MAGNETIC QUANTUM CURRENTS IN THE PRESENCE OF A NEUMANN WALL

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ABSTRACT. The Schrödinger operator with constant magnetic on a half-place with Neumann boundary conditions is considered. Low energy currents flowing along the boundary are analyzed and used to establish a Limiting Absorption Principle for the electrically perturbed operator.

1. CONTEXT AND MOTIVATION

1.1. **Definition of the main operator.** In this paper, we consider the Hamitonian with constant magnetic field on the half-plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and Neumann boundary condition:

$$\mathscr{L}_h := D_x^2 + (hD_y - x)^2 \,, \quad D = -i\partial \,, \quad h > 0 \,, \tag{1.1}$$

acting on the domain

$$\text{Dom}\,(\mathscr{L}_h) = \Big\{ \psi \in L^2(\mathbb{R}^2_+) : D_x\psi \in L^2(\mathbb{R}^2_+) \,, (hD_y - x)\psi \in L^2(\mathbb{R}^2_+) \,, \\ \big(D_x^2 + (hD_y - x)^2\big)\,\psi \in L^2(\mathbb{R}^2_+) \text{ and } \forall y \in \mathbb{R} \,, \partial_x\psi(0,y) = 0 \Big\} \,.$$

This operator appears in many contexts. For instance, it plays a major role in the study of surface superconductivity, see [14], and it also acquired a life of its own over the years, see [24]. The present paper is concerned with the semiclassical spectral analysis of the operator $\mathscr{L}_h$, that is to say that we mostly focus on spectral results for the operator $\mathscr{L}_h$ as $h \to 0$. It might seem surprising at first sight that y is the only variable affected by the semiclassical parameter h in this model, but this can be explained as follows. By performing the rescaling

$$x = h^{-\frac{1}{2}}\mathsf{x} \,, \quad y = h^{\frac{1}{2}}\mathsf{y} \,,$$

in (1.1), we notice that the operator $\mathscr{L}_h$ is unitarily equivalent to

$$B^{-1}\left(D_{\mathsf{x}}^2 + (D_{\mathsf{y}} - B\mathsf{x})^2\right) \,, \quad \text{where } B = h^{-1} \,,$$

which shows that the semiclassical limit corresponds to the *large magnetic field regime*. Such partially semiclassical scaling has already been used by numerous authors in spectral theory of magnetic Schrödinger operators, as, e.g., in [3, 4] and in [25], where magnetic models *à la* Iwatsuka (see [20]) are considered. This paper is also fitted in the semiclassical framework in order to give a lighter formulation of the results and to use the power of the theory of pseudo-differential operators.

1.2. **A short bibliography.** Magnetic quantum Hall currents have attracted a lot of attention from the mathematical community over the last decades, see, e.g., [7, 10, 12, 13, 15, 17, 16, 21], this list being non exhaustive. Quantum Hall devices can be modeled by a magnetic Laplacian describing the planar motion of an electron submitted to a constant transverse magnetic field. The electron is confined to unbounded regions of the plane by potential barriers or Dirichlet boundary conditions. The edge, be it the edge of the electrostatic confining potential or the Dirichlet boundary of the spatial domain, creates edge currents, whereas for Iwatsuka Hamiltonians (see [20]), quantum currents are generated by changes in the strength of the magnetic field, see, e.g., [11, 18]. Some authors have also observed that the role of the "edge" could be played by a discontinuity of the magnetic field, see [19] or [2] where the dispersion curves are studied in detail.

In translationally invariant unbounded straight-edge geometries such as the half-plane or the infinite strip, the unperturbed quantum Hall Hamiltonian is fibered and edge currents occur at energies associated with fibers whose group velocity is non-zero, see [7, 10, 15, 17, 16, 21]. In this case, the existence of edge currents can be linked to the absolutely continuous spectral nature of the spectrum through the use of Mourre's positive commutator method, see, e.g. [5, 6], or also [25] (where the absolute continuity is established by elementary means for Iwatsuka Hamiltonians). Similarly, limit absorption for the quantum Hall Hamiltonian in $\mathbb{R}^2_+$ at all energies except Landau levels, was derived in [23] from the monotonicity of the dispersion relations.

Notice that quantum Hall currents in two-edge geometries such as the infinite strip, propagate in opposite directions along the left and right edges of the band, respectively, see [16]. This can be understood from the fact that, unlike for the case of one-edge geometries, the dispersion functions of the corresponding quantum Hamiltonian are no longer monotonic, see [17, 16]. Moreover, these functions being ”symmetric” about their minimum value, the net current flowing across any line orthogonal to the strip, is zero.

This picture is quite reminiscent of the one of the present paper where none of the dispersion functions of the Neumann Laplacian in $\mathbb{R}^2_+$ is monotonic. Nevertheless, in contrast to the quantum Hall Hamiltonian in the band, the dispersion relations of the Neumann Laplacian are not symmetric about their minimum, which motivates for a closer look into its transport and spectral properties.

1.3. **Basic results on low energy currents.** In this section, we describe some elementary results about the low energy currents associated with the Hamiltonian $\mathscr{L}_h$. Due to the translational invariance of this operator in the y-direction, we shall see that its discrete spectrum is empty.

1.3.1. *Fiber decomposition.* Let us define the (unitary) semiclassical Fourier transform with respect to y as

$$\mathscr{F}_h\varphi(x,\xi)=\frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} e^{-i\xi y/h}\varphi(x,y)\mathrm{d}y\,.$$

Then, $\mathscr{L}_h$ being invariant in the y-direction, we have the direct integral decomposition

$$\mathscr{F}_h\mathscr{L}_h\mathscr{F}_h^{-1}=\int_{\mathbb{R}}^{\oplus}\ell_\xi\,\mathrm{d}\xi\,,$$

where for all $\xi\in\mathbb{R}$, ℓ_ξ is the Neumann realization in $L^2(\mathbb{R}_+)$ of the differential operator

$$D_x^2+(\xi-x)^2\,.$$

The operator ℓ_ξ is sometimes referred as the "de Gennes operator”. Since $x\mapsto(\xi-x)^2$ is unbounded as x goes to infinity, ℓ_ξ has a compact resolvent and we denote by $\mu_j(\xi)$

its j-th eigenvalue, $j \in \mathbb{N} := \{1, 2, \dots\}$. The well-known properties of this operator are recalled in the following proposition (see the original paper [9], and [14, Section 3.2], [24, Section 2.4]).

Proposition 1.1. *For all $j \in \mathbb{N}$, it holds true that:*

i. *$\xi \mapsto \mu_j(\xi)$ is analytic in $\mathbb{R}$ and has a unique minimum denoted by ξ_{j-1}, which is non-degenerate.*
ii. *μ_j decreases on $(-\infty, \xi_{j-1})$ and increases on $(\xi_{j-1}, +\infty)$.*
iii. *$\Theta_{j-1} = \xi_{j-1}^2$, where $\Theta_{j-1} := \mu_j(\xi_{j-1})$.*
iv. *$\lim_{\xi \to +\infty} \mu_j(\xi) = 2j - 1$ and $\lim_{\xi \to -\infty} \mu_j(\xi) = +\infty$.*
v. *$\Theta_0 \in (0, 1)$ and $\Theta_{j-1} \in (2j - 3, 2j - 1)$.*

Since the μ_j's are non-constant by Proposition 1.1.iv, the spectrum of $\mathscr{L}_h$ is purely absolutely continuous (see [20]). Moreover we have

$$\mathsf{sp}(\mathscr{L}_h) = [\Theta_0, +\infty)\,,$$

according to Proposition 1.1.v (see [26, Theorem XIII.85]).

1.3.2. *Basic properties of low energy magnetic quantum currents.* Let us define the current operator in the direction y by

$$\mathscr{J}_h := [\mathscr{L}_h, iy] = 2h(hD_y - x)\,,$$

and the current operator with energy concentration in I by

$$\mathscr{J}_{h,I} := \mathbb{1}_I(\mathscr{L}_h)\mathscr{J}_h\mathbb{1}_I(\mathscr{L}_h)\,.$$

The operator $\mathscr{J}_{h,I}$ is bounded on $E_{h,I} := \mathsf{range}(\mathbb{1}_I(\mathscr{L}_h))$ and symmetric.

Put

$$\lambda_{\min}(h, I) := \inf \mathsf{sp}\left(\mathscr{J}_{h,I}\right)\,, \quad \lambda_{\max}(h, I) := \sup \mathsf{sp}\left(\mathscr{J}_{h,I}\right)\,.$$

We recall from the min-max principle that

$$\lambda_{\min}(h, I) = \inf_{\substack{\psi \in E_{h,I} \\ \psi \neq 0}} \frac{\langle \mathscr{J}_{h,I}\psi, \psi\rangle}{\|\psi\|^2_{L^2(\mathbb{R}^2_+)}}\,, \quad \lambda_{\max}(h, I) = \sup_{\substack{\psi \in E_{h,I} \\ \psi \neq 0}} \frac{\langle \mathscr{J}_{h,I}\psi, \psi\rangle}{\|\psi\|^2_{L^2(\mathbb{R}^2_+)}}\,.$$

Then, the spectral radius of $\mathscr{J}_{h,I}$, given by

$$\rho\left(\mathscr{J}_{h,I}\right) = \max\left(|\lambda_{\min}(h, I)|, |\lambda_{\max}(h, I)|\right)\,,$$

is the maximal strength of the (absolute value of the) current carried by the energy window I.

Definition 1.2. Let $e \in [\Theta_0, \Theta_1)$ and consider the equation

$$\mu_1(\xi) = e\,. \tag{1.2}$$

i. When $e \in (\Theta_0, 1)$, the equation (1.2) has two solutions $\mu_-^{-1}(e) < \xi_0 < \mu_+^{-1}(e)$.
ii. When $e = \Theta_0$, the equation (1.2) has only one solution $\mu_\pm^{-1}(\Theta_0) = \xi_0$.
iii. When $e \in (1, \Theta_1)$, the equation (1.2) has one solution $\mu_-^{-1}(e) < \xi_0$ and, by convention, we set $\mu_+^{-1}(e) = +\infty$.

For all $e \in [\Theta_0, \Theta_1)$, we define the *algebraic current* as

$$c(e) = \mu_1'(\mu_-^{-1}(e)) + \mu_1'(\mu_+^{-1}(e))\,, \tag{1.3}$$

where we have set $\mu_1'(+\infty) = 0$.

In the sequel, it will be assumed at some stage of the analysis that the following conjecture (which is supported by numerical simulations carried out by M. P. Sundqvist with Mathematica) is verified.

Conjecture 1.3. *We have* $\mu_1^{(3)}(\xi_0) < 0$.

Lemma 1.4. *Assume that Conjecture 1.3 is true. Then, there exists* $e_\star \in (\Theta_0, 1)$ *such that for all* $e \in (\Theta_0, e_\star)$*, we have* $c(e) < 0$.

For $e \in [\Theta_0, \Theta_1)$, we set

$$I_\delta := [e - \delta, e + \delta] .$$

Proposition 1.5. *Assume that* $\mathbb{R}_+ \ni h \mapsto \delta(h)$ *tends to* 0 *as* $h \downarrow 0$*. Then, we have*

$$h^{-1}\lambda_{\min}(I_\delta, h) = \mu_1'(\mu_-^{-1}(e)) + \mathcal{O}(\delta(h)) , \quad h^{-1}\lambda_{\max}(I_\delta, h) = \mu_1'(\mu_+^{-1}(e)) + \mathcal{O}(\delta(h)) .$$

In particular, the spectral radius reads

$$\rho\left(h^{-1}\mathscr{J}_{I_\delta,h}\right) = \max\left(|\mu_1'(\mu_-^{-1}(e))|, |\mu_1'(\mu_+^{-1}(e))|\right) + o(1) .$$

Assume in addition that Conjecture 1.3 holds. Then, for all $e \in (\Theta_0, e_\star)$ *the current of maximal strength is negative provided* h *is sufficiently small, i.e., we have* $\rho\left(\mathcal{J}_{I_\delta,h}\right) = |\lambda_{\min}(I_\delta, h)|$.

Moreover, the low energy states are localized in the vicinity of the Neumann boundary $x = 0$, as can be seen from the following Agmon type estimate.

Proposition 1.6. *Put*

$$J_e := (-\infty, e) , \quad e < 1 .$$

Then, for all $h > 0$*, all* $K > 0$ *and all* $\psi \in E_{h,J_e}$*, we have*

$$\int_{\mathbb{R}^2_+} e^{Kx}|\psi(x,y)|^2 \mathrm{d}x\mathrm{d}y \leqslant C\|\psi\|^2_{L^2(\mathbb{R}^2_+)} ,$$

for some constant $C > 0$ *depending only on* e *and* K.

1.4. **Mourre estimate and applications.** When the perturbation V is switched off, the presence of a positive edge current with energy concentration in I is, by definition of the current operator $\mathscr{J}_{h,I}$, tied to the existence of a positive local commutator estimate (or Mourre estimate) for the operator $\mathscr{L}_h$ in the same energy interval. This motivates for a closer look into the problem of designing a Mourre inequality for the perturbed magnetic Laplacian

$$\mathscr{L}_{h,V} := D_x^2 + (hD_y - x)^2 + h^\gamma V(x, y, h) ,$$

for some $\gamma > 0$ and some suitably smooth (in a sense that will be made precise further) bounded real-valued perturbation V. Such an estimate will prove useful for characterizing the spectrum of $\mathscr{L}_{h,V}$ in the corresponding energy interval. We are more specifically interested in energy intervals lying in the vicinity of Θ_0, in which the unperturbed current $\mathscr{J}_{h,I}$ tends to vanish. For that purpose we introduce the following spectral window located above Θ_0, whose size depends on $h > 0$:

$$I_\delta := [e - \delta, e + \delta] , \qquad e := \Theta_0 + ah^\alpha \in (\Theta_0, 1) , \quad \delta := bh^\beta , \quad 0 < b < a , \quad (1.4)$$

where

$$0 \leqslant \alpha < 1 , \quad \beta > 2\alpha$$

and

$$a \in (0, 1 - \Theta_0) \text{ when } \alpha = 0 . \qquad (1.5)$$

1.4.1. *Statement of the Mourre estimate.* In view if designing a Mourre estimate, *i.e.*, a positive commutator estimate for the perturbed operator $\mathscr{L}_{h,V}$ on the "sliding" energy window I_δ, where the strength of current is very weak, we pick a smooth compactly supported function f_h satisfying

$$\forall \xi \in (\mu_\mp^{-1}(e \pm bh^\alpha), \mu_\mp^{-1}(e \mp bh^\alpha))\,, \quad f_h(\xi) = \mp 1\,,$$

where $\mu_\pm^{-1}(E)$ denotes the largest/smallest solution of the equation $\mu_1(\xi) = E$. Due to the quadratic behavior of μ_1 near its minimum, we can take $f_h \in \mathscr{S}_{\frac{\alpha}{2}}(1)$, where, as in [27, Section 4.4], we write for any (order) function g,

$$\mathscr{S}_\gamma(g) := \{\psi \in \mathscr{C}^\infty(\mathbb{R}^d) : \forall \alpha \in \mathbb{N}^d\,, \exists C_\alpha > 0\,, \quad |\partial^\alpha \psi| \leqslant C_\alpha h^{-\gamma|\alpha|} g\}\,.$$

Then, the function $(y, \eta) \mapsto y f_h(\eta)$ being in $\mathscr{S}_{\frac{\alpha}{2}}(\langle y \rangle)$, we set

$$\mathscr{A}_h := y f_h(hD_y) + f_h(hD_y) y = \mathrm{Op}_h^w\left(y f_h(\eta)\right)\,.$$

For the sake of notational simplicity, we drop the dependence on h and write f instead of f_h in the sequel.

Theorem 1.7 (Mourre estimate)**.** *Let $V(\cdot,\cdot,h) \in L^\infty(\mathbb{R}^2_+)$, $h > 0$, be real-valued, let $\alpha \geq 0$, let $\beta > 2\alpha$, and let I_δ be defined by (1.4)-(1.5). Assume that $V(x,\cdot,h) \in \mathscr{S}_{\frac{\alpha}{2}}(\langle y \rangle^{-1})$ uniformly in (x,h) and pick $\gamma \geqslant \beta$ so large that $\gamma > 1 + 2\alpha$. Then, there exist $\tilde{c}_0 > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$, we have:*

$$\forall \phi \in \mathrm{range}\, \mathbb{1}_{I_\delta}(\mathscr{L}_{h,V})\,, \quad \langle [\mathscr{L}_{h,V}, i\mathscr{A}_h]\phi, \phi \rangle \geqslant \tilde{c}_0 h^{1+\alpha} \|\phi\|^2\,.$$

1.4.2. *Application.* One of the main consequences of the above Mourre estimate is the following Limiting Absorption Principle (LAP) for the operator $\mathscr{L}_{h,V}$ in I_δ, and subsequently the absolute continuity of its spectrum in I_δ. This is detailed in Section 4.

Corollary 1.8. *Let V and γ be the same as in Theorem 1.7. Then, there exist $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and all bounded operator $\mathscr{C}_h$ such that $\mathscr{C}_h \mathscr{A}_h$ and $\mathscr{A}_h \mathscr{C}_h$ are bounded uniformly in h, we have for all $z \in I_\delta \times \mathbb{R} \setminus \{0\}$,*

$$\|\mathscr{C}_h (\mathscr{L}_{h,V} - z)^{-1} \mathscr{C}_h\| \leqslant C h^{\min(-2+3\alpha, -1-\alpha)}\,. \tag{1.6}$$

In particular, it holds true that:

(i) *For all $z \in I_\delta \times \mathbb{R} \setminus \{0\}$,*

$$\|\langle y \rangle^{-1} (\mathscr{L}_{h,V} - z)^{-1} \langle y \rangle^{-1}\| \leqslant C h^{\min(-2+3\alpha, -1-\alpha)}\,.$$

(ii) *The spectrum of $\mathscr{L}_{h,V}$ lying in I_δ is absolutely continuous.*

Remark 1.9. For all $\alpha \in (0, 1)$, it follows from Corollary 1.8 that the spectrum of the operator $\mathscr{L}_{h,V}$ is purely absolutely continuous spectrum in $(\Theta_0 + ah^\alpha, 1)$.

1.5. **Structure of the article.** The paper is organized as follows. In Section 2 we present basic considerations on low energy magnetic quantum currents and we prove Propositions 1.5 and 1.6. Section 3 contains the proof of Theorem 1.7 and Corollary 1.8. They essentially boil down to the results of Section 4, where a LAP is derived (in a more general setting than the one considered in this article) by revisiting the Mourre theory and estimating carefully the involved constants (which is crucial here since the spectral window of interest also depends on h).

2. Basic properties of low energy currents

2.1. **Proof of Lemma 1.4.** We have

$$\mu_1(\xi) = \mu_1(\xi_0) + \frac{\mu_1''(\xi_0)}{2}(\xi-\xi_0)^2 + \frac{\mu_1^{(3)}(\xi_0)}{6}(\xi-\xi_0)^3 + r(\xi-\xi_0)\,, \tag{2.1}$$

from the analyticity of μ_1 at ξ_0, where r is an analytic function at 0 such that $r(p) \underset{p\to 0}{=} \mathscr{O}(p^4)$. Since $\mu_1''(\xi_0) > 0$, the Morse lemma then yields that the equation

$$\mu_1(\xi) = \mu_1(\xi_0) + \varepsilon \tag{2.2}$$

has two distinct solutions, provided $\varepsilon > 0$ is sufficiently small. Putting $\xi = \xi_0 + p$ in the above equation, this amounts to saying that there exist two solutions $p_-(\varepsilon) < p_+(\varepsilon)$ such that

$$p_\pm(\varepsilon) = \pm\sqrt{\varepsilon}\sqrt{\frac{2}{\mu_1''(\xi_0)}} + u_\pm(\varepsilon)\,,$$

where $u_\pm(\varepsilon) = o(\sqrt{\varepsilon})$. Namely, by substituting the right-hand side of the above equality for $\xi - \xi_0$ in (2.1), and then using (2.2), we get through standard computations that

$$u_\pm(\varepsilon) = -\frac{\mu_1^{(3)}(\xi_0)}{3\mu_1''(\xi_0)^2}\varepsilon + o(\varepsilon)\,.$$

As a consequence we have

$$\mu_1'(\xi_0 + p_-(\varepsilon)) + \mu_1'(\xi_0 + p_+(\varepsilon)) = \frac{4\mu_1^{(3)}(\xi_0)}{3\mu_1''(\xi_0)}\varepsilon + o(\varepsilon)\,,$$

which is negative whenever ε is sufficiently small.

2.2. **Proof of Proposition 1.5.**

2.2.1. *Preliminaries.*

Lemma 2.1. *For all $\psi \in E_{h,I_\delta}$, it holds true that*

$$\mathscr{F}_h\psi(x,\xi) = \mathbb{1}_{I_\delta}(\mu_1(\xi))\langle \mathscr{F}_h\psi(\cdot,\xi), u_1(\cdot,\xi)\rangle_{L^2(\mathbb{R}_+)} u_1(x,\xi)\,, \quad (x,\xi) \in \mathbb{R}^2_+.$$

Proof. Since $\mathbb{1}_{I_\delta}(\mathscr{L}_h)\psi = \psi$, we have

$$\mathscr{F}_h \mathbb{1}_{I_\delta}(\mathscr{L}_h)\mathscr{F}_h^{-1}\mathscr{F}_h\psi = \mathbb{1}_{I_\delta}(\mathscr{F}_h\mathscr{L}_h\mathscr{F}_h^{-1})\mathscr{F}_h\psi = \mathscr{F}_h\psi\,.$$

Plugging this into the following decomposition of $\mathscr{F}_h\psi(\cdot,\xi)$ on the Hilbertian basis $(u_j(\cdot,\xi))_{j\geqslant 1}$ of $L^2(\mathbb{R}+)$,

$$\mathscr{F}_h\psi(x,\xi) = \sum_{j\geqslant 1}\langle \mathscr{F}_h\psi(\cdot,\xi), u_j(\cdot,\xi)\rangle_{L^2(\mathbb{R}_+)} u_j(x,\xi)\,,$$

and taking into account that $\mathbb{1}_{I_\delta}(\mathscr{F}_h\mathscr{L}_h\mathscr{F}_h^{-1})u_j(\cdot,\xi) = \mathbb{1}_{I_\delta}(\mu_j(\xi))u_j(\cdot,\xi)$, we get the desired result upon remembering that $I_\delta \cap (\Theta_1, +\infty) = \emptyset$. □

Lemma 2.2. *For all $\psi \in E_{h,I_\delta}$, we have*

$$\langle \mathscr{J}_{I_\delta,h}\psi,\psi\rangle = h\int_{\mathbb{R}} \mathbb{1}_{I_\delta}(\mu_1(\xi))\mu_1'(\xi)|\langle \mathscr{F}_h\psi(\cdot,\xi), u_1(\cdot,\xi)\rangle_{L^2(\mathbb{R}_+)}|^2 \mathrm{d}\xi\,.$$

Proof. In light of Lemma 2.1, we have

$$\langle \mathscr{J}_{I_\delta,h}\psi,\psi\rangle = 2h\int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathbb{1}_{I_\delta}(\mu_1(\xi))(\xi-x)|u_1(x,\xi)|^2|\langle \mathscr{F}_h\psi(\cdot,\xi),u_1(\cdot,\xi)\rangle_{L^2(\mathbb{R}_+)}|^2\mathrm{d}x\mathrm{d}\xi\,,$$

by the Parseval formula. Lemma 2.2 follows from this and the Feynman-Hellmann formula

$$\mu_1'(\xi) = 2\int_{\mathbb{R}_+}(\xi-x)|u_1(x,\xi)|^2\mathrm{d}x\,,$$

upon applying Fubini's theorem. ☐

Armed with Lemma 2.2, we are now in position to prove Proposition 1.5.

2.2.2. *Estimation of the current.* With reference to Lemma 2.2, we have for all $\psi\in E_{h,I_\delta}$,

$$\langle \mathscr{J}_{I_\delta,h}\psi,\psi\rangle = h\int_{\mathbb{R}} \mathbb{1}_{I_\delta}(\mu_1(\xi))\mu_1'(\xi)|\langle \mathscr{F}_h\psi(\cdot,\xi),u_1(\cdot,\xi)\rangle_{L^2(\mathbb{R}_+)}|^2\mathrm{d}\xi\,.$$

Therefore, we infer from the analyticity of μ_1 in I_δ that

$$\begin{aligned} h^{-1}\langle \mathscr{J}_{I_\delta,h}\psi,\psi\rangle = &\int_{\mu_1^{-1}(e-\delta,e+\delta)\cap\{\xi<\xi_0\}} \mu_1'(\mu_-^{-1}(e))|\langle \mathscr{F}_h\psi(\cdot,\xi),u_1(\cdot,\xi)\rangle_{L^2(\mathbb{R}_+)}|^2\mathrm{d}\xi \\ &+\int_{\mu_1^{-1}(e-\delta,e+\delta)\cap\{\xi>\xi_0\}} \mu_1'(\mu_+^{-1}(e))|\langle \mathscr{F}_h\psi(\cdot,\xi),u_1(\cdot,\xi)\rangle_{L^2(\mathbb{R}_+)}|^2\mathrm{d}\xi + \mathcal{O}(\delta)\|\psi\|^2\,. \end{aligned} \tag{2.3}$$

Bearing in mind that $(\xi-\xi_0)\mu_1'(\xi)\geqslant 0$ for all $\xi\in\mu_1^{-1}(e-\delta,e+\delta)$, the min-max theorem then yields that

$$h^{-1}\lambda_{\min}(\mathscr{J}_{I_\delta,h})\geqslant \mu_1'(\mu_-^{-1}(e))+\mathcal{O}(\delta)$$

and

$$h^{-1}\lambda_{\max}(\mathscr{J}_{I_\delta,h})\leqslant \mu_1'(\mu_+^{-1}(e))+\mathcal{O}(\delta)\,.$$

The converse inequalities can be shown upon considering appropriate trial states. For instance, for $\lambda_{\min}(\mathscr{J}_{I_\delta,h})$, it is enough to take $\psi_h\in E_{h,I_\delta}$ such that

$$\mathscr{F}_h\psi_h(x,\xi) = \delta^{-1}(2\pi)^{-\frac14}u_1(x,\mu_-^{-1}(e))e^{-\frac{\left(\xi-\mu_-^{-1}(e)\right)^2}{2\delta^4}}\mathbb{1}_{I_\delta}(\mu_1(\xi))\,.$$

Indeed, by substituting the above expression for $\mathscr{F}_h\psi(\cdot,\xi)$ in (2.3) we get two integral terms. Using the analyticity of $\xi\mapsto u_1(\cdot,\xi)$ at $\mu_-^{-1}(e)$ we see that the first term reads $\mu_1'(\mu_-^{-1}(e))+\mathcal{O}(\delta(h))$, whereas, since $\xi-\mu_-^{-1}(e)>\xi_0-\mu_-^{-1}(e)$ for all $\xi>\xi_0$, the second term is bounded, up to some multiplicative constant that is independent of h, by $\delta(h)^{-1}e^{-\frac{(\xi_0-\mu_-^{-1}(e))^2}{2\delta(h)^4}}$. Finally, the desired result follows readily from this and $\|\psi_h\|_{L^2(\mathbb{R}^2_+)}=1$.

2.3. **Proof of Proposition 1.6.** Since $e\in(\Theta_0,1)$, we have

$$\mu_1^{-1}(J_e) = (\mu_-^{-1}(e),\mu_+^{-1}(e)) \qquad\text{and}\qquad \mu_j^{-1}(J_e)=\emptyset,\ j\geqslant 2. \tag{2.4}$$

Further, for $\psi\in E_{h,J_e}$ we have $\mathbb{1}_{J_e}(\mu_1(\xi))\mathscr{F}_h\psi(\cdot,\xi)=\mathscr{F}_h\psi(\cdot,\xi)$ for all $\xi\in\mu_1^{-1}(J_e)$, and consequently

$$\ell_\xi\mathscr{F}_h\psi(\cdot,\xi) = \mu_1(\xi)\mathscr{F}_h\psi(\cdot,\xi)\,,$$

by arguing as in Lemma 2.1. Next, by Agmon's theorem (see for instance [1], [14, Section 7.2] and [24, Section 4.2]), the following localization formula holds for any $\chi \in W^{1,\infty}(\mathbb{R}_+)$ and all $\xi \in \mu_1^{-1}(J_e)$,

$$\mathrm{Re}\,\langle \ell_\xi \mathscr{F}_h\psi(\cdot,\xi), \chi^2 \mathscr{F}_h\psi(\cdot,\xi)\rangle = q_\xi(\chi \mathscr{F}_h\psi(\cdot,\xi)) - \|\chi' \mathscr{F}_h\psi(\cdot,\xi)\|^2_{L^2(\mathbb{R}_+)}\,,$$

where q_ξ denotes the quadratic form associated with the operator ℓ_ξ. It follows from this that

$$\int_{\mathbb{R}_+} \left((\xi-x)^2\chi(x)^2 - \chi'(x)^2 - \chi(x)^2\mu_1(\xi)\right)|\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x = -\|(\chi\mathscr{F}_h\psi)'(\cdot,\xi)\|^2_{L^2(\mathbb{R}_+)} \leqslant 0\,.$$

Therefore, since $(\xi-x)^2 \geqslant \frac{x^2}{2} - \xi^2$ and $\mu_1(\xi) \leqslant e$, we get that

$$\int_{\mathbb{R}_+} \left(\left(\frac{x^2}{2} - \xi^2 - e\right)\chi(x)^2 - \chi'(x)^2\right)|\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x \leqslant 0\,,\ \xi \in \mu_1^{-1}(J_e).$$

In light of (2.4), this entails that

$$\int_{\mathbb{R}_+} \left(\left(\frac{x^2}{2} - C_e\right)\chi(x)^2 - \chi'(x)^2\right)|\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x \leqslant 0\,,$$

where $C_e := \max\{\mu_-^{-1}(e)^2, \mu_+^{-1}(e)^2\} + e$.

Thus, taking $\chi(x) = e^{\Phi(x)}$ in the above estimate, where Φ is a non-negative K-Lipschitzian function, we obtain that

$$\int_{\mathbb{R}_+} \left(\frac{x^2}{2} - C_e - \Phi'^2(x)\right)|e^{\Phi(x)}\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x \leqslant 0\,,$$

and hence that

$$\int_{x_{e,K}}^{+\infty} \left(\frac{x^2}{2} - C_e - K^2\right)|e^{\Phi(x)}\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x \leqslant -\int_0^{x_{e,K}} \left(\frac{x^2}{2} - C_e - K^2\right)|e^{\Phi(x)}\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x\,,$$

where $x_{e,K} := \sqrt{2(C_e+K^2+1)} > 0$. Therefore, there exists a constant $C>0$, depending only on e, K and $\|\Phi\|_{L^\infty(0,x_{e,K})}$, such that

$$\int_{x_{e,K}}^{+\infty} |e^{\Phi(x)}\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x \leqslant C\int_0^{x_{e,K}} |\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x\,,$$

from where we get

$$\int_{\mathbb{R}_+} |e^{\Phi(x)}\mathscr{F}_h\psi(x,\xi)|^2\mathrm{d}x \leqslant C\|\mathscr{F}_h\psi(\cdot,\xi)\|^2_{L^2(\mathbb{R}_+)}\,,$$

upon substituting $C + e^{\|\Phi\|_{L^\infty(0,x_{e,K})}}$ for C. Now, integrating the above estimate with respect to ξ and applying the Parseval formula, we obtain that

$$\int_{\mathbb{R}_+^2} |e^{\Phi(x)}\psi(x,y)|^2\mathrm{d}x\mathrm{d}y \leqslant C\|\psi\|^2_{L^2(\mathbb{R}_+^2)}\,. \tag{2.5}$$

Next, for $n \in \mathbb{N}$, put

$$\Phi_n(s) := K \times \begin{cases} s & \text{for } 0 \leqslant s \leqslant n\,, \\ 2n - s & \text{for } n \leqslant s \leqslant 2n\,, \\ 0 & \text{for } s \geqslant 2n\,. \end{cases}$$

Notice that $\|\Phi_n\|_{L^\infty(0,x_{e,K})} = K x_{e,K}$ for all $n \geqslant x_{e,K}$, in such a way that (2.5) holds with $\Phi = \Phi_n$, where the constant C is independent of n.

Finally, the result follows from this upon sending n to infinity and applying Fatou's lemma.

3. Mourre estimates and limiting absorption

3.1. **A Mourre estimate for the unperturbed operator.** Since $V(x,\cdot,h) \in \mathscr{S}_{\frac{\alpha}{2}}(\langle y\rangle^{-1})$ and $\mathscr{A}_h \in \mathrm{Op}_h^w \mathscr{S}_{\frac{\alpha}{2}}(\langle y\rangle)$, the following lemma is a direct consequence of the composition theorem of pseudo-differential operators (see [27, Theorem 4.18]) and the Calderon-Vaillancourt theorem (see [27, Theorem 4.23]).

Lemma 3.1. *The pseudo-differential operator $[V(x,\cdot,h),\mathscr{A}_h]$ is bounded from $L^2(\mathbb{R}_y)$ to $L^2(\mathbb{R}_y)$. More precisely, there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0,h_0)$ and all $x > 0$, we have*

$$\|[V(x,\cdot,h),\mathscr{A}_h]\|_{\mathscr{L}^2(L^2(\mathbb{R}_y))} \leqslant C h^{1-\alpha}\,.$$

In particular, this operator extends to a bounded operator on $L^2(\mathbb{R}^2_+)$.

Remark 3.2. The expression of the above commutator can be explicitly calculated at the cost of rather tedious computations.

Nevertheless we do not follow this path here as the pseudo-differential approach that we used in the derivation of Lemma 3.1 is more straightforward.

Lemma 3.3. *The following commutator is well defined on* $\mathrm{Dom}(\mathscr{L}_h)$ *and*

$$[\mathscr{L}_h, i\mathscr{A}_h] = 4h(hD_y - x)f(hD_y)\,.$$

Proof. Since

$$\begin{aligned} [\mathscr{L}_h, i\mathscr{A}_h] &= [(hD_y - x)^2, i\mathscr{A}_h] = i[(hD_y - x)^2, yf(hD_y) + f(hD_y)y] \\ &= i[(hD_y - x)^2, y]f(hD_y) + ihf(hD_y)[(hD_y - x)^2, y], \end{aligned} \tag{3.1}$$

we get the desired result upon recalling that $[(hD_y - x)^2, y] = -2ih(hD_y - x)$. $\square$

For $d := bh^\alpha$, put $I_d := [e-d, e+d]$. Since $\beta > 2\alpha$, we have $0 < d < \delta$ whenever $h \in (0,1)$, and consequently I_δ is a proper subset of I_d: $\overline{I}_\delta \subset I_d$.

Remark 3.4. Notice for further use that due to the above embedding, we can apply the results of Section 4 where the interval I_δ (resp., I_d) is substituted for I (resp., J).

Proposition 3.5. *There exist $\tilde{c}_0 > 0$ and $h_0 > 0$ such that for all $h \in (0,h_0)$, we have*

$$\mathbb{1}_{I_d}(\mathscr{L}_h)[\mathscr{L}_h, i\mathscr{A}_h]\mathbb{1}_{I_d}(\mathscr{L}_h) \geqslant \tilde{c}_0 h^{1+\alpha} > 0\,.$$

Proof. By the Parseval formula, we get from Lemma 3.3 that for all $u \in \mathrm{Ran}(\mathbb{1}_{I_d}(\mathscr{L}_h))$,

$$\langle [\mathscr{L}_h, i\mathscr{A}_h]u, u\rangle_{L^2(\mathbb{R}_+\times\mathbb{R})} = 4h\langle (\eta - x)f(\eta)\mathscr{F}_h u, \mathscr{F}_h u\rangle_{L^2(\mathbb{R}_+\times\mathbb{R}_\eta)}\,.$$

Moreover, analogously to Lemma 2.1 we have

$$\mathscr{F}_h u(x,\eta) = \mathbb{1}_{\mu_1(\eta)\in I_d} U(\eta) u_1(x,\eta)\,,$$

where

$$U(\eta) = \langle \mathscr{F}_h u(\cdot,\eta), u_1(\cdot,\eta)\rangle_{L^2(\mathbb{R}_+)}\,,$$

in such a way that

$$\langle [\mathscr{L}_h, i\mathscr{A}_h]u, u\rangle_{L^2(\mathbb{R}_+\times\mathbb{R})} = 4h\int_{\mathbb{R}_+}\int_{\mathbb{R}} (\eta - x)f(\eta)\mathbb{1}_{\mu_1(\eta)\in I_d}|U(\eta)|^2|u_1(x,\eta)|^2 \mathrm{d}\eta\mathrm{d}x\,.$$

Therefore, applying the Feynman-Hellmann formula, we obtain that

$$\langle [\mathscr{L}_h, i\mathscr{A}_h]u, u\rangle_{L^2(\mathbb{R}_+\times\mathbb{R})} = 2h\int_{\mu_1(\eta)\in I_d} f(\eta)\mu_1'(\eta)|U(\eta)|^2\mathrm{d}\eta\,.$$

Moreover, we have $\int_{\mu_1(\eta)\in I_d} f(\eta)\mu_1'(\eta)|U(\eta)|^2\mathrm{d}\eta \geqslant \int_{\mu_1(\eta)\in I_d} |\mu_1'(\eta)||U(\eta)|^2\mathrm{d}\eta$ from the definition of f, and consequently

$$\langle [\mathscr{L}_h, i\mathscr{A}_h]u, u\rangle_{L^2(\mathbb{R}_+\times\mathbb{R})} \geqslant \tilde{c}_0 h^{1+\alpha}\int_{\mu_1(\eta)\in I_d} |U(\eta)|^2\mathrm{d}\eta \geqslant \tilde{c}_0 h^{1+\alpha}\|u\|^2\,,$$

from the quadratic behavior of μ_1 at its minimum, expressed in Proposition 1.1.iii. □

One of the benefits of a Mourre estimate is its stability under perturbation. Having established a Mourre inequality for the unperturbed operator $\mathscr{L}_h$ in the above proposition, we turn now to extending this estimate to the case of the perturbed operator $\mathscr{L}_h + V$.

3.2. **The case of the perturbed operator.** Since the proof of a Mourre estimate for the perturbed operator essentially boils down to the existence of a Mourre inequality for unperturbed operator, we preliminarily establish the following property of the spectral decomposition associated with $\mathscr{L}_h$.

3.2.1. *Spectral decomposition associated with $\mathscr{L}_h$.*

Lemma 3.6. *Let $\phi \in \mathrm{range}\,\mathbb{1}_{I_\delta}(\mathscr{L}_{h,V})$. Then, ϕ decomposes as*

$$\phi = \phi_1 + \phi_2\,,\qquad \phi_1 = \mathbb{1}_{I_d}(\mathscr{L}_h)\phi\,,\quad \phi_2 = \mathbb{1}_{\mathbb{R}\setminus I_d}(\mathscr{L}_h)\phi\,,$$

and we have

$$\|\phi_2\| \leqslant c_h\|\phi\|\,,\quad c_h = d^{-1}(\delta + h^\gamma\|V\|_\infty)\,.$$

Proof. Since $\phi_2 = \mathbb{1}_{\mathbb{R}\setminus I_d}(\mathscr{L}_h)\mathbb{1}_{I_\delta}(\mathscr{L}_{h,V})\phi$, we have

$$\begin{aligned}\phi_2 &= (\mathscr{L}_h - e)^{-1}\mathbb{1}_{\mathbb{R}\setminus I_d}(\mathscr{L}_h)(\mathscr{L}_h - e)\mathbb{1}_{I_\delta}(\mathscr{L}_{h,V})\phi\\ &= (\mathscr{L}_h - e)^{-1}\mathbb{1}_{\mathbb{R}\setminus I_d}(\mathscr{L}_h)(\mathscr{L}_{h,V} - e - h^\gamma V)\mathbb{1}_{I_\delta}(\mathscr{L}_{h,V})\phi\,,\end{aligned}$$

which immediately yields that $\|\phi_2\| \leqslant d^{-1}(\delta + h^\gamma\|V\|_\infty)\|\phi\|$. □

3.2.2. *Proof of Theorem 1.7.* For all $\phi \in \operatorname{range} \mathbb{1}_{I_\delta}(\mathscr{L}_{h,V})$, we have

$$\langle [\mathscr{L}_{h,V}, i\mathscr{A}_h]\phi, \phi\rangle = \langle [\mathscr{L}_h, i\mathscr{A}_h]\phi, \phi\rangle + h^\gamma \langle [V, i\mathscr{A}_h]\phi, \phi\rangle . \tag{3.2}$$

The second term on the right-hand-side of the above line can be estimated with the aid of Lemma 3.1: since $\gamma - \alpha + 1 > 1 + \alpha$, we get that

$$h^\gamma \left|\langle [V, i\mathscr{A}_h]\phi, \phi\rangle\right| \leqslant h^\gamma \|[V, i\mathscr{A}_h]\|_{\mathscr{L}(L^2(\mathbb{R}^2_+))} \|\phi\|^2 \leqslant C h^{\gamma-\alpha+1} \|\phi\|^2 \leqslant \frac{\tilde{c}_0}{4} h^{1+\alpha} \|\phi\|^2 , \tag{3.3}$$

provided h is small enough. In the first term on right-hand-side of (3.2), the commutator $[\mathscr{L}_h, i\mathscr{A}_h]$ is acting on the state $\phi \in \operatorname{range} \mathbb{1}_{I_\delta}(\mathscr{L}_{h,V})$, which decomposes according to Lemma 3.6, giving:

$$\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi, \phi\rangle = \langle [\mathscr{L}_h, i\mathscr{A}_h]\phi_1, \phi_1\rangle + \langle [\mathscr{L}_h, i\mathscr{A}_h]\phi_2, \phi_2\rangle + 2\mathrm{Re}\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi_1, \phi_2\rangle .$$

Therefore, we have

$$\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi, \phi\rangle \geqslant \tilde{c}_0 h^{1+\alpha} \|\phi_1\|^2 - |\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi_2, \phi_2\rangle| - 2|\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi_1, \phi_2\rangle| ,$$

by Proposition 3.5, and hence

$$\begin{aligned}\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi, \phi\rangle \geqslant \tilde{c}_0 h^{1+\alpha} \|\phi_1\|^2 &- 4h\|f\|_\infty \|\phi_2\| \|(hD_y - x)\phi_2\| \\ &- 8h\|f\|_\infty \|(hD_y - x)\phi_1\| \|\phi_2\| ,\end{aligned}$$

from Lemma 3.3. Further, taking into account that

$$\|(hD_y - x)\phi_1\|^2 \leqslant \|\phi_1\| \|\mathscr{L}_h \phi_1\| \leqslant (e+d)\|\phi_1\|^2$$

and that

$$\begin{aligned}\|(hD_y - x)\phi_2\|^2 = \langle \phi_2, \mathscr{L}_h \phi_2\rangle = \langle \phi_2, \mathscr{L}_h \phi\rangle &= \langle \phi_2, (\mathscr{L}_h + h^\gamma V)\phi\rangle - \langle \phi_2, h^\gamma V \phi\rangle \\ &\leqslant (e + \delta + h^\gamma \|V\|_\infty) \|\phi_2\| \|\phi\| ,\end{aligned}$$

we get that

$$\begin{aligned}\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi, \phi\rangle &\geqslant \tilde{c}_0 h^{1+\alpha} \|\phi_1\|^2 \\ &- 4h(e + \delta + h^\gamma \|V\|_\infty)^{\frac12} \|f\|_\infty \|\phi_2\|^{\frac32} \|\phi\|^{\frac12} - 8h(e+d)^{\frac12} \|f\|_\infty \|\phi_1\| \|\phi_2\| .\end{aligned}$$

From this and Lemma 3.6 it then follows that

$$\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi, \phi\rangle \geqslant \left(\tilde{c}_0 h^{1+\alpha} - \left(8(e + d + h^\gamma \|V\|_\infty)^{\frac12} \|f\|_\infty (1 + c_h^{\frac12}) + c_h h^\alpha\right) c_h h\right) \|\phi\|^2 .$$

Thus, bearing in mind that $d = h^\alpha$ in such a way that c_h (which scales like $h^{\beta-\alpha}$) can be made arbitrarily small relative to h^α (as we have $\beta > 2\alpha$) in the asymptotic regime $h \downarrow 0$, we infer from the above estimate that

$$\langle [\mathscr{L}_h, i\mathscr{A}_h]\phi, \phi\rangle \geqslant \frac{3\tilde{c}_0}{4} h^{1+\alpha} \|\phi\|^2 ,$$

whenever h is sufficiently small. Putting this together with (3.2)-(3.3), we get the result of Theorem 1.7 upon replacing $\frac{\tilde{c}_0}{2}$ by $\tilde{c}_0$.

3.3. **Proof of Corollary 1.8.** By Theorem 1.7, the self-adjoint operator $\mathscr{L}_{h,V}$ satisfies a Mourre estimate of type (4.1) on the interval I_δ, associated with the local commutator $\mathscr{A}_h$ and a constant c_0 of size $h^{1+\alpha}$. Next, since $f(hD_y)$ is a bounded operator by definition of f, Lemmas 3.1 and 3.3 ensure us that the pair $(\mathscr{L}_{h,V}, \mathscr{A}_h)$ fulfills the assumption (A) of Section 4.1 with a constant c_1 of order $\mathscr{O}(h^{1-\alpha})$. Moreover, we have

$$\begin{aligned}[][\mathscr{L}_h, \mathscr{A}_h], \mathscr{A}_h] &= -4ih[(hD_y - x)f(hD_y), yf_h(hD_y) + f_h(hD_y)y] \\ &= -4ih\left(-2ih(hD_y - x)f'(hD_y)f(hD_y) - 2ihf^2(hD_y)\right),\end{aligned}$$

by direct computation. We deduce from this and the boundedness of the operators $f(hD_y)$ and $f'(hD_y)$ that $[[\mathscr{L}_h, \mathscr{A}_h], \mathscr{A}_h](\mathscr{L}_h + i)^{-1}$ is bounded and that its norm is of order $\mathscr{O}(h^{2-\alpha})$. Moreover, since $V(x, \cdot, h) \in \mathscr{S}_{\frac{\alpha}{2}}(\langle y\rangle^{-1})$ by assumption and $\mathscr{A}_h \in \mathrm{Op}_h^w \mathscr{S}_{\frac{\alpha}{2}}(\langle y\rangle)$, we can check upon arguing in the same fashion as in the derivation of Lemma 3.1 that $(\mathscr{L}_{h,V}, \mathscr{A}_h)$ satisfies the assumption (B) of Section 4.1, where the constant c_2 is of order $\mathscr{O}(h^{2-\alpha})$.

Therefore, with reference to Remark 3.4 and Section 4, the statement of Corollary 1.8 follows from Theorem 4.4 and the fact that we can actually track the powers of h in Lemma 4.2 and Theorem 4.4. As a matter of fact, by applying Lemma 4.2 with $I = I_\delta$ and $J = I_d$, we get that

$$(\mathrm{dist}(I, J^c))^{-1} \simeq h^{-\alpha}, \quad \varepsilon_1 \simeq h^{-1+4\alpha}, \quad \varepsilon_0 \simeq h^{-1+4\alpha} \quad \text{and} \quad \tilde{c}_0 \gtrsim h^{1+\alpha},$$

where the notation $a(h) \gtrsim b(h)$ means that there exist $h_0 > 0$ and $c > 0$ such that $a(h) \geqslant cb(h)$ for all $h \in (0, h_0)$, and $a(h) \simeq b(h)$ stands for $a(h) \gtrsim b(h)$ and $b(h) \gtrsim a(h)$. Similarly, Theorem 4.4 with $I = I_\delta$ and $J = I_d$, yields that

$$K_1 = \mathscr{O}(h^{-\frac{1}{2}-\frac{\alpha}{2}}), \quad K_2 = \mathscr{O}(h^{1-2\alpha}), \quad K = \mathscr{O}(h^{-1-\alpha}), \quad C(\varepsilon_0) = \mathscr{O}(h^{-2+3\alpha}).$$

Finally, since $\min(-2+3\alpha, -3/2+\alpha, -1-\alpha) = \min(-2+3\alpha, -1-\alpha)$ for all $\alpha \in [0, 1)$, we obtain that

$$C = \mathscr{O}(h^{\min(-2+3\alpha, -1-\alpha)}),$$

which gives (1.6) by sending ε to 0 in Theorem 4.4. Then, (i) follows from this by taking $\mathscr{C}_h = \langle y\rangle^{-1}$, which is allowed since the operators $\mathscr{C}_h\mathscr{A}_h$ and $\mathscr{A}_h\mathscr{C}_h$ are bounded by the composition theorem for pseudodifferential operators and the Calderón-Vaillancourt theorem. Finally, (ii) classically follows from (i), by using for instance [8, Prop. 4.1].

4. Limiting absorption revisited

In this section we build a LAP for a self-adjoint operator from a Mourre estimate. The derivation of this result is inspired by [22] and [8, Section 4.3] but we provide here a different approach based on coercivity estimates.

4.1. **Assumptions.** We consider two self-adjoint operators $\mathscr{L}$ and $\mathscr{A}$ satisfying

$$\mathbb{1}_J(\mathscr{L})\mathscr{B}\mathbb{1}_J(\mathscr{L}) \geqslant c_0\mathbb{1}_J(\mathscr{L}), \quad \mathscr{B} := [\mathscr{L}, i\mathscr{A}], \quad c_0 > 0, \tag{4.1}$$

where I and J are two fixed intervals such that $I \subset\subset J$.
Moreover, we assume that

(A) $[\mathscr{L}, \mathscr{A}](\mathscr{L} + i)^{-1}$ is bounded:

$$\exists c_1 > 0, \ \|[\mathscr{L}, \mathscr{A}](\mathscr{L} + i)^{-1}\| \leqslant c_1.$$

(B) $[[\mathscr{L},\mathscr{A}],\mathscr{A}](\mathscr{L}+i)^{-1}$ is bounded:

$$\exists c_2>0,\ \|[[\mathscr{L},\mathscr{A}],\mathscr{A}](\mathscr{L}+i)^{-1}\|\leqslant c_2.$$

4.2. **Limiting absorption through coercivity estimates.** We consider $\varepsilon \geqslant 0$ and $z\in\mathbb{C}$ such that $\operatorname{Re} z\in I$ and $\operatorname{Im} z\geqslant 0$. For the sake of notational simplicity we write $z\in I\times[0,+\infty)$ in the sequel. Set

$$\mathscr{L}_{z,\varepsilon}:=\mathscr{L}-z-i\varepsilon\mathscr{B},\ \operatorname{Dom}\mathscr{L}_{z,\varepsilon}:=\operatorname{Dom}\mathscr{L}\,.$$

It is apparent that the family $(\mathscr{L}_{z,\varepsilon})_{\varepsilon\in\mathbb{R}}$ is analytic of type (A) in the sense of Kato. Moeroever, $\mathscr{L}_{z,\varepsilon}$ is bijective provided z is not on the imaginary axis.

Lemma 4.1. *Let $M>0$. Then, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and all $z\in I\times[M,+\infty)$, the operator $\mathscr{L}_{z,\varepsilon}$ is bijective and satisfies the estimate:*

$$\|\mathscr{L}_{z,\varepsilon}^{-1}\|\leqslant\frac{4}{\operatorname{Im} z}\leqslant\frac{4}{M}\,.$$

Proof. Since $\mathscr{L}_{z,\varepsilon}=\mathscr{L}-\operatorname{Re} z-i\operatorname{Im} z-i\varepsilon\mathscr{B}$, we have

$$\|\mathscr{L}_{z,\varepsilon}u\|\geqslant\frac{1}{2}\|(\mathscr{L}-\operatorname{Re} z)u\|+\frac{1}{2}\operatorname{Im} z\|u\|-\varepsilon\|\mathscr{B}u\|\,,$$

and hence

$$2\|\mathscr{L}_{z,\varepsilon}u\|\geqslant\|(\mathscr{L}-\operatorname{Re} z)u\|+\operatorname{Im} z\|u\|-2c_1\varepsilon\|(\mathscr{L}+i)u\|\,.$$

It follows from this that

$$2\|\mathscr{L}_{z,\varepsilon}u\|\geqslant\|(\mathscr{L}-\operatorname{Re} z)u\|+\operatorname{Im} z\|u\|-2c_1\varepsilon(\|(\mathscr{L}-\operatorname{Re} z)u\|+\|(i+\operatorname{Re} z)u\|)\,,$$

and consequently

$$2\|\mathscr{L}_{z,\varepsilon}u\|\geqslant(1-2c_1\varepsilon)\|(\mathscr{L}-\operatorname{Re} z)u\|+\left(\operatorname{Im} z-2c_1\varepsilon\max_{x\in I}|i+x|\right)\|u\|\,.$$

Next, choosing ε so small in the above line that $\varepsilon\leqslant\min\left(\frac{1}{2c_1},\frac{\max_{x\in I}|i+x|}{4c_1}\right)$, we get that

$$2\|\mathscr{L}_{z,\varepsilon}u\|\geqslant\frac{\operatorname{Im} z}{2}\|u\|\,,$$

which shows that $\mathscr{L}_{z,\varepsilon}$ is injective with closed range. Arguing as above with the adjoint operator $\mathscr{L}_{z,\varepsilon}^*=\mathscr{L}_{\bar{z},-\varepsilon}$ instead of $\mathscr{L}_{z,\varepsilon}$, we get that $\mathscr{L}_{z,\varepsilon}^*$ is injective as well. Therefore, $\mathscr{L}_{z,\varepsilon}$ has a dense range and the conclusion follows. □

Lemma 4.2. *Let $M>0$, set $B:=I\times[0,M]$ and put*

$$\varepsilon_1:=\left(\frac{\sup_B|\ell+i|}{\operatorname{dist}(I,J^c)}+1\right)^{-1}\min\left(\frac{c_0}{4c_1^2\sup_J|\ell+i|},\frac{1}{2c_1}\right)$$

and

$$\varepsilon_0:=\min\left(\varepsilon_1,\frac{2\sup_J|\ell+i|}{c_0\left(\frac{\sup_B|\ell+i|}{\operatorname{dist}(I,J^c)}+1\right)\left(1+\frac{4c_1}{c_0}\sup_J|\ell+i|\right)}\right)\,.$$

Then, for all $\varepsilon>0$ and all $z\in B$, the following estimates hold.

(a)

$$\forall u \in \mathrm{Dom}(\mathscr{L})\,, \quad \|\mathbb{1}_{J^c}(\mathscr{L})\mathscr{L}_{z,\varepsilon}u\| \geqslant C(\varepsilon,z)\|(\mathscr{L}+i)\mathbb{1}_{J^c}(\mathscr{L})u\| - c_1\varepsilon \sup_J |\ell+i|\|\mathbb{1}_J(\mathscr{L})u\|\,,$$

where

$$C(\varepsilon,z) := \left(1 - c_1\varepsilon\left(1+\frac{|z+i|}{\mathrm{dist}(I,J^c)}\right)\right)\left(1+\frac{|z+i|}{\mathrm{dist}(I,J^c)}\right)^{-1}.$$

Moreover, we have $C(\varepsilon,z) \geqslant \frac{1}{2}\left(\frac{\sup_B|\ell+i|}{\mathrm{dist}(I,J^c)}+1\right)^{-1}$ *for* $0<\varepsilon\leqslant \frac{1}{2c_1}\left(\frac{\sup_B|\ell+i|}{\mathrm{dist}(I,J^c)}+1\right)^{-1}$.

(b)

$$\forall u \in \mathrm{Dom}(\mathscr{L})\,, \quad \|\mathscr{L}_{z,\varepsilon}u\| \geqslant D_1(\varepsilon,z)\|\mathbb{1}_J(\mathscr{L})u\|$$

where

$$D_1(\varepsilon,z) := \frac{\mathrm{Im}\, z + c_0\varepsilon - \frac{(c_1\varepsilon)^2\sup_J|\ell+i|}{C(\varepsilon,z)}}{1+\frac{c_1\varepsilon}{C(\varepsilon,z)}}.$$

Moreover, we have $D_1(\epsilon,z) \geqslant \frac{c_0\varepsilon}{2}\left(1-\frac{c_1^2\varepsilon\sup_J|\ell+i|}{c_0C(\varepsilon,z)}\right) \geqslant \frac{c_0\varepsilon}{4}$ *provided* $0<\varepsilon\leqslant\varepsilon_1$.

(c)

$$\forall u \in \mathrm{Dom}(\mathscr{L})\,, \quad \|\mathscr{L}_{z,\varepsilon}u\| \geqslant D_2(\varepsilon,z)\|(\mathscr{L}+i)\mathbb{1}_{J^c}(\mathscr{L})u\|\,,$$

where

$$D_2(\varepsilon,z) = \frac{C(\varepsilon,z)}{1+c_1\varepsilon D_1(\varepsilon,z)^{-1}\sup_J|\ell+i|}.$$

Moreover, we have $D_2(\varepsilon,z) \geqslant \frac{\left(\frac{\sup_B|\ell+i|}{\mathrm{dist}(I,J^c)}+1\right)^{-1}}{2\left(1+\frac{4c_1}{c_0}\sup_J|\ell+i|\right)}$ *whenever* $0<\varepsilon\leqslant\varepsilon_1$.

(d) *In particular* $\mathscr{L}_{z,\varepsilon}$ *is bijective and*

$$\|(\mathscr{L}+i)\mathscr{L}_{z,\varepsilon}^{-1}\| \leqslant D_3(\varepsilon,z)^{-1},$$

where

$$D_3(\varepsilon,z) := \frac{1}{\sqrt{2}}\min\left(\frac{D_1(\varepsilon,z)}{\sup_J|\ell+i|}, D_2(\varepsilon,z)\right)$$

satisfies $D_3(\varepsilon,z) \geqslant \frac{c_0\varepsilon}{4\sqrt{2}\sup_J|\ell+i|}$ *provided* $0<\varepsilon\leqslant\varepsilon_0$.
Moreover, for $\varepsilon=0$, $\mathscr{L}_{z,\varepsilon}$ *is bijective for* $\mathrm{Im}\, z>0$.

(e)

$$\forall u \in \mathrm{Dom}(\mathscr{L})\,, \forall \varepsilon\in(0,\varepsilon_2), \quad |\langle \mathscr{L}_{z,\varepsilon}u,u\rangle| + \|\mathscr{L}_{z,\varepsilon}u\|^2 \geqslant \tilde{c}_0\varepsilon\|u\|^2\,,$$

where

$$\varepsilon_2 := \min\left(\frac{c_0\sup_J|\ell+i|}{2c_1^2C(\varepsilon,z)}, \frac{2D_2(\varepsilon,z)^2}{c_0}, \varepsilon_1\right)$$

and

$$\tilde{c}_0 := \frac{c_0}{2}\left(1+\frac{2\left(1+\frac{4c_1}{c_0}\sup_J|\ell+i|\right)}{\left(\frac{\sup_B|\ell+i|}{\mathrm{dist}(I,J^c)}+1\right)^{-1}}+\frac{4c_1}{c_0}\right)^{-1}.$$

Proof. (a) We have

$$\|\mathbb{1}_{J^c}(\mathscr{L})\mathscr{L}_{z,\varepsilon}u\| \geqslant \|\mathbb{1}_{J^c}(\mathscr{L})(\mathscr{L}-z)u\| - \varepsilon\|\mathscr{B}u\| \geqslant \|\mathbb{1}_{J^c}(\mathscr{L})(\mathscr{L}-z)u\| - c_1\varepsilon\|(\mathscr{L}+i)u\|\,,$$

by (A) and thus

$$\|\mathbb{1}_{J^c}(\mathscr{L})\mathscr{L}_{z,\varepsilon}u\| \geqslant \|\mathbb{1}_{J^c}(\mathscr{L})(\mathscr{L}-z)u\| - c_1\varepsilon\sup_J|\ell+i|\|\mathbb{1}_J(\mathscr{L})u\| - c_1\varepsilon\|(\mathscr{L}+i)\mathbb{1}_{J^c}(\mathscr{L})u\|,$$

from the orthogonal decomposition of $(\mathscr{L}+i)u$. This entails that

$$\|\mathbb{1}_{J^c}(\mathscr{L})\mathscr{L}_{z,\varepsilon}u\| \geqslant (1-c_1\varepsilon)\|\mathbb{1}_{J^c}(\mathscr{L})(\mathscr{L}-z)u\| - c_1\varepsilon\sup_J|\ell+i|\|\mathbb{1}_J(\mathscr{L})u\| - c_1\varepsilon|z+i|\|\mathbb{1}_{J^c}(\mathscr{L})u\|\,. \quad (4.2)$$

Further, since

$$\|\mathbb{1}_{J^c}(\mathscr{L})(\mathscr{L}-z)u\| \geqslant \mathrm{dist}(I,J^c)\|\mathbb{1}_{J^c}(\mathscr{L})u\| \quad (4.3)$$

and

$$\|\mathbb{1}_{J^c}(\mathscr{L})(\mathscr{L}-z)u\| \geqslant \|(\mathscr{L}+i)\mathbb{1}_{J^c}(\mathscr{L})u\| - |i+z|\|\mathbb{1}_{J^c}(\mathscr{L})u\|\,,$$

we obtain that

$$\|\mathbb{1}_{J^c}(\mathscr{L})(\mathscr{L}-z)u\| \geqslant \left(\frac{|i+z|}{\mathrm{dist}(I,J^c)}+1\right)^{-1}\|(\mathscr{L}+i)\mathbb{1}_{J^c}(\mathscr{L})u\|\,. \quad (4.4)$$

Now, plugging (4.3) and (4.4) into (4.2), we get (a).

(b) Since

$$\begin{aligned}&-\mathrm{Im}\,\langle\mathscr{L}_{z,\varepsilon}u,\mathbb{1}_J(\mathscr{L})u\rangle\\ &= \mathrm{Im}\,z\|\mathbb{1}_J(\mathscr{L})u\|^2 + \varepsilon\langle\mathbb{1}_J(\mathscr{L})\mathscr{B}\mathbb{1}_J(\mathscr{L})u,u\rangle + \varepsilon\mathrm{Re}\,\langle\mathscr{B}\mathbb{1}_{J^c}(\mathscr{L})u,\mathbb{1}_J(\mathscr{L})u\rangle \quad (4.5)\\ &\geqslant \mathrm{Im}\,z\|\mathbb{1}_J(\mathscr{L})u\|^2 + c_0\varepsilon\|\mathbb{1}_J(\mathscr{L})u\|^2 - c_1\varepsilon\|\mathbb{1}_J(\mathscr{L})u\|\|(\mathscr{L}+i)\mathbb{1}_{J^c}(\mathscr{L})\|\end{aligned}$$

by (A) and

$$\|(\mathscr{L}+i)\mathbb{1}_{J^c}(\mathscr{L})\| \leqslant C(\varepsilon,z)^{-1}\left(\|\mathbb{1}_{J^c}(\mathscr{L})\mathscr{L}_{z,\varepsilon}u\| + c_1\varepsilon\sup_J|\ell+i|\|\mathbb{1}_J(\mathscr{L})u\|\right)\,,$$

from (a), we get that

$$\begin{aligned}-\mathrm{Im}\,\langle\mathscr{L}_{z,\varepsilon}u,\mathbb{1}_J(\mathscr{L})u\rangle &\geqslant \mathrm{Im}\,z\|\mathbb{1}_J(\mathscr{L})u\|^2 + c_0\varepsilon\|\mathbb{1}_J(\mathscr{L})u\|^2\\ &- C(\varepsilon,z)^{-1}c_1\varepsilon\|\mathbb{1}_J(\mathscr{L})u\|\left(\|\mathbb{1}_{J^c}(\mathscr{L})\mathscr{L}_{z,\varepsilon}u\| + c_1\varepsilon\sup_J|\ell+i|\|\mathbb{1}_J(\mathscr{L})u\|\right)\,. \quad (4.6)\end{aligned}$$

An application of the Cauchy-Schwarz inequality on the left-hand-side of the above inequality then yields

$$\left(1+\frac{c_1\varepsilon}{C(\varepsilon,z)}\right)\|\mathscr{L}_{z,\varepsilon}u\|\|\mathbb{1}_J(\mathscr{L})u\| \geqslant \left(\mathrm{Im}\,z + c_0\varepsilon - \frac{(c_1\varepsilon)^2\sup_J|\ell+i|}{C(\varepsilon,z)}\right)\|\mathbb{1}_J(\mathscr{L})u\|^2\,,$$

which entails (b).

(c) This statement follows readily from (a) and (b).

(d) We have $\|\mathscr{L}_{z,\varepsilon}u\| \geqslant 2^{-\frac{1}{2}}\min(D_1(\varepsilon,z),D_2(\varepsilon,z))\|u\|$ by (b) and (c), hence $\mathscr{L}_{z,\varepsilon}$ is injective with closed range. Since the same is true for its adjoint $\mathscr{L}_{\overline{z},-\varepsilon}$, the operator $\mathscr{L}_{z,\varepsilon}$ is bijective and $\|\mathscr{L}_{z,\varepsilon}^{-1}\| \leqslant \frac{\sqrt{2}}{\min(D_1(\varepsilon,z),D_2(\varepsilon,z))}$.

Next, we have

$$\|(\mathscr{L}+i)\mathbb{1}_J(\mathscr{L})u\| \leqslant \sup_J|\ell+i|\|\mathbb{1}_J(\mathscr{L})u\| \leqslant D_1^{-1}(\varepsilon,z)\sup_J|\ell+i|\|\mathscr{L}_{z,\varepsilon}u\|\,,$$

from (b). Putting this together with (c) we obtain that

$$\|\mathscr{L}_{z,\varepsilon}u\| \geqslant \frac{1}{\sqrt{2}}\min\left(\frac{D_1(\varepsilon,z)}{\sup_J|\ell+i|},D_2(\varepsilon,z)\right)\|(\mathscr{L}+i)u\|\,.$$

(e) By combining the identity

$$-\mathrm{Im}\,\langle\mathscr{L}_{z,\varepsilon}u,u\rangle+\mathrm{Im}\,\langle\mathscr{L}_{z,\varepsilon}u,\mathbb{1}_{J^c}(\mathscr{L})u\rangle=-\mathrm{Im}\,\langle\mathscr{L}_{z,\varepsilon}u,\mathbb{1}_J(\mathscr{L})u\rangle\,,$$

with (4.6), we get that

$$\begin{aligned}&-\mathrm{Im}\,\langle\mathscr{L}_{z,\varepsilon}u,u\rangle+\mathrm{Im}\,\langle\mathscr{L}_{z,\varepsilon}u,\mathbb{1}_{J^c}(\mathscr{L})u\rangle\\&\quad\geqslant\left(c_0\varepsilon-\frac{(c_1\varepsilon)^2\sup_J|\ell+i|}{C(\varepsilon,z)}\right)\|\mathbb{1}_J(\mathscr{L})u\|^2-C(\varepsilon,z)^{-1}c_1\varepsilon\|\mathbb{1}_J(\mathscr{L})u\|\|\mathscr{L}_{z,\varepsilon}u\|\,,\end{aligned}$$

and consequently

$$\begin{aligned}&|\langle\mathscr{L}_{z,\varepsilon}u,u\rangle|+\|\mathscr{L}_{z,\varepsilon}u\|\|\mathbb{1}_{J^c}(\mathscr{L})u\|+C(\varepsilon,z)^{-1}c_1\varepsilon\|\mathbb{1}_J(\mathscr{L})u\|\|\mathscr{L}_{z,\varepsilon}u\|\\&\qquad\geqslant\left(c_0\varepsilon-\frac{(c_1\varepsilon)^2\sup_J|\ell+i|}{C(\varepsilon,z)}\right)\|\mathbb{1}_J(\mathscr{L})u\|^2\,.\end{aligned}$$

From this, (b) and (c), it then follows that

$$\begin{aligned}&|\langle\mathscr{L}_{z,\varepsilon}u,u\rangle|+\left(D_2(\varepsilon,z)^{-1}+D_1(\varepsilon,z)^{-1}C(\varepsilon,z)c_1\varepsilon\right)\|\mathscr{L}_{z,\varepsilon}u\|^2\\&\qquad\geqslant\left(c_0\varepsilon-\frac{(c_1\varepsilon)^2\sup_J|\ell+i|}{C(\varepsilon,z)}\right)\|\mathbb{1}_J(\mathscr{L})u\|^2\,.\end{aligned}$$

Next, with reference to (c) we may add $\|\mathscr{L}_{z,\varepsilon}u\|^2$ on the left-hand-side of the above estimate and $D_2(\varepsilon,z)^2\|\mathbb{1}_{J^c}(\mathscr{L})u\|^2$ on its right-hand-side. We obtain that

$$\begin{aligned}&|\langle\mathscr{L}_{z,\varepsilon}u,u\rangle|+\left(1+D_2(\varepsilon,z)^{-1}+D_1(\varepsilon,z)^{-1}C(\varepsilon,z)c_1\varepsilon\right)\|\mathscr{L}_{z,\varepsilon}u\|^2\\&\qquad\geqslant\left(c_0\varepsilon-\frac{(c_1\varepsilon)^2\sup_J|\ell+i|}{C(\varepsilon,z)}\right)\|\mathbb{1}_J(\mathscr{L})u\|^2+D_2(\varepsilon,z)^2\|\mathbb{1}_{J^c}(\mathscr{L})u\|^2\,.\end{aligned}$$

As a consequence we have for all $\varepsilon\in(0,\varepsilon_2]$,

$$|\langle\mathscr{L}_{z,\varepsilon}u,u\rangle|+\left(1+D_2(\varepsilon,z)^{-1}+D_1(\varepsilon,z)^{-1}C(\varepsilon,z)c_1\varepsilon\right)\|\mathscr{L}_{z,\varepsilon}u\|^2\geqslant\frac{c_0\varepsilon}{2}\|u\|^2\,.$$

Bearing in mind that $C(\epsilon,z)\in(0,1)$, this entails that

$$|\langle\mathscr{L}_{z,\varepsilon}u,u\rangle|+\left(1+\frac{2\left(1+\frac{4c_1}{c_0}\sup_J|\ell+i|\right)}{\left(\frac{\sup_B|\ell+i|}{\mathrm{dist}(I,J^c)}+1\right)^{-1}}+\frac{4c_1}{c_0}\right)\|\mathscr{L}_{z,\varepsilon}u\|^2\geqslant\frac{c_0\varepsilon}{2}\|u\|^2\,,$$

which yields the desired result.

□

Having established Lemma 4.2, we can now state the following technical result.

Lemma 4.3. *For all bounded self-adjoint operator $\mathscr{C}$, we have*

$$\|\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C}\| \leqslant \left(\frac{1}{\tilde{c}_0\varepsilon}\right)^{\frac{1}{2}} (\|\mathscr{C}\| + \|\mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C}\|^{\frac{1}{2}})$$

and

$$\|\mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}\| \leqslant \left(\frac{1}{\tilde{c}_0\varepsilon}\right)^{\frac{1}{2}} (\|\mathscr{C}\| + \|\mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C}\|^{\frac{1}{2}}) .$$

Proof. Taking $u = \mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C}\varphi$ in (e), we obtain the first estimate. The second one follows from this and the fact that $\mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}$ is the adjoint of $\mathscr{L}_{\overline{z},-\varepsilon}^{-1}\mathscr{C}$. □

Armed with Lemma 4.3, we are in position to prove the main result of this section.

Theorem 4.4 (Limiting Absorption Principle)**.** *Let $\mathscr{L}$ and $\mathscr{A}$ fulfill the conditions of Section 4.1. Then, for any bounded self-adjoint operator $\mathscr{C}$ such that $\mathscr{C}\mathscr{A}$ and $\mathscr{A}\mathscr{C}$ are bounded, it holds true for all $\varepsilon \in (0, \min(1, \varepsilon_0)]$ that*

$$\sup_{\operatorname{Im} z>0\,,\operatorname{Re} z\in I} \|\mathscr{C}(\mathscr{L} - z - i\varepsilon\mathscr{B})^{-1}\mathscr{C}\| \leq C,$$

where

$$C := C(\varepsilon_0) + (K_1 + K_2)(1 + C(\varepsilon_0)^{\frac{1}{2}}) \int_0^1 \frac{\mathrm{d}t}{t^{\frac{1}{2}}} + \sqrt{2}K^{\frac{1}{2}}(K_1 + K_2) \int_0^1 \frac{|\ln(t)|^{\frac{1}{2}}}{t^{\frac{1}{2}}}\mathrm{d}t,$$

$$K_1 := \frac{2}{\sqrt{\tilde{c}_0}} \max(\|\mathscr{C}\mathscr{A}\|, \|\mathscr{A}\mathscr{C}\|)\,, \quad K_2 := \frac{4\sqrt{2}c_2 \sup_J |\ell + i|}{c_0}\|\mathscr{C}\|\,,$$

$$K := (K_1 + K_2)\, \|\mathscr{C}\| \left(1 + \frac{2\sup_J |\ell + i|^{\frac{1}{2}}}{\sqrt{c_0}}\right)$$

and $C(\varepsilon_0)$ is a positive constant satisfying

$$C(\varepsilon_0) \leqslant \frac{4\sqrt{2}\sup_J |\ell + i|}{c_0}\|\mathscr{C}\|^2\varepsilon_0^{-1} .$$

Moreover, we have

$$\sup_{\operatorname{Im} z>0\,,\operatorname{Re} z\in I} \|\mathscr{C}(\mathscr{L} - z)^{-1}\mathscr{C}\| \leqslant C .$$

Proof. Let us differentiate $F(\varepsilon) := \mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C}$ w.r.t. ε. We obtain that

$$\begin{aligned} F'(\varepsilon) &= \mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}[\mathscr{L}, \mathscr{A}]\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C} \\ &= \mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}[\mathscr{L}_{z,\varepsilon}, \mathscr{A}]\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C} - \varepsilon\mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}[[\mathscr{L}, \mathscr{A}], \mathscr{A}]\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C} \\ &= \mathscr{C}\mathscr{A}\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C} - \mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{A}\mathscr{C} - \varepsilon\mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}[[\mathscr{L}, \mathscr{A}], \mathscr{A}]\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C}. \end{aligned}$$

Bearing in mind that $\mathscr{A}\mathscr{C}$ and $\mathscr{C}\mathscr{A}$ are bounded, we refer to (B), (d) and Lemma 4.3, and deduce from the above estimate that

$$\begin{aligned} \|F'(\varepsilon)\| &\leqslant \|\mathscr{C}\mathscr{A}\|\|\mathscr{L}_{z,\varepsilon}^{-1}\mathscr{C}\| + \|\mathscr{A}\mathscr{C}\|\|\mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}\| + c_2\varepsilon\|\mathscr{C}\|\|\mathscr{C}\mathscr{L}_{z,\varepsilon}^{-1}\|\|(\mathscr{L} + i)\mathscr{L}_{z,\varepsilon}^{-1}\| \\ &\leqslant \left(\frac{2}{\sqrt{\tilde{c}_0\varepsilon}} \max(\|\mathscr{C}\mathscr{A}\|, \|\mathscr{A}\mathscr{C}\|) + c_2\varepsilon\|\mathscr{C}\|D_3(\varepsilon, z)^{-1}\right) (\|\mathscr{C}\| + \|F\|^{\frac{1}{2}}) \\ &\leqslant \left(K_1\varepsilon^{-\frac{1}{2}} + K_2\right) (\|\mathscr{C}\| + \|F\|^{\frac{1}{2}}) . \end{aligned} \tag{4.7}$$

Further, since $\|F(\varepsilon)\| \leqslant \|\mathscr{C}\|^2 D_3(\varepsilon,z)^{-1} \leqslant \frac{4\sup_J|\ell+i|}{c_0}\|\mathscr{C}\|^2\varepsilon^{-1}$ for all $\varepsilon \in (0,\varepsilon_0]$, by (d), we infer from (4.7) upon possibly substituting 1 for ε_0, that

$$\begin{aligned}\|F'(\varepsilon)\| &\leqslant \left(K_1\varepsilon^{-\frac{1}{2}}+K_2\right)\|\mathscr{C}\|\left(1+\varepsilon^{-\frac{1}{2}}\frac{2\sup_J|\ell+i|^{\frac{1}{2}}}{\sqrt{c_0}}\right)\\ &\leqslant (K_1+K_2)\,\|\mathscr{C}\|\left(1+\frac{2\sup_J|\ell+i|^{\frac{1}{2}}}{\sqrt{c_0}}\right)\varepsilon^{-1}.\end{aligned}$$

Integrating the above estimate over $(\varepsilon,\varepsilon_0)$ then yields

$$\|F(\varepsilon)\| \leqslant K|\ln(\varepsilon)| + \|F(\varepsilon_0)\|\,, \varepsilon \in (0,\varepsilon_0].$$

Plugging this into (4.7), we obtain that

$$\|F'(\varepsilon)\| \leqslant (K_1+K_2)\varepsilon^{-\frac{1}{2}}\left(\|\mathscr{C}\| + K^{\frac{1}{2}}|\ln(\varepsilon)|^{\frac{1}{2}} + \|F(\varepsilon_0)\|^{\frac{1}{2}}\right),$$

which, upon integrating over $(\varepsilon,\varepsilon_0)$, leads to

$$\|F(\varepsilon)\| \leq \|F(\varepsilon_0)\| + (K_1+K_2)\left(\left(\|\mathscr{C}\| + \|F(\varepsilon_0)\|^{\frac{1}{2}}\right)\int_0^1 \frac{\mathrm{d}t}{t^{\frac{1}{2}}} + K^{\frac{1}{2}}\int_0^1 \frac{|\ln t|^{\frac{1}{2}}}{t^{\frac{1}{2}}}\mathrm{d}t\right).$$

This and the estimate

$$\begin{aligned}\|F(\varepsilon_0)\| &= \|\mathscr{C}(\mathscr{L}+i)^{-1}(\mathscr{L}+i)\mathscr{L}_{z,\varepsilon_0}^{-1}\mathscr{C}\|\\ &\leq \|(\mathscr{L}+i)\mathscr{L}_{z,\varepsilon_0}^{-1}\|\|\mathscr{C}\|^2\\ &\leq \frac{4\sqrt{2}\sup_J|\ell+i|}{c_0}\|\mathscr{C}\|^2\varepsilon_0^{-1},\end{aligned}$$

arising from (d), yield the desired result with $C(\varepsilon_0) = \|F(\varepsilon_0)\|$.

□

Acknowledgments. Nicolas Raymond and Éric Soccorsi are deeply grateful to the CIRM where this work was completed. Éric Soccorsi is partially supported by the Agence Nationale de la Recherche (ANR) under grant ANR-17-CE40-0029.

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