definition will be considered in this paper. Polymers, plastics, rubber and oil are all viscoelastic materials that are used in biology, medicine, chemical engineering and other fields. CO fractional viscoelasticity models do not account for the evolution of the microstructure of viscoelastic materials during deformation, which changes the mechanical properties of deforming materials. On the other hand, in VO fractional models with a fractional time-derivative of time-dependent order, the variation of the fractional order is used for describing the change of the mechanical properties of the material under various loading conditions. For this reason, they can accurately predict the viscoelastic behaviour of soft materials, see e.g., [11, 26, 32]. In these models, the fractional order can be any $(0, 1)$-valued function of the time variable, but in practice, it is fitted to piecewise constant time domain data, see, e.g. [1]. This is quite reminiscent of time-fractional multi-state regime-switching option pricing models. Indeed, although the CO time-fractional Black-Scholes equation can account for the nonlocal properties of the assets' prices, see e.g. [3], it cannot describe changes in market states. Hence, regime-switching VO time-fractional models have been proposed in [5, 28] to price options. Since each market state is described by a CO time fractional model, the fractional order of the VO model is piecewise constant, see, e.g. [5][Eq. (1)].

When solving a problem with time-dependent fractional derivatives, one can distinguish two major cases:

The highest order of the time derivative is a constant: This covers parabolic and hyperbolic situations; see [29, 37, 36, 38, 39, 40]. In this framework, the VO time derivatives can be interpreted as Volterra operators of lower order. Consequently, they can be incorporated into the right-hand side of the governing differential equation. The well-posedness of the setting can be shown using a generalised Grönwall lemma, see [13, Lemma 7.1.1] or [4, Lemma 1]. We refer the reader to [9, 30] for a discrete version of this lemma. Up to now, the most general article addressing this proof technique is [31], which also includes nonlinear functions of VO fractional derivatives.

The highest order of the time derivative is time-dependent: Kochubei [18, 19] developed a general FC for

$$D_{\varphi}f(t) := \frac{d}{dt}\left(\varphi * f\right)(t) - \varphi(t)f(0),$$

which is based on the Laplace transform. The integral kernel has to obey some conditions constraining φ to be completely monotone (cf. [12, 21]). This is very restrictive for the choice of the kernels and, therefore, less interesting. The authors of [12] present a few examples of practical relevance (exponential, Mittag-Leffler, and error function-type transitions). Until now, there is no general theory that covers this situation. One would like to have the existence and uniqueness of a solution for $\beta(t)$, which is a piecewise smooth function. This case remains an open problem.

Highlights of the paper. We consider the time-fractional diffusion problem (1.5) with the time-dependent highest order $\beta(t)$ of the fractional derivative in time. We assume that $\beta(t)$ is a piecewise constant function with a finite number of steps. In Section 3, we design a new proof technique to establish the well-posedness of the problem (stated in Theorem 2.2) by using Fourier analysis and the interpretation of a solution for CO FC in terms of the Mittag-Leffler function. We want to point out that $\beta(t)$ does not need to be monotone, and the convolution kernel is not positive definite. Our proof technique is limited to a finite number of steps/jumps of $\beta(t)$. Until now, we have not found a way to treat a continuously varying $\beta(t)$.

1.1. **Formulation of the problem.** Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N} := \{1, 2, \ldots\}$, be an open bounded domain with Lipschitz boundary $\partial\Omega$. We consider a general second-order linear differential operator $\mathscr{L}$ defined by

$$\mathscr{L}(u) = -\text{div}\ (\mathbf{A}(x)\nabla u) + c(x)u, \tag{1.1}$$

where

$$\mathbf{A}(x) = \left(a_{i,j}(x)\right)_{i,j=1,\ldots,d},\ a_{ij} \in L^{\infty}(\Omega, \mathbb{R}),\ \mathbf{A}^T = \mathbf{A},\ c \in L^{\infty}(\Omega, \mathbb{R}).$$

Set $\mathscr{H} := L^2(\Omega)$. We assume that there exist two constants $\alpha \in (0, \infty)$ and $c_0 \in [0, \alpha\kappa^2)$, where κ is the Ω-related constant of the classical Poincaré inequality $\|\nabla u\|_{\mathscr{H}^d} \geq \kappa \|u\|_{\mathscr{H}}$ for all $u \in \mathscr{V} := H_0^1(\Omega)$, the closure of $C_0^{\infty}(\Omega)$ in the first order Sobolev space $H^1(\Omega)$, such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \alpha \left|\boldsymbol{\xi}\right|^2, \quad \text{for a.e. } x \in \Omega \text{ and all } \boldsymbol{\xi} = (\xi_i)_{i=1,\ldots,d} \in \mathbb{R}^d, \tag{1.2}$$

and

$$c(x) \geq -c_0 \text{ for a.e. } x \in \Omega. \tag{1.3}$$

Moreover, for $T \in (0,\infty)$ fixed and for $\beta : (0,T) \to (0,1)$, we define the variable-order fractional integral operator ${}_0I_t^{\beta(t)}$, and the variable-order fractional Caputo operator $\frac{\partial^{\beta(t)}}{\partial t^{\beta(t)}}$ as follows [6, 20, 34, 41]

$$ {}_0I_t^{\beta(t)} u(t) := \frac{1}{\Gamma(\beta(t))} \int_0^t \frac{u(s)}{(t-s)^{1-\beta(t)}} \mathrm{d}s, $$

$$ \partial_t^{\beta(t)} u(t) := {}_0I_t^{1-\beta(t)} u'(t) = \frac{1}{\Gamma(1-\beta(t))} \int_0^t \frac{u'(s)}{(t-s)^{\beta(t)}} \mathrm{d}s. \tag{1.4} $$

In this contribution, we examine the existence and uniqueness issue for the solution to the following initial-boundary value problem (IBVP)

$$ \begin{cases} \left(\partial_t^{\beta(t)} u\right)(x,t) + \mathscr{L} u(x,t) &= f(x,t), \qquad (x,t) \in \Omega \times (0,T), \\ u(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x), \qquad x \in \Omega, \end{cases} \tag{1.5} $$

in the special case where the function β is piecewise constant. Here u_0 (resp., f) is a suitable initial condition (resp., source term) that will be made precise further.

2. Results

2.1. **Definitions and notations.** In what follows, the usual norm in $\mathscr{H}$ or $\mathscr{H}^d$ is denoted by $\|\cdot\|$. We introduce the following bilinear form

$$ \ell(u,v) := \sum_{i,j=1}^{d} \int_\Omega \left(a_{i,j}(x)\partial_{x_i} u(x) \partial_{x_j} v(x) dx + c(x)u(x)v(x)\right) dx, \ u,v \in \mathscr{V}. $$

Evidently, ℓ is continuous in $\mathscr{V} \times \mathscr{V}$, as we have

$$ \begin{aligned} |\ell(u,v)| &\leq \max_{1\leq i,j\leq d} \|a_{i,j}\|_{L^\infty(\Omega)} \|\nabla u\| \|\nabla v\| + \|c\|_{L^\infty(\Omega)} \|u\| \|v\| \\ &\leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \ u,v \in \mathscr{V}, \end{aligned} $$

where $C = \max_{1\leq i,j\leq d} \|a_{i,j}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)}$. Further, it readily follows from (1.2)-(1.3) and from the Poincaré inequality that ℓ is $\mathscr{V}$-coercive:

$$ \ell(u,u) \geq \alpha \|\nabla u\|^2 - c_0 \|u\|^2 \geq (\alpha\kappa^2 - c_0) \|u\|^2_{H^1(\Omega)}, \ u \in \mathscr{V}. $$

Therefore, the linear operator $\mathcal{L}$ is associated with $(\ell, \mathscr{V}, \mathscr{H})$ in the sense of [7, Chap. VI, Section 3.2.5] is selfadjoint (and positive) in $\mathscr{H}$, and acts on its dense domain $D(\mathcal{L}) = \{u \in \mathscr{V} : \mathscr{L}u \in \mathscr{H}\}$ as $\mathcal{L}u = \mathscr{L}u$. Moreover, since $\mathscr{V}$ is compactly embedded in $\mathscr{H}$, the resolvent of $\mathcal{L}$ is compact, and consequently, the spectrum of $\mathcal{L}$ is discrete. We denote by λ_n, $n \geq 1$, the eigenvalues of $\mathcal{L}$, arranged in non-decreasing order and repeated with the (finite) multiplicity (see, e.g., [27, Theorem XIII.64]):

$$ 0 < \lambda_1 \leq \lambda_2 \leq \cdots $$

and by $\{X_n,\ n \geq 1\}$ an orthonormal basis in $\mathscr{H}$ of eigenfunctions such that $\mathcal{L}X_n = \lambda_n X_n$.
We recall that

$$D(\mathcal{L}) = \left\{ v \in \mathscr{H} :\ \sum_{n=1}^{\infty} \lambda_n^2 \left| \langle v, X_n \rangle \right|^2 < \infty \right\},$$

is a Banach space with respect to the norm

$$\|v\|_{D(\mathcal{L})} := \left(\sum_{n=1}^{\infty} \lambda_n^2 \left| \langle v, X_n \rangle \right|^2 \right)^{1/2}.$$

In the next definition, we formulate how the solution to (1.5) should be understood. We set $I := (0,T)$ and we recall that the space $W^{1,1}(I,\mathscr{H})$ consists of functions $u \in L^1(I,\mathscr{H})$ satisfying $\partial_t u \in L^1(I,\mathscr{H})$.

Definition 2.1. *Let $u_0 \in D(\mathcal{L})$ and let $f \in L^1(I,\mathscr{H})$. Then, a solution to the IBVP* (1.5) *is any function*

$$u \in L^1(I, D(\mathcal{L})) \cap W^{1,1}(I,\mathscr{H})$$

satisfying the two following conditions simultaneously:

i) $\partial_t^{\beta} u(x,t) + \mathcal{L}u(x,t) = f(x,t)$ *for a.e.* $(x,t) \in \Omega \times I$,
ii) $\lim_{t\downarrow 0} \|u(\cdot,t) - u_0\| = 0$.

2.2. **Main result.** Let $M \in \mathbb{N}$. Given $\beta_j \in (0,1)$, $j = 0,1,\ldots,M-1$, and $t_0 := 0 < t_1 < t_2 < \ldots < t_M := T$, we consider $\beta : \overline{I} \to (0,1)$ such that

$$\beta(t) := \beta_j,\ t \in [t_j, t_{j+1}),\ j = 0,1,\ldots,M-1.$$

Moreover, we put $I_j := (t_j, t_{j+1})$, $j = 0,\ldots,M-1$. Then, the main result of this paper can be stated as follows.

Theorem 2.2. *Let $u_0 \in D(\mathcal{L})$, let $f \in W^{1,1}\left(\cup_{j=0}^{M-1} I_j, \mathscr{H}\right)$, and assume that for all $j = 0,\ldots,M-1$,*

$$\exists \varepsilon_j \in (0, 1-\beta_j) \text{ such that } \left\| (\cdot - t_j)^{\beta_j + \varepsilon_j} f' \right\|_{L^\infty(I_j,\mathscr{H})} < \infty. \tag{2.1}$$

Then, the IBVP (1.5) *admits a unique solution*

$$u \in \mathcal{C}^0(\overline{I}, D(\mathcal{L})) \cap W^{1,1}(I,\mathscr{H})$$

in the sense of Definition 2.1. Moreover, there exists a positive constant C, depending only on Ω, T and $\{(t_j,\beta_j,\varepsilon_j),\ j = 0,\ldots,M-1\}$, such that

$$\begin{aligned} &\|u\|_{\mathcal{C}^0(\overline{I},D(\mathcal{L}))} + \|u\|_{W^{1,1}(I,\mathscr{H})} \\ \leq &C \left(\|u_0\|_{D(\mathcal{L})} + \sum_{j=0}^{M-1} \left(\|f\|_{W^{1,1}(I_j,\mathscr{H})} + \left\| (\cdot - t_j)^{\beta_j+\varepsilon_j} f' \right\|_{L^\infty(I_j,\mathscr{H})} \right) \right). \end{aligned} \tag{2.2}$$

2.3. **Outline and comments.** Making use of an auxiliary result stated in Proposition 3.1, of which the proof is postponed to Section 4, we will prove the main result of this article, Theorem 2.2, in Section 3.

Theorem 2.2 claims existence of a unique solution u in the sense of Definition 2.1 to the IBVP (1.5) provided that the initial state $u_0 \in D(\mathcal{L})$ and the source term $f \in W^{1,1}\left(\cup_{j=0}^{M-1} I_j, \mathscr{H}\right)$ satisfies the condition (2.1). According to it, the first-order time derivative $f'(\cdot,t)$ should not blow up faster as $t \downarrow t_j$, $j=0,\dots,M-1$, than the power function $(t-t_j)^{-\kappa_j}$ for some $\kappa_j \in (0,1)$. Such a condition is a key element to the proof of Theorem 2.2. The reason is as follows.

For all $j=0,\dots,M-1$, $u_{|I_j}$ is characterised as a solution on the interval I_j to a time-fractional PDE of constant order β_j, with source term f_j. Namely, we have $f_0=f$, while f_j-f is expressed in terms of the first-order time derivative of the functions $u_{|I_k}$, $k=0,\dots,j-1$, when $j=1,\dots,M-1$, see (3.3) and (3.7). In the peculiar case where $j=1$ in (3.3), we have

$$f_1'(\cdot,t) = f'(\cdot,t) + \frac{\beta_1}{\Gamma(1-\beta_1)} \int_0^{t_1} (t-s)^{-1-\beta_1} u'(\cdot,s) \mathrm{d}s,\ t \in I_1,$$

so we need a good control on $u'_{|I_0}$ in order to guarantee that $f_1'(\cdot,t) \in L^1(I_1,\mathscr{H})$. This is achieved by enforcing the condition (2.1) with $j=0$ on f, see (4.15).

Finally, we point out that the statement of Theorem 2.2 is no longer valid when M goes to infinity. This can be understood from the identity (3.8) and the estimate (3.11) below (where the notation v_j stands for $u_{|I_j}$), as the constant in the inequality (2.2) blows up when M becomes infinitely large.

3. Proof of Theorem 2.2

The proof of Theorem 2.2 relies on the Fourier method, results of CO FC and an auxiliary result (Proposition 3.1). We split the proof into three parts, which can be found in the next subsections.

3.1. **The Fourier method.** Using the Fourier method, we build the solution to the IBVP (1.5). Namely, assuming that u is a solution to (1.5) in the sense of Definition (2.1), we write

$$u(\cdot,t) = \sum_{n=1}^{\infty} u_n(t) X_n,\ t \in I,$$

where $u_n(t) := \langle u(\cdot,t), X_n \rangle_{\mathscr{H}}$. Evidently, for each $n \in \mathbb{N}$, u_n is the solution to the following fractional differential system (FDS)

$$\begin{cases} \partial_t^{\beta} u_n + \lambda_n u_n = f_{0,n}, & t \in I, \\ u_n(0) = u_{0,n}, \end{cases} \tag{3.1}$$

with source term $f_{0,n}(t) := \langle f(\cdot,t), X_n \rangle_{\mathscr{H}}$ and initial state $u_{0,n} := \langle u_0, X_n \rangle_{\mathscr{H}}$. Therefore, the function $v_{j,n} := u_{n|\overline{I_j}}$, $j = 0, 1, \ldots, M-1$, $n \in \mathbb{N}$, solves

$$\begin{cases} \displaystyle\int_{t_j}^{t} \frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)} v'_{j,n}(s)\mathrm{d}s + \lambda_n v_{j,n}(t) = f_{j,n}(t), \quad t \in I_j, \\ v_{j,n}(t_j) = \upsilon_{j,n}, \end{cases} \tag{3.2}$$

where

$$f_{j,n}(t) := f_{0,n}(t) - \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)} v'_{k,n}(s)\mathrm{d}s, \quad t \in I_j, \tag{3.3}$$

and

$$\upsilon_{0,n} := u_{0,n}, \quad \upsilon_{j,n} := v_{j-1,n}(t_j), \quad j = 1, \ldots, M-1. \tag{3.4}$$

Here and in the remaining part of this text, any finite sum over $k = 0$ to $j-1$ is taken equal to zero when $j = 0$.

As a consequence, for all $j = 0, 1, \ldots, M-1$ and all $n \in \mathbb{N}$, we get by substituting $t - t_j$ for t in [17, Theorem 5.15], that the solution to (3.2) reads

$$v_{j,n}(t) := \upsilon_{j,n} E_{\beta_j,1}(-\lambda_n (t-t_j)^{\beta_j}) + \int_{t_j}^{t} (t-s)^{-1+\beta_j} E_{\beta_j,\beta_j}(-\lambda_n (t-s)^{\beta_j}) f_{j,n}(s)\mathrm{d}s, \ t \in \overline{I_j}, \tag{3.5}$$

where $E_{\alpha,\beta}$ denotes the two-parameter Mittag-Leffler function defined by $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$, see [24, Eq. (1.56)]. By [24, Theorem 1.6], we have for $\alpha \in (0,2)$ and $\beta > 0$ that

$$\exists C_E > 0, \ \forall z \in [0,\infty), \ |E_{\alpha,\beta}(-z)| \le \frac{C_E}{1+z} \le C_E. \tag{3.6}$$

3.2. **Auxiliary result.** With reference to (3.3) and (3.5), we set

$$v_j(\cdot,t) := \sum_{n=1}^{\infty} v_{j,n}(t) X_n, \quad f_j(\cdot,t) := \sum_{n=1}^{\infty} f_{j,n}(t) X_n, \quad t \in \overline{I_j}, \quad \text{for } j = 0, 1, \ldots, M-1, \tag{3.7}$$

and we note from this that

$$u_{|\overline{I_j}} = \sum_{n=1}^{\infty} u_{n|\overline{I_j}} X_n = \sum_{n=1}^{\infty} v_{j,n} X_n = v_j.$$

The proof of Theorem 2.2 essentially relies on the following technical result.

Proposition 3.1. *Suppose that u_0 and f satisfy the assumptions of Theorem 2.2. Then, putting*

$$\mathcal{F}_j := \|u_0\|_{D(\mathscr{L})} + \sum_{k=0}^{j} \left(\|f\|_{W^{1,1}(I_k,\mathscr{H})} + \left\| (\cdot - t_k)^{\beta_k+\varepsilon_k} f' \right\|_{L^\infty(I_k,\mathscr{H})} \right), \quad j = 0, \ldots, M-1, \tag{3.8}$$

we have for all $j = 0, \ldots, M-1$:

(i) $f_j \in W^{1,1}(I_j, \mathscr{H})$ and the estimates

$$\|f_j\|_{W^{1,1}(I_j,\mathscr{H})} \le C_j \mathcal{F}_j, \tag{3.9}$$

$$\left\| f'_j(\cdot,t) \right\| \le C_j \mathcal{F}_j (t-t_j)^{-\beta_j-\varepsilon_j}, \quad t \in I_j. \tag{3.10}$$

(ii) $v_j \in \mathcal{C}^0(\overline{I_j}, D(\mathcal{L})) \cap W^{1,1}(I_j, \mathscr{H})$ *and the estimates*

$$\|v_j\|_{\mathcal{C}^0(\overline{I_j}, D(\mathcal{L}))} + \|v_j\|_{W^{1,1}(I_j,\mathscr{H})} \leq C_j \mathcal{F}_j, \tag{3.11}$$

$$\left\| v_j'(\cdot, t) \right\| \leq C_j \mathcal{F}_j (t - t_j)^{-1+\beta_j}, \quad t \in I_j. \tag{3.12}$$

Here and in the remaining part of this text, C_j *denotes a generic positive constant depending only on* $\{(I_k, \beta_k, \varepsilon_k),\ k = 0, \ldots, j\}$*, which may change from line to line.*

The proof of Proposition 3.1 being quite tedious, we postpone it to Section 4.

We notice from Proposition 3.1 that $v_j \in D(\mathcal{L})$ for all $j = 0, \ldots, M-1$ (this follows from the identity $v_0 = u_0$ and the assumption $u_0 \in D(\mathcal{L})$ when $j = 0$, and from Proposition 3.1(ii), namely from $v_j \in \mathcal{C}^0(\overline{I_j}, D(\mathcal{L}))$, when $j = 1, \ldots, M-1$, since $v_{j-1}(\cdot, t_j) = v_j$ in this case). Moreover, we have $v_j(\cdot, t_j) = v_j$ for all $j = 0, \ldots, M-1$, by virtue of (3.5)-(3.7), and consequently

$$v_{j-1}(t_j) = v_j(t_j),\ j = 1, \ldots, M-1. \tag{3.13}$$

Now, since $v_j \in \mathcal{C}^0(\overline{I_j}, D(\mathcal{L}))$ for all $j = 0, \ldots, M-1$, by Proposition 3.1(ii), (3.13) and the identity $v_j = u_{|\overline{I_j}}$ then yield that

$$u \in \mathcal{C}^0(\overline{I}, D(\mathcal{L})). \tag{3.14}$$

3.3. **End of the proof of Theorem 2.2.** Let us first prove that $u' \in L^1(I, \mathscr{H})$. To this end, taking into account that $u \in L^1(I, \mathscr{H})$, according to (3.14) and the embedding $\mathcal{C}^0(\overline{I}, D(\mathcal{L})) \subset L^1(I, \mathscr{H})$, we find for every $\varphi \in \mathcal{C}_0^\infty(I)$ and a.e. $x \in \Omega$, that

$$\begin{aligned}
\langle u'(x,\cdot), \varphi \rangle_{\mathcal{C}_0^\infty(I)' \times \mathcal{C}_0^\infty(I)} &= -\langle u(x,\cdot), \varphi' \rangle_{\mathcal{C}_0^\infty(I)' \times \mathcal{C}_0^\infty(I)} \\
&= -\sum_{j=0}^{M-1} \int_{t_j}^{t_{j+1}} v_j(x,t)\varphi'(t)\mathrm{d}t \\
&= \sum_{j=0}^{M-1} \int_{t_j}^{t_{j+1}} v_j'(x,t)\varphi(t)\mathrm{d}t - \sum_{j=0}^{M-1} \left(v_j(x,t_{j+1})\varphi(t_{j+1}) - v_j(x,t_j)\varphi(t_j) \right).
\end{aligned}$$

Here we used that $v_j \in W^{1,1}(I_j, \mathscr{H})$ for all $j = 0, \ldots, M-1$, as stated in Proposition 3.1(ii). Further, bearing in mind that $v_j \in \mathcal{C}^0(\overline{I_j}, \mathscr{H})$ and that $\varphi(0) = \varphi(T) = 0$, we see that

$$\sum_{j=0}^{M-1} \left(v_j(x,t_{j+1})\varphi(t_{j+1}) - v_j(x,t_j)\varphi(t_j) \right) = 0,\ x \in \Omega.$$

Thus, we have

$$\langle u'(x,\cdot), \varphi \rangle_{\mathcal{C}_0^\infty(I)', \mathcal{C}_0^\infty(I)} = \sum_{j=0}^{M-1} \int_{t_j}^{t_{j+1}} v_j'(x,t)\varphi(t)\mathrm{d}t,\ x \in \Omega,$$

for all $\varphi \in \mathcal{C}_0^\infty(I)$, from where we get that

$$u'(x,t) = \sum_{j=0}^{M-1} \chi_{I_j}(t) v_j'(x,t),\ x \in \Omega,\ t \in I, \tag{3.15}$$

where χ_I denotes the characteristic function of I. As a consequence we have $u' \in L^1(I, \mathscr{H})$ from Proposition 3.1(ii), and hence $u \in \mathcal{C}^0(\overline{I}, D(\mathcal{L})) \cap W^{1,1}(I, \mathscr{H})$ by (3.14). Next, for all $j = 0, \ldots, M-1$ we have

$$\int_{t_j}^t \frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)} v_j'(x,s)\mathrm{d}s + \mathcal{L}v_j(x,t) = f_j(x,t),\ x \in \Omega,\ t \in \overline{I_j},$$

from the first equation of (3.1) and (3.7), whence

$$\int_0^t \frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)} u'(x,s)\mathrm{d}s + \mathcal{L}u(x,t) = f(x,t),\ x \in \Omega,\ t \in \overline{I_j},$$

by (3.3) and (3.15). Moreover, we have $u(\cdot,0) = u_0$ from (3.5)-(3.7) and the identities $u_{|\overline{I_0}} = v_0$ and $v_0 = u_0$. This proves that u is a solution to the IBVP (1.5) in the sense of Definition 2.1. There exists at most one solution to (1.5) since the solution to (3.2) on I_j, $j = 0, \ldots, M-1$, is unique according to [17, Theorem 4.3].

Finally, since $u(\cdot,t) = \sum_{j=0}^{M-1} \chi_{\overline{I_j}}(t) v_j(\cdot,t)$ for all $t \in \overline{I}$, we have

$$\|u\|_{\mathcal{C}^0(\overline{I},D(\mathcal{L}))} = \max\{\|v_j\|_{\mathcal{C}^0(\overline{I_j},D(\mathcal{L}))},\ j = 0, \ldots, M-1\}, \tag{3.16}$$

and

$$\|u\|_{W^{1,1}(I,\mathscr{H})} = \sum_{j=0}^{M-1} \|v_j\|_{W^{1,1}(I_j,\mathscr{H})}, \tag{3.17}$$

according to (3.15), so we obtain (2.2) by combining (3.11) with (3.16)-(3.17).

4. Proof of Proposition 3.1

The proof is by induction on j. Given $j \in \{1, \ldots, M-1\}$, the induction hypothesis $(\mathrm{IH})_j$ is that the claim of Proposition 3.1 (and in particular the estimates (3.9) – (3.12)) holds when $k = 0, 1, \ldots, j-1$ is substituted for j in its statement.

As the method of the proof is basically the same for $j = 0$ and for $j \in \{1, \ldots, M-1\}$, we start with the inductive step, only pointing out in the initial step the specifics of the proof for $j = 0$ when needed.

4.1. **Inductive step.**
Let $j \in \{1, \ldots, M-1\}$ be fixed. Assuming $(\mathrm{IH})_j$, we aim to prove the statement of Proposition 3.1.

4.1.1. *Proof of (i).*
We split the proof into two main steps.

Step 1. We start by showing that $f_j \in L^1(I_j, \mathscr{H})$ satisfies the estimate

$$\|f_j\|_{L^1(I_j,\mathscr{H})} \leq C_j \mathcal{F}_j. \tag{4.1}$$

To this purpose we refer to (3.3) and (3.7), and obtain that

$$f_j(\cdot,t) = \sum_{n=1}^{\infty} \left(f_{0,n}(t) - \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)} v'_{k,n}(s) \mathrm{d}s \right) X_n, \; t \in I_j. \tag{4.2}$$

Next, for all $N \in \mathbb{N}$ we have

$$\begin{aligned} &\left\| \sum_{n=1}^{N} \left(f_{0,n}(t) - \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)} v'_{k,n}(s) \mathrm{d}s \right) X_n \right\| \\ &\leq \left\| \sum_{n=1}^{N} f_{0,n}(t) X_n \right\| + \frac{1}{\Gamma(1-\beta_j)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t-s)^{-\beta_j} \left\| \sum_{n=1}^{N} v'_{k,n}(s) X_n \right\| \mathrm{d}s \\ &\leq \|f(\cdot,t)\| + \frac{1}{\Gamma(1-\beta_j)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t-s)^{-\beta_j} \|v'_k(s)\| \, \mathrm{d}s, \end{aligned}$$

and $(t-s)^{-\beta_j} \leq (t-t_{k+1})^{-\beta_j}$ whenever $s \leq t_{k+1} \leq t_j < t$ and $k = 0, \ldots, j-1$. This and (4.2) yield that

$$\|f_j(\cdot,t)\| \leq \|f(\cdot,t)\| + \frac{1}{\Gamma(1-\beta_j)} \sum_{k=0}^{j-1} (t-t_{k+1})^{-\beta_j} \|v'_k\|_{L^1(I_k,\mathscr{H})}, \; t \in I_j.$$

Thus, we get, by integrating with respect to t over I_j and using that $(t_{j+1}-t_{k+1})^{1-\beta_j} - (t_j - t_{k+1})^{1-\beta_j} \leq t_{j+1}^{1-\beta_j}$ for all $k = 0, \ldots, j-1$, that

$$\|f_j\|_{L^1(I_j,\mathscr{H})} \leq \|f\|_{L^1(I_j,\mathscr{H})} + \frac{t_{j+1}^{1-\beta_j}}{\Gamma(2-\beta_j)} \sum_{k=0}^{j-1} \|v'_k\|_{L^1(I_k,\mathscr{H})}.$$

Now, applying $(\mathrm{IH})_j$, and more precisely (3.11) where $k = 0, \ldots, j-1$ is substituted for j, we find that

$$\|f_j\|_{L^1(I_j,\mathscr{H})} \leq \|f\|_{L^1(I_j,\mathscr{H})} + \frac{t_{j+1}^{1-\beta_j}}{\Gamma(2-\beta_j)} \sum_{k=0}^{j-1} C_k \mathcal{F}_k,$$

and (4.1) follows from this and the estimates $\mathcal{F}_k \leq \mathcal{F}_{j-1}$ for $k = 0, \ldots, j-1$.

Step 2. Having established (4.1), we turn now to the proof of (3.10). This can be achieved by differentiating (4.2) with respect to $t \in I_j$: we obtain that

$$f'_j(\cdot,t) = f'(\cdot,t) + \frac{\beta_j}{\Gamma(1-\beta_j)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t-s)^{-1-\beta_j} v'_k(\cdot,s) \mathrm{d}s,$$

and hence that

$$\|f'_j(\cdot,t)\| \leq \|f'(\cdot,t)\| + \frac{\beta_j}{\Gamma(1-\beta_j)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t-s)^{-1-\beta_j} \|v'_k(\cdot,s)\| \, \mathrm{d}s. \tag{4.3}$$

The main difficulty lies in handling the term $k = j-1$ in the above summation. We distinguish in two cases. In the first one, when $k = 0, \dots, j-2$, we get upon using that $(t-s)^{-1-\beta_j} \leq (t_j - t_{j-1})^{-\beta_j - 1}$ for all $t \in I_j$ and $s \in I_k$, that

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} (t-s)^{-1-\beta_j} \left\| v_k'(\cdot, s) \right\| \mathrm{d}s &= \int_{t_k}^{t_{k+1}} (t-s)^{-1-\beta_j} (s-t_k)^{-1+\beta_k} \left\| (s-t_k)^{1-\beta_k} v_k'(\cdot, s) \right\| \mathrm{d}s \\
&\leq (t_j - t_{j-1})^{-1-\beta_j} \left(\int_{t_k}^{t_{k+1}} (s-t_k)^{-1+\beta_k} \mathrm{d}s \right) \left\| (\cdot - t_k)^{1-\beta_k} v_k' \right\|_{L^\infty(I_k, \mathscr{H})} \\
&\leq \beta_k^{-1} (t_j - t_{j-1})^{-1-\beta_j} (t_{k+1} - t_k)^{\beta_k} \left\| (\cdot - t_k)^{1-\beta_k} v_k' \right\|_{L^\infty(I_k, \mathscr{H})}. \quad (4.4)
\end{aligned}$$

In the second case, when $k = j-1$, we have $(t-s)^{-\beta_j - \varepsilon_j} \leq (t-t_j)^{-\beta_j - \varepsilon_j}$ and $(t-s)^{-1+\varepsilon_j} \leq (t_j - s)^{-1+\varepsilon_j}$ for all $t \in I_j$ and $s \in I_{j-1}$, whence

$$\begin{aligned}
&\int_{t_{j-1}}^{t_j} (t-s)^{-1-\beta_j} \left\| v_{j-1}'(\cdot, s) \right\| \mathrm{d}s \\
&= \int_{t_{j-1}}^{t_j} (t-s)^{-\beta_j - \varepsilon_j} (t-s)^{-1+\varepsilon_j} (s - t_{j-1})^{-1+\beta_{j-1}} (s - t_{j-1})^{1-\beta_{j-1}} \left\| v_{j-1}'(s) \right\| \mathrm{d}s \\
&\leq (t-t_j)^{-\beta_j - \varepsilon_j} \left(\int_{t_{j-1}}^{t_j} (t_j - s)^{-1+\varepsilon_j} (s - t_{j-1})^{-1+\beta_{j-1}} \mathrm{d}s \right) \left\| (\cdot - t_{j-1})^{1-\beta_{j-1}} v_{j-1}' \right\|_{L^\infty(I_{j-1}, \mathscr{H})}.
\end{aligned}$$

Further, by performing the change of variable $r = \frac{s - t_{j-1}}{t_j - t_{j-1}}$ in the integral of the above line, we see that

$$\begin{aligned}
\int_{t_{j-1}}^{t_j} (t_j - s)^{-1+\varepsilon_j} (s - t_{j-1})^{-1+\beta_{j-1}} \mathrm{d}s &= (t_j - t_{j-1})^{-1+\beta_{j-1}+\varepsilon_j} \left(\int_0^1 (1-r)^{-1+\varepsilon_j} r^{-1+\beta_{j-1}} \mathrm{d}r \right) \\
&= (t_j - t_{j-1})^{-1+\beta_{j-1}+\varepsilon_j} \mathcal{B}(\beta_{j-1}, \varepsilon_j),
\end{aligned}$$

where $\mathcal{B}$ denotes the beta function

$$\mathcal{B}(r_1, r_2) := \int_0^1 s^{r_1 - 1} (1-s)^{r_2 - 1} \mathrm{d}s, \quad r_1 \in (0, \infty),\ r_2 \in (0, \infty).$$

As a consequence, we have for all $t \in I_j$,

$$\begin{aligned}
&\int_{t_{j-1}}^{t_j} (t-s)^{-1-\beta_j} \left\| v_{j-1}'(\cdot, s) \right\| \mathrm{d}s \\
&\leq (t_j - t_{j-1})^{-1+\beta_{j-1}+\varepsilon_j} \mathcal{B}(\beta_{j-1}, \varepsilon_j) \left\| (\cdot - t_{j-1})^{1-\beta_{j-1}} v_{j-1}' \right\|_{L^\infty(I_{j-1}, \mathscr{H})} (t-t_j)^{-\beta_j - \varepsilon_j}.
\end{aligned}$$

Putting this together with (4.3)-(4.4), and applying (3.12) where $k = 0, \dots, j-1$ is substituted for j, in accordance with $(\mathrm{IH})_j$, we find for a.e. $t \in I_j$ that the difference

$$(t-t_j)^{\beta_j + \varepsilon_j} \left\| f_j'(\cdot, t) \right\| - (t-t_j)^{\beta_j + \varepsilon_j} \left\| f'(\cdot, t) \right\|$$

is upper bounded by the constant

$$\frac{\beta_j}{\Gamma(1-\beta_j)} \left(\frac{(t_{j+1} - t_j)^{\beta_j + \varepsilon_j}}{(t_j - t_{j-1})^{1+\beta_j}} \sum_{k=0}^{j-2} \frac{C_k (t_{k+1} - t_k)^{\beta_k}}{\beta_k} \mathcal{F}_k + \frac{C_{j-1} \mathcal{B}(\beta_{j-1}, \varepsilon_j)}{(t_j - t_{j-1})^{1-\beta_{j-1}-\varepsilon_j}} \mathcal{F}_{j-1} \right) \leq C_j \mathcal{F}_{j-1}.$$

Now, (3.10) follows readily from this and (2.1). Moreover, using (4.1) and recalling that $\beta_j + \varepsilon_j \in (0,1)$, we get (3.9) by integrating (3.10) over I_j.

We turn now to showing (ii), but prior to that, we notice for further use from (3.9) and from the continuous embedding $W^{1,1}(I_j, \mathscr{H}) \subset \mathcal{C}^0(\overline{I_j}, \mathscr{H})$, that

$$\|f_j(\cdot, t)\| \leq C_j \mathcal{F}_j, \quad t \in \overline{I_j}. \tag{4.5}$$

4.1.2. *Proof of (ii).* The proof of the second claim, (ii), of Proposition 3.1 being quite lengthy, we break it into two parts.

Step 1. The first step is to show that $v_j \in \mathcal{C}^0(\overline{I_j}, D(\mathcal{L}))$ satisfies $\|v_j\|_{\mathcal{C}^0(\overline{I_j}, D(\mathcal{L}))} \leq C_j \mathcal{F}_j$. To do that, we recall from (i) that the function $t \mapsto f_{j,n}(t) = \langle f_j(\cdot, t), X_n \rangle \in W^{1,1}(I_j)$, for all $n \in \mathbb{N}$. Then, bearing in mind that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right), \ t > 0.$$

and integrating by parts, we obtain for all $t \in \overline{I_j}$ that

$$\begin{aligned}
&\int_{t_j}^t (t-s)^{-1+\beta_j} E_{\beta_j,\beta_j}(-\lambda_n (t-s)^{\beta_j}) f_{j,n}(s) \mathrm{d}s \\
&\qquad = \frac{1}{\lambda_n} \left(E_{\beta_j,1}(0) f_{j,n}(t) - E_{\beta_j,1}(-\lambda_n (t-t_j)^{\beta_j}) f_{j,n}(t_j) + \int_{t_j}^t E_{\beta_j,1}(-\lambda_n (t-s)^{\beta_j}) f'_{j,n}(s) \mathrm{d}s \right).
\end{aligned}$$

As a consequence, we have for all $N \in \mathbb{N}$ that

$$\begin{aligned}
&\left\| \sum_{n=1}^N \left(\int_{t_j}^t (t-s)^{-1+\beta_j} E_{\beta_j,\beta_j}(-\lambda_n (t-s)^{\beta_j}) f_{j,n}(s) \mathrm{d}s \right) X_n \right\|_{D(\mathcal{L})} \\
&\leq \left\| \sum_{n=1}^N \left(E_{\beta_j,1}(0) f_{j,n}(t) - E_{\beta_j,1}(-\lambda_n (t-t_j)^{\beta_j}) f_{j,n}(t_j) \right) X_n \right\| \\
&\quad + \left\| \sum_{n=1}^N \left(\int_{t_j}^t E_{\beta_j,1}(-\lambda_n (t-s)^{\beta_j}) f'_{j,n}(s) \mathrm{d}s \right) X_n \right\|, \quad t \in \overline{I_j}.
\end{aligned} \tag{4.6}$$

The first term on the right-hand side of (4.6) is simply handled by (3.6), as follows:

$$\begin{aligned}
&\left\| \sum_{n=1}^N \left(E_{\beta_j,1}(0) f_{j,n}(t) - E_{\beta_j,1}(-\lambda_n (t-t_j)^{\beta_j}) f_{j,n}(t_j) \right) X_n \right\| \\
&\leq \left\| \sum_{n=1}^N E_{\beta_j,1}(0) f_{j,n}(t) X_n \right\| + \left\| \sum_{n=1}^N E_{\beta_j,1}(-\lambda_n (t-t_j)^{\beta_j}) f_{j,n}(t_j) X_n \right\| \\
&\leq C_E \left(\left(\sum_{n=1}^N |f_{j,n}(t)|^2 \right)^{1/2} + \left(\sum_{n=1}^N |f_{j,n}(t_j)|^2 \right)^{1/2} \right) \\
&\leq C_E \left(\|f_j(\cdot, t)\| + \|f_j(\cdot, t_j)\| \right).
\end{aligned} \tag{4.7}$$

As for the second term, we get through standard computations that

$$\left\| \sum_{n=1}^N \left(\int_{t_j}^t E_{\beta_j,1}(-\lambda_n (t-s)^{\beta_j}) f'_{j,n}(s) \mathrm{d}s \right) X_n \right\| \leq \int_{t_j}^t \left\| \sum_{n=1}^N E_{\beta_j,1}(-\lambda_n (t-s)^{\beta_j}) f'_{j,n}(s) X_n \right\| \mathrm{d}s$$

$$\leq C_E \int_{t_j}^{t} \left(\sum_{n=1}^{N} \left| f'_{j,n}(s) \right|^2 \right)^{1/2} \mathrm{d}s$$
$$\leq C_E \int_{t_j}^{t} \left\| f'_j(s) \right\| \mathrm{d}s.$$

This and (4.6)-(4.7) then yield that for all $t \in \overline{I_j}$,

$$\left\| \sum_{n=1}^{\infty} \left(\int_{t_j}^{t} (t-s)^{-1+\beta_j} E_{\beta_j,\beta_j}(-\lambda_n (t-s)^{\beta_j}) f_{j,n}(s) \mathrm{d}s \right) X_n \right\|_{D(\mathcal{L})}$$
$$\leq C_E \left(2 \left\| f_j \right\|_{\mathcal{C}^0(\overline{I_j},\mathscr{H})} + \left\| f'_j \right\|_{L^1(I_j,\mathscr{H})} \right). \quad (4.8)$$

Now, taking into account that

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} v_{j,n} E_{\beta_j,1}(-\lambda_n (t-t_j)^{\beta_j}) X_n \right\|_{D(\mathcal{L})} &= \left(\sum_{n=1}^{\infty} \lambda_n^2 \left| v_{j,n} \right|^2 E_{\beta_j,1}(-\lambda_n (t-t_j)^{\beta_j})^2 \right)^{\frac{1}{2}} \\ &\leq C_E \left(\sum_{n=1}^{\infty} \lambda_n^2 \left| v_{j,n} \right|^2 \right)^{\frac{1}{2}} \\ &\leq C_E \left\| v_j \right\|_{D(\mathcal{L})}, \quad t \in \overline{I_j}, \end{aligned} \quad (4.9)$$

and recalling (3.5)-(3.7), we deduce from (4.8) that

$$\left\| v_j(\cdot,t) \right\|_{D(\mathcal{L})} \leq C_E \left(\left\| v_j \right\|_{D(\mathcal{L})} + 2 \left\| f_j \right\|_{\mathcal{C}^0(\overline{I_j},\mathscr{H})} + \left\| f'_j \right\|_{L^1(I_j,\mathscr{H})} \right), \quad t \in \overline{I_j}. \quad (4.10)$$

Next, bearing in mind that $v_j = v_{j-1}(\cdot, t_j)$ since $j \geq 1$ here, we infer from $(\mathrm{IH})_j$ (namely, upon substituting $j-1$ for j in (3.11)) that

$$\left\| v_j \right\|_{D(\mathcal{L})} \leq C_{j-1} \mathcal{F}_{j-1}. \quad (4.11)$$

By inserting this, (3.9) and (4.5) into (4.10), we obtain that

$$\left\| v_j(\cdot,t) \right\|_{D(\mathcal{L})} \leq C_j \mathcal{F}_j, \quad t \in \overline{I_j}. \quad (4.12)$$

Further, we see from (3.5) that for all fixed $N \in \mathbb{N}$, the function $t \mapsto \sum_{n=1}^{N} v_{j,n}(t) X_n \in \mathcal{C}^0(\overline{I_j}, D(\mathcal{L}))$. Moreover, with reference to (3.5)-(3.7), it is clear from (4.8)-(4.9) that the series $\sum_{n=1}^{\infty} v_{j,n}(t) X_n$ converges to $v_j(\cdot,t)$ in $D(\mathcal{L})$, uniformly in $t \in \overline{I_j}$. Therefore, we have

$$v_j \in \mathcal{C}^0(\overline{I_j}, D(\mathcal{L})) \quad (4.13)$$

and the estimate

$$\left\| v_j \right\|_{\mathcal{C}^0(\overline{I_j}, D(\mathcal{L}))} \leq C_j \mathcal{F}_j, \quad (4.14)$$

according to (4.12).

Step 2. Having established (4.13)-(4.14), we turn now to proving the estimate (3.12). For this purpose, we differentiate (3.5) with respect to $t \in I_j$, and obtain that

$$\begin{aligned} v'_{j,n}(t) &= (f_{j,n}(t_j) - \lambda_n v_{j,n})(t-t_j)^{-1+\beta_j} E_{\beta_j,\beta_j}(-\lambda_n (t-t_j)^{\beta_j}) \\ &\quad + \int_{t_j}^{t} (t-s)^{-1+\beta_j} E_{\beta_j,\beta_j}(-\lambda_n (t-s)^{\beta_j}) f'_{j,n}(s) \mathrm{d}s, \ n \in \mathbb{N}. \end{aligned} \tag{4.15}$$

Next, using (3.6), we get for all $N \in \mathbb{N}$ and all $t \in I_j$ that

$$\begin{aligned} &\left\| \sum_{n=1}^{N} \left(f_{j,n}(t_j) - \lambda_n v_{j,n})(t-t_j)^{-1+\beta_j} E_{\beta_j,\beta_j}(-\lambda_n (t-t_j)^{\beta_j} \right) X_n \right\| \\ &\le \left\| \sum_{n=1}^{N} f_{j,n}(t_j)(t-t_j)^{-1+\beta_j} E_{\beta_j,\beta_j} \left(-\lambda_n (t-t_j)^{\beta_j} \right) X_n \right\| \\ &\quad + \left\| \sum_{n=1}^{N} \lambda_n v_{j,n}(t-t_j)^{-1+\beta_j} E_{\beta_j,\beta_j} \left(-\lambda_n (t-t_j)^{\beta_j} \right) X_n \right\| \\ &\le C_E (t-t_j)^{-1+\beta_j} \left(\left(\sum_{n=1}^{N} |f_{j,n}(t_j)|^2 \right)^{1/2} + \left(\sum_{n=1}^{N} \lambda_n^2 |v_{j,n}|^2 \right)^{1/2} \right) \\ &\le C_E (t-t_j)^{-1+\beta_j} \left(\|f_j(\cdot,t_j)\| + \|v_j\|_{D(\mathcal{L})} \right) \end{aligned}$$

and that

$$\begin{aligned} &\left\| \sum_{n=1}^{N} \left(\int_{t_j}^{t} (t-s)^{-1+\beta_j} E_{\beta_j,\beta_j}(-\lambda_n (t-s)^{\beta_j}) f'_{j,n}(s) \mathrm{d}s \right) X_n \right\| \\ &\le \int_{t_j}^{t} (t-s)^{-1+\beta_j} \left\| \sum_{n=1}^{N} E_{\beta_j,\beta_j}(-\lambda_n (t-s)^{\beta_j}) f'_{j,n}(s) X_n \right\| \mathrm{d}s \\ &\le C_E \int_{t_j}^{t} (t-s)^{-1+\beta_j} \left(\sum_{n=1}^{N} \left| f'_{j,n}(s) \right|^2 \right)^{1/2} \mathrm{d}s \\ &\le C_E \int_{t_j}^{t} (t-s)^{-1+\beta_j} \left\| f'_j(\cdot,s) \right\| \mathrm{d}s. \end{aligned}$$

Therefore, it follows from (3.5)-(3.7) and (4.15) that for all $t \in I_j$,

$$\left\| v'_j(\cdot,t) \right\| \le C_E \left((t-t_j)^{-1+\beta_j} \left(\|f_j(\cdot,t_j)\| + \|v_j\|_{D(\mathcal{L})} \right) + \int_{t_j}^{t} (t-s)^{-1+\beta_j} \left\| f'_j(\cdot,s) \right\| \mathrm{d}s \right). \tag{4.16}$$

Moreover, since $\beta_j + \varepsilon_j \in (0,1)$, we apply (3.10) and get for all $t \in I_j$ that

$$\begin{aligned} &\int_{t_j}^{t} (t-s)^{-1+\beta_j} \left\| f'_j(\cdot,s) \right\| \mathrm{d}s \\ &\le \left\| (\cdot - t_j)^{\beta_j+\varepsilon_j} f'_j \right\|_{L^\infty(I_j,\mathscr{H})} \int_{t_j}^{t} (t-s)^{-1+\beta_j} (s-t_j)^{-\beta_j-\varepsilon_j} \mathrm{d}s \\ &\le C_j \mathcal{F}_j \mathcal{B}(1-\beta_j-\varepsilon_j, \beta_j)(t-t_j)^{-\varepsilon_j} \\ &\le C_j \mathcal{F}_j \mathcal{B}(1-\beta_j-\varepsilon_j, \beta_j)(t_{j+1}-t_j)^{1-\beta_j-\varepsilon_j}(t-t_j)^{-1+\beta_j}, \end{aligned}$$

so we obtain (3.12) by plugging the above estimate, (4.5) and (4.11) into (4.16). Finally, (3.11) follows straightforwardly from (3.12) and (4.14).

4.2. **Base step.** Firstly, since $f_0 = f_{|I_0}$, it is clear from the assumption $f_{|I_0} \in W^{1,1}(I_0, \mathscr{H})$ and from (2.1) that the claims of (i) in Proposition 3.1, hold for $j = 0$.

Secondly, taking $j = 0$ in the derivation of (4.10), we get that for all $t \in I_0$,

$$\|v_0(\cdot, t)\|_{D(\mathcal{L})} \leq C_E \left(\|u_0\|_{D(\mathcal{L})} + 2 \|f\|_{\mathcal{C}^0(\overline{I_0}, \mathscr{H})} + \|f'\|_{L^1(I_0, \mathscr{H})} \right). \tag{4.17}$$

Similarly, since

$$\int_0^t (t-s)^{-1+\beta_0} \|f'(\cdot, s)\| \, \mathrm{d}s \leq \mathcal{B}(1 - \beta_0 - \varepsilon_0, \beta_0) t_1^{1-\beta_0-\varepsilon_0} \left\| (\cdot)^{\beta_0+\varepsilon_0} f' \right\|_{L^\infty(I_j, \mathscr{H})} t^{-1+\beta_0},$$

for all $t \in I_0$, we find upon mimicking the proof of (4.16) that

$$\|v_0'(\cdot, t)\| \leq C_E \left(\|u_0\|_{D(\mathcal{L})} + \|f(\cdot, 0)\| + \mathcal{B}(1 - \beta_0 - \varepsilon_0, \beta_0) t_1^{-1+\beta_0} \left\| (\cdot)^{\beta_0+\varepsilon_0} f' \right\|_{L^\infty(I_0, \mathscr{H})} \right) t^{-\varepsilon_0}. \tag{4.18}$$

Therefore, the estimates (3.11)-(3.12) for $j = 0$ follow from (4.17)-(4.18) and the continuity of the embedding $W^{1,1}(I_0, \mathscr{H}) \subset \mathcal{C}^0(\overline{I_0}, \mathscr{H})$.

This completes the proof of the proposition.

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